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Research Article

A Study on the *k*-Mersenne and *k*-Mersenne-Lucas Sequences

Engin Özkan[1*], Bayram Şen[2], Hakan Akkuş[3] and Mine Uysal[4]

[1]*Department of Mathematics, Faculty of Sciences, Marmara University, İstanbul, Türkiye*
[2,3]*Department of Mathematics, Graduate School of Natural and Applied Sciences, Erzincan Binali Yıldırım University, Erzincan, Türkiye*
[4]*Department of Mathematics, Faculty of Sciences and Arts, Erzincan Binali Yıldırım University, Erzincan, Türkiye*
[*]*Corresponding author*

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Abstract

In this study, we examine an application of $k-$Mersenne and $k-$Mersenne-Lucas sequences. We present the Catalan transforms of these sequences and give the properties of these Catalan transforms. Catalan transforms of $k-$Mersenne and $k-$Mersenne-Lucas sequences have terms according to different values of k, and some of them are associated with the sequences in OEIS. We obtain the generating functions of the Catalan transforms of $k-$Mersenne and $k-$Mersenne-Lucas sequences. Moreover, we apply the Hankel transform to the Catalan transforms of these sequences. Finally, the determinants of the created matrices are calculated by applying the Hankel transform to the terms of the Catalan transforms of these sequences.

1. Introduction

Number sequences constitute an important subject area in mathematics. These sequences usually consist of consecutive numbers that follow a certain rule. The Fibonacci sequence is one of the most well-known and frequently encountered number sequences. These sequences have intrigued scientists for a long time. Fibonacci and Lucas numbers and many of their applications have been studied and many properties of these sequences have been investigated [1]. New sequences are obtained by changing the recurrence relation and initial conditions of the generalized Fibonacci sequence.
In [2], Falcon defined $k-$Fibonacci sequence. This paper is an extension of the work of Falcon [3]. In [3], Falcon gave an application of the Catalan transform to the k-Fibonacci sequences.
Mersenne and Mersenne-Lucas numbers are a generalization of these numbers, similar to the Fibonacci sequence. Many researchers have studied Mersenne number sequences and their different applications have been investigated [4–17].

Definition 1.1. *The Mersenne numbers M_n are defined by the following recurrence relation for $n \in \mathbb{N}$,*

$$M_{n+2} = 3M_{n+1} - 2M_n$$

with the initial conditions $M_0 = 0$, $M_1 = 1$ [16].

Definition 1.2. *The Mersenne-Lucas numbers m_n are defined by the following recurrence relation for $n \in \mathbb{N}$*

$$m_{n+2} = 3m_{n+1} - 2m_n,$$

with the initial conditions $m_0 = 2$, $m_1 = 3$, [17].

Definition 1.3. *[16] The Binet formula for Mersenne numbers is defined by*

$$M_n = 2^n - 1.$$

Definition 1.4. *[17] The Binet formula for Mersenne-Lucas numbers is defined by*

$$m_n = 2^n + 1.$$

Email addresses and ORCID numbers: **engin.ozkan@marmara.edu.tr, 0000-0002-4188-7248 (E.Özkan), bayram.s29@hotmail.com, 0009-0009-5570-4981 (B.Şen), hakan.akkus@ogr.ebyu.edu.tr, 0000-0001-9716-9424 (H. Akkuş), mine.uysal@erzincan.edu.tr, 0000-0002-2362-3097 (M. Uysal)**
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Catalan and Hankel transforms of some number sequences have been investigated by some authors and their properties have been examined [18–24].

In this paper, the Catalan transforms of these sequences are examined in detail. We give the generating functions for the Catalan transformations of the sequences. Finally, the Hankel transforms of the newly obtained sequences and the determinants of these transforms are discussed.

