http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 74, Number 1, Pages 92–102 (2025)
https://doi.org/10.31801/cfsuasmas.1566336
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: October 14, 2024; Accepted: December 25, 2024

Approximation properties of convolution operators via statistical convergence based on a power series

Ramazan DİNAR[1] and Tuğba YURDAKADİM[2]
[1,2]Bilecik Şeyh Edebali University, Department of Mathematics, Bilecik 11230, TÜRKİYE

Abstract. In this study, our main goal is to obtain approximation properties of convolution operators for multivariables via a special method which is not included in any other methods given before, also known as P-statistical convergence. We present the P-statistical rate of this approximation and provide examples of convolution operators. It is noteworthy to express that one can not approximate f by earlier results for our examples. Therefore, our results fill an important gap in the existing literature. Furthermore, we also present a P-statistical approximation result in the space of periodic continuous functions of period 2π, for short C^*.

2020 Mathematics Subject Classification. Primary 40A35; Secondary 41A25, 41A35, 40C05.
Keywords. Power series method, statistical convergence, convolution operators, Korovkin type approximation, rate of convergence

1. Introduction and Background

In the theory of analysis, dealing with the approximation of a given function by other functions which are good and simple is an important problem. Consider a set consisting of functions with bad properties and a subset $Y \subset X$ where Y is a dense subset with good properties. Then one can write any function f of X as a limit of functions of Y. Actually this problem has been studied by Weierstrass [27] and he has shown that every continuous real valued function on $[a, b]$ can be written as a limit of polynomials, i.e., the set of all algebraic polynomials constructs a dense subset of $C[a, b]$. The proof of this theorem is long and hard to follow. Therefore giving a simpler alternative proof to this theorem turns out to be an attractive aim for other mathematicians. Bernstein [5] is the first who has given the simplest proof by using Bernstein polynomials. Then Bohman [6], Korovkin [19] and Popoviciu [20] have extended this by positive and linear operators, independently. The effect of positive and linear operators is well-known in approximation theory, functional analysis, statistics, computer engineering and image processing. One of the important class of such operators is convolution operators. Besides this, the limits used are classical limits in the setting of this theory. But if the classical limit fails, what can be done? The main goal of using summability theory is to still give a limit to a divergent sequence. Since it is effective in such cases, Gadjiev and Orhan [17] have combined summability and approximation theories. Then many results of approximation theory have been extended by statistical convergence, summation process, ideal convergence, power series, indeed general summability methods [2], [3], [11], [12], [22], [25]. In 2003, Srivastava and Gupta obtained approximation properties of operators of some summation-integral forms using classical convergence [23]. Later, in 2008, Duman obtained the approximation properties of convolution operators of integral form using A-statistical convergence defined by an infinite matrix $A = (a_{jn})$ instead of classical convergence [12]. Many mathematicians have also studied them with the use of different types of convergence in both single and multivariable cases, and they are still being researched nowadays. For example, in 2017, Atlıhan, Yurdakadim and Taş investigated the approximation

[1] ramazan.dinar13@gmail.com; 0009-0006-2095-2086
[2] tugba.yurdakadim@bilecik.edu.tr-Corresponding author; 0000-0003-2522-6092.

properties of convolution operators in the multivariable case by using summation process [2]. In 2022, Çınar and Yıldız studied these operators via P-statistical summation process [8]. In this paper, our goal is to obtain approximation results for convolution operators both for one variable and for multivariables via P-statistical convergence defined by a power series. We have already known that neither statistical convergence nor P-statistical convergence implies each other [25]. We also present the P-statistical rate of this approximation. Furthermore we provide examples as an application of our results. Following the similar technique used here, we present a P-statistical approximation result in C^*, consisting of 2π periodic and continuous functions on $\mathbb{R}$.
Before recalling the basic concepts it is noteworthy to mention that approximation theory, studying positive linear operators, relaxing the positivity, linearity and assigning a limit when the classical limit fails is important since these results have applications in image processing, computer engineering, physics, statistics, computer aided geometric design, deep learning and 3D-modelling.

Now we pause to collect basic concepts which are the main tools of our study.
If

$$\delta(G) := \lim_k \frac{1}{k} \#\{n \leq k : n \in G\}$$

exists then it is called as the density of $G \subseteq \mathbb{N}$ where $\#$ is the number of the elements of enclosed set and $\mathbb{N}$ is natural numbers. If $\delta(G_\varepsilon) = 0$ for all $\varepsilon > 0$ where $G_\varepsilon = \{n \in \mathbb{N} : |u_n - l| \geq \varepsilon\}$ then it is said that $u = (u_n)$ statistically converges to l [14], [16], [21]. Let (p_n) be a real number sequence such that $p_1 > 0$, $p_n \geq 0$ for $n \geq 2$ and $p(t) := \sum_{n=1}^{\infty} p_n t^{n-1}$ with radius of convergence $R \in (0, \infty]$. Then we define the method of power series by the following:

Let

$$C_P := \left\{ f : (-R, R) \to \mathbb{R} | \lim_{0<t\to R^-} \frac{f(t)}{p(t)} \quad \text{exists} \right\},$$

$$C_{P_p} := \left\{ u = (u_n) | p_u(t) := \sum_{n=1}^{\infty} p_n t^{n-1} u_n \quad \text{with radius of convergence} \geq R \quad \text{and} \quad p_u \in C_p \right\}$$

and

$$P - \lim u = \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{n=1}^{\infty} p_n t^{n-1} u_n.$$

Here $P - \lim$ is a functional from C_{P_p} to $\mathbb{R}$ for short P and we say that u is $P-$convergent [7], [18].