Definition 1.5. *Let k be any positive number, then the $k-$Mersenne sequence is recursively given for $n \in \mathbb{N}$ as*

$$M_{k,n+2} = 3kM_{k,n+1} - 2M_{k,n},$$

where $M_{k,0} = 0$, $M_{k,1} = 1$. From now on we will show the sequence as $M_{k,n}$. When $k = 1$, Mersenne sequence is obtained [16]. The characteristic equation of the sequence is as follows

$$x^2 - 3kx + 2 = 0.$$

Its characteristic roots are

$$x_1 = \frac{3k + \sqrt{9k^2 - 8}}{2} \text{ and}$$

$$x_2 = \frac{3k - \sqrt{9k^2 - 8}}{2}.$$

Characteristic roots prove the following properties to be true.

$$x_1 + x_2 = 3k,$$
$$x_1 x_2 = 2,$$
$$x_1 - x_2 = \sqrt{9k^2 - 8}.$$

Binet's formula for $M_{k,n}$ is

$$M_{k,n} = \frac{{x_1}^n - {x_2}^n}{x_1 - x_2}.$$

Definition 1.6. *Let k be any positive number, the $k-$Mersenne-Lucas number sequence is recursively given for $n \in \mathbb{N}$ as*

$$m_{k,n+2} = 3km_{k,n+1} - 2m_{k,n},$$

where $m_{k,0} = 2, m_{k,1} = 3k$ [9].From now on we will show the sequence as $m_{k,n}$. When $k = 1$, Mersenne-Lucas sequence is obtained. The characteristic equation of the sequence is as follows

$$x^2 - 3kx + 2 = 0.$$

Its characteristic roots are

$$x_1 = \frac{3k + \sqrt{9k^2 - 8}}{2}$$

and

$$x_2 = \frac{3k - \sqrt{9k^2 - 8}}{2}.$$

Binet's formula for $m_{k,n}$ is

$$m_{k,n} = {x_1}^n + {x_2}^n.$$

2. Catalan Transforms for the $k-$Mersenne and $k-$Mersenne Lucas Sequences

In this section, we present the Catalan transform of these sequences and give the properties of these Catalan transforms. Catalan transforms of k-Mersenne and k-Mersenne-Lucas sequences have terms according to different values of k, and some of them are associated with the sequences in OEIS.

Definition 2.1. *The n^{th} Catalan number is introduced by Barry [8]. For $n \geq 0$, it is*

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

or

$$C_n = \frac{2n!}{(n+1)!n!}.$$

The Catalan number generating function is

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

Some Catalan numbers are in the order $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots$.

Using the definition of Catalan numbers, the Catalan transform of these sequences can be defined as follows.

Definition 2.2. *Let $k>0, n>0$ and $CM_{k,0}=0$;*

$$CM_{k,n}=\sum_{i=0}^{n}\frac{i}{2n-i}\binom{2n-i}{n-i}M_{k,i}$$

equation is called Catalan transforms in $k-$Mersenne sequences. Based on this definition, the terms of the Catalan transforms of the $k-$Mersenne sequence are as follows.

$$\begin{aligned}
CM_{k,1}&=\sum_{i=0}^{1}\frac{i}{2-i}\binom{2-i}{1-i}M_{k,i}=\binom{1}{0}M_{k,1}=1,\\
CM_{k,2}&=\sum_{i=0}^{2}\frac{i}{4-i}\binom{4-i}{2-i}M_{k,i}=\frac{1}{3}\binom{3}{1}M_{k,1}+\binom{2}{0}M_{k,2}=3k+1,\\
CM_{k,3}&=\sum_{i=0}^{3}\frac{i}{6-i}\binom{6-i}{3-i}M_{k,i}=9k^2+6k,\\
CM_{k,4}&=\sum_{i=0}^{4}\frac{i}{8-i}\binom{8-i}{4-i}M_{k,i}=27k^3+27k^2+3k-1,\\
CM_{k,5}&=\sum_{i=0}^{5}\frac{i}{10-i}\binom{10-i}{5-i}M_{k,i}=81k^4+108k^3+27k^2-6k,\\
CM_{k,6}&=\sum_{i=0}^{6}\frac{i}{12-i}\binom{12-i}{6-i}M_{k,i}=243k^5+405k^4+162k^3-18k^2-6k+6,\\
CM_{k,7}&=\sum_{i=0}^{7}\frac{i}{14-i}\binom{14-i}{7-i}M_{k,i}=729k^6+1458k^5+2025k^4-864k^2+36k+84.
\end{aligned}$$

We can write the found $CM_{k,n}$ sequences as the product of the $nx1$-dimensional $M_{k,n}$ matrix and the lower triangular matrix C.

$$\begin{bmatrix} CM_{k,1}\\ CM_{k,2}\\ CM_{k,3}\\ CM_{k,4}\\ CM_{k,5}\\ CM_{k,6}\\ \vdots \end{bmatrix}=\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 5 & 5 & 3 & 1 & & \\ 14 & 14 & 9 & 4 & 1 & \\ 42 & 42 & 28 & 14 & 5 & 1\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}\begin{bmatrix} M_{k,1}\\ M_{k,2}\\ M_{k,3}\\ M_{k,4}\\ M_{k,5}\\ M_{k,6}\\ \vdots \end{bmatrix}$$

In this case, the 1^{st} and 2^{nd} columns of the C matrix are filled with Catalan numbers, and the elements of this matrix validate a recurrence relation. The Table 2.1 shows the coefficients of the terms of the Catalan $k-$Mersenne sequence, which we call the Catalan triangle.

$CM_{k,1}$	1					
$CM_{k,2}$	3	1				
$CM_{k,3}$	9	6				
$CM_{k,4}$	27	27	3	-1		
$CM_{k,5}$	81	108	27	-6		
$CM_{k,6}$	243	405	162	-18	-6	6
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2.1: Catalan triangle of $k-$Mersenne sequence

The results of some Catalan transforms are presented below.

$CM_1=\{0,\ 1,\ 4,\ 15,\ 56,\ 210,\ 792,\ \ldots\}$ It is indexed as A001791 in OEIS.

$CM_2=\{0, 1,\ 7,\ 48,\ 329,\ 2256,\ 15474,\ \ldots\}$

$CM_3=\{0,\ 1,\ 10,\ 99,\ 980,\ 9702,\ 96054,\ \ldots\}$

$CM_4=\{0,\ 1,\ 13,\ 168,\ 2171,\ 28056,\ 362524,\ \ldots\}$

$CM_5=\{0,\ 1,\ 16,\ 255,\ 4064,\ 64770,\ 1032276,\ \ldots\}$

$CM_6=\{0,\ 1,\ 19,\ 360,\ 6821,\ 129240,\ 2464882,\ \ldots\}$

Definition 2.3. *Let $k>0, n>0$ and $Cm_{k,0}=0$;*

$$Cm_{k,n} = \sum_{i=0}^{n} \frac{i}{2n-i}\binom{2n-i}{n-i} m_{k,i}$$

equation is called Catalan transform in k-Mersenne-Lucas sequences. Based on this definition, the Catalan transform of the k-Mersenne-Lucas sequence are found below.

$$\begin{aligned}
Cm_{k,1} &= \sum_{i=0}^{1} \frac{i}{2-i}\binom{2-i}{1-i} m_{k,i} = \binom{1}{0} m_{k,1} = 3k, \\
Cm_{k,2} &= \sum_{i=0}^{2} \frac{i}{4-i}\binom{4-i}{2-i} m_{k,i} = \frac{1}{3}\binom{3}{1} m_{k,1} + \binom{2}{0} m_{k,2} = 9k^2+3k-4, \\
Cm_{k,3} &= \sum_{i=0}^{3} \frac{i}{6-i}\binom{6-i}{3-i} m_{k,i} = 27k^3+18k^2-12k-8, \\
Cm_{k,4} &= \sum_{i=0}^{4} \frac{i}{8-i}\binom{8-i}{4-i} m_{k,i} = 81k^4+81k^3-27k^2-39k-12, \\
Cm_{k,5} &= \sum_{i=0}^{5} \frac{i}{10-i}\binom{10-i}{5-i} m_{k,i} = 243k^5+324k^4-27k^3-162k^2-60k-24, \\
Cm_{k,6} &= \sum_{i=0}^{6} \frac{i}{12-i}\binom{12-i}{6-i} m_{k,i} = 729k^6+1215k^5+162k^4-594k^3-306k^2-78k-72, \\
Cm_{k,7} &= \sum_{i=0}^{7} \frac{i}{14-i}\binom{14-i}{7-i} m_{k,i} = 2187k^7+4374k^6+1458k^5-1944k^4-1458k^3-324k^2-192k-240.
\end{aligned}$$

We can write the found $Cm_{k,n}$ sequences as the product of the $nx1$-dimensional $m_{k,n}$ matrix and the lower triangular matrix C.

$$\begin{bmatrix} Cm_{k,1} \\ Cm_{k,2} \\ Cm_{k,3} \\ Cm_{k,4} \\ Cm_{k,5} \\ Cm_{k,6} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 5 & 5 & 3 & 1 & & \\ 14 & 14 & 9 & 4 & 1 & \\ 42 & 42 & 28 & 14 & 5 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} m_{k,1} \\ m_{k,2} \\ m_{k,3} \\ m_{k,4} \\ m_{k,5} \\ m_{k,6} \\ \vdots \end{bmatrix}$$

In this case, the 1^{st} and 2^{nd} columns of the C matrix are filled with Catalan numbers, and the elements of this matrix validate a recurrence relation. The Table 2.2 shows the coefficients of the terms of the Catalan transform of the k-Mersenne sequence, which we call the Catalan triangle.

$Cm_{k,1}$	3						
$Cm_{k,2}$	9	3	-4				
$Cm_{k,3}$	27	18	-12	-8			
$Cm_{k,4}$	81	81	-27	-39	-12		
$Cm_{k,5}$	243	324	-27	-162	-60	-24	
$Cm_{k,6}$	729	1215	162	-594	-306	-78	-72
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2.2: Catalan triangle of $k-$Mersenne-Lucas sequence

3. The Generating Functions of the Catalan Transform of the $k-$ Mersenne and $k-$Mersenne-Lucas Sequences

In this section, we obtain the generating functions of the Catalan transforms of k-Mersenne and k-Mersenne-Lucas sequences.

Theorem 3.1. *The generating function of the k-Mersenne sequence is*

$$M(x) = \frac{x}{2x^2-3kx+1}$$

[9].

Theorem 3.2. *The generating function of the Catalan transform of the k-Mersenne sequences is*

$$T(x)=\frac{1-\sqrt{1-4x}}{4-4x-3k+(3k-2)\sqrt{1-4x}}.$$

Proof. The generating function of k-Mersenne and Catalan sequences are as follows, respectively,

$$M(x)=\frac{x}{2x^2-3kx+1}$$

and

$$C(x)=\frac{1-\sqrt{1-4x}}{2x}$$

If this information and the properties of the composition process in the functions are used, the desired result is achieved. Thus, the following equations for the generating function of the Catalan transform of the k-Mersenne sequence are obtained.

$$\begin{aligned}
T(x)=M(x)*C=\ T(x*C(x)) &= \frac{\frac{1-\sqrt{1-4x}}{2}}{2\left(\frac{1-\sqrt{1-4x}}{2}\right)^2-\frac{3k\left(1-\sqrt{1-4x}\right)}{2}+1}\\
&= \frac{\frac{1-\sqrt{1-4x}}{2}}{\frac{1-2\sqrt{1-4x}+1-4x}{2}-\frac{3k-3k\sqrt{1-4x}}{2}+1}\\
&= \frac{\frac{1-\sqrt{1-4x}}{2}}{\frac{1-2\sqrt{1-4x}+1-4x-3k+\ 3k\sqrt{1-4x}\ +\ 2}{2}}\\
&= \frac{1-\sqrt{1-4x}}{4-4x-3k+(3k-2)\sqrt{1-4x}}.
\end{aligned}$$

□

Theorem 3.3. *The generating function of the $k-$Mersenne-Lucas sequence is*

$$m(x)=\frac{2-3kx}{2x^2-3kx+1}$$

Theorem 3.4. *The generating function of the Catalan transform of the k-Mersenne-Lucas number is*

$$L(x)=\frac{4-3k+3k\sqrt{1-4x}}{4-4x-3k+(3k-2)\sqrt{1-4x}}.$$

Proof. The generating function of k-Mersenne-Lucas and Catalan sequences are as follows respectively,

$$m(x)=\frac{2-3kx}{2x^2-3kx+1}$$

and

$$C(x)=\frac{1-\sqrt{1-4x}}{2x}.$$

Thus, the following equations for the generating function of the Catalan transform of the k-Mersenne-Lucas sequence is obtained.

$$\begin{aligned}
L(x)=m(x)*C=m(x*C(x)) &= \frac{2-3k\left(\frac{1-\sqrt{1-4x}}{2}\right)}{2\left(\frac{1-\sqrt{1-4x}}{2}\right)^2-\frac{3k\left(1-\sqrt{1-4x}\right)}{2}+1}\\
&= \frac{2-3k\left(\frac{1-\sqrt{1-4x}}{2}\right)}{\frac{1-2\sqrt{1-4x}+1-4x}{2}-\frac{3k-3k\sqrt{1-4x}}{2}+1}\\
&= \frac{\frac{4-3k+3k\sqrt{1-4x}}{2}}{\frac{1-2\sqrt{1-4x}+1-4x-3k+\ 3k\sqrt{1-4x}+2}{2}}\\
&= \frac{4-3k+3k\sqrt{1-4x}}{4-4x-3k+(3k-2)\sqrt{1-4x}}.
\end{aligned}$$

□

4. Hankel Transform

In this section, we apply the Hankel transform to the Catalan transforms of these sequences. In addition, the determinants of the created matrices are calculated by applying the Hankel transform to the terms of the Catalan transforms of these sequences.

Definition 4.1. *Let $R = (r_1, r_2, r_3, r_4, \ldots)$ be a sequence of real numbers. The Hankel transform of R is the sequence of determinants* $H_{\mathrm{n}} = \det\left[\mathrm{r}_{\mathrm{i+j-2}}\right]$ *. That is,*

$$H_n = \begin{vmatrix} r_0 & r_1 & r_2 & r_3 & \cdots \\ r_1 & r_2 & r_3 & r_4 & \cdots \\ r_2 & r_3 & r_4 & r_5 & \cdots \\ r_3 & r_4 & r_5 & r_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

the n^{th} order Hankel determinant of R is the upper left nxn subdeterminant of H.

Based on this definition, we apply the Hankel transform to the Catalan transforms of $k-$Mersenne and $k-$ Mersenne-Lucas sequences, respectively, as follows.

$$HCM_{k,1} = |1| = 1,$$

$$HCM_{k,2} = \begin{vmatrix} 1 & 3k+1 \\ 3k+1 & 9k^2+6k \end{vmatrix} = -1,$$

$$HCM_{k,3} = -1,$$

and

$$HCm_{k,1} = det\,|3k| = 3k,$$

$$HCm_{k,2} = \begin{vmatrix} 3k & 9k^2+3k-4 \\ 9k^2+3k-4 & 27k^3+18k^2-12k-8 \end{vmatrix} = 27k^2-16,$$

$$HCm_{k,3} = 270k^4+297k^3-304k^2-240k-128.$$

5. Conclusion

In this study, we presented the Catalan transform of $k-$Mersenne and $k-$Mersenne-Lucas sequences and obtained the generating functions of the Catalan transform of these sequences. Catalan transform of $k-$Mersenne and $k-$ Mersenne-Lucas sequences have terms according to different values of k, and some of them are associated with the sequences in OEIS. Moreover, we applied the Hankel transform to the Catalan transforms of $k-$Mersenne and $k-$Mersenne-Lucas sequences. Finally, the determinants of the created matrices are calculated by applying the Hankel transform to the terms of the Catalan transform of the $k-$Mersenne and $k-$Mersenne-Lucas sequences. If this study is examined, the same results can be found for other sequences. For example, the Catalan and Hankel transforms of Perrin and Padovan sequences.

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