The regularity of this method, i.e, if $P - \lim u$ is the same as the limit of u for every convergent sequence $u = (u_n)$, equals to

$$\lim_{t\to R^-} \frac{p_n t^{n-1}}{p(t)} = 0$$

for every $n \in \mathbb{N}$ [7].

Then combining the statistical convergence and power series, a novel concept of convergence known as P-statistical convergence has been introduced in [25] and some results have been investigated by this concept [9], [26].
Now we are ready to recall this concept of convergence.
If

$$\delta_P(G) := \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{n\in G} p_n t^{n-1}$$

exists then it is called as P-density of G where P is regular. It can be easily seen that $\delta_P(G) \in [0, 1]$ provided that it exists [25]. If $\delta_P(G_\varepsilon) = 0$ for every $\varepsilon > 0$, i.e,

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{n\in G_\varepsilon} p_n t^{n-1} = 0$$

then we say that u P-statistically converges to l and denote by $st_P - \lim u = l$ where P is regular and u is a real number sequence [25].

Also, it is already known that $\left(C[a,b], ||.||_{C[a,b]}\right)$ is a Banach space
where

$$||f|| := ||f||_{C[a,b]} = \sup_{x\in[a,b]} |f(x)|, \quad f \in C[a,b].$$

The following operators of convolution type for one variable:

$$T_n(f;x) = \int_a^b f(y)K_n(y-x)\,dy, \quad n \in \mathbb{N}, \quad x \in [a,b] \tag{1}$$

where $f \in C[a,b], a < b, a, b \in \mathbb{R}$ have been considered via P-statistical summation process in [8] .
One can easily see that T_n are linear, also suppose that K_n are continuous functions on $[a-b, b-a]$ and $K_n(u) \geq 0$ for every $u \in [a-b, b-a]$, for every $n \in \mathbb{N}$. Hence, T_n given by (1) are positive and linear. If one takes $a_{kj}^{(n)} = I$, identity matrix, for all $n \in \mathbb{N}$ in [8], the following results can be obtained immediately. For the completeness, we find it useful to recall the following:

Theorem 1. *[25] Let P be regular, L_n be positive and linear operators for each $n \in \mathbb{N}$ on $C[0,1]$ and $e_i(y) = y^i$, $i = 0,1,2$.*
Then

$$st_P - \lim_n ||L_n(e_i) - e_i|| = 0,$$

$i = 0,1,2$ *implies that*

$$st_P - \lim_n ||L_n(f) - f|| = 0$$

for any $f \in C[0,1]$.

It is worth for mentioning that this theorem is the P-statistical version of the well-known theorem given by Gadjiev and Orhan in 2002 [17].
Indeed, this result have been obtained in [22] as follows since $L_n(\varphi;x) = L_n(e_2;x) - 2xL_n(e_1;x) + x^2L_n(e_0;x)$ provided that L_n are positive and linear for every $n \in \mathbb{N}$ where $\varphi(y) := (y-x)^2$ for every $x \in [a,b]$.

Lemma 1. *[22] If*

$$st_P - \lim_n ||L_n(e_0) - e_0|| = 0$$

and

$$st_P - \lim_n ||L_n(\varphi)|| = 0$$

then

$$st_P - \lim_n ||L_n(f) - f|| = 0$$

holds for all $f \in C[a,b]$, where P is regular and L_n are positive and linear operators for each $n \in \mathbb{N}$.

Theorem 2. *[8] Let P be regular and (T_n) be given by (1). If*

$$st_P - \lim_n \int_{-\delta}^{\delta} K_n(y)\,dy = 1$$

and

$$st_P - \lim_n \Big(\sup_{|y|\geq\delta} K_n(y)\Big) = 0$$

hold for a fixed $\delta \in (0, \frac{b-a}{2})$ then

$$st_P - \lim_n ||T_n(f) - f||_\delta = 0$$

holds for every $f \in C[a,b]$ where

$$||f||_\delta := \sup_{a+\delta\leq x\leq b-\delta} |f(x)|.$$

Now, we provide examples such that the earlier results can not be used but we still have the opportunity to approximate f by the above results.

Example 1. *Define the sequences* (p_n) *and* (u_n) *as follows:*

$$p_n = \begin{cases} 1 & , \quad n = 2k \\ 0 & , \quad n = 2k+1 \end{cases} , \qquad u_n = \begin{cases} 0 & , \quad n = 2k \\ 1 & , \quad n = 2k+1 \end{cases} .$$

Notice that P *is regular and* $st_P - \lim u_n = 0$.
Then let T_n *on* $C[a,b]$ *be constructed by*

$$T_n(f;x) = \frac{n(1+u_n)}{\sqrt{\pi}} \int_a^b f(y) e^{-n^2(y-x)^2}\, dy. \tag{2}$$

Here

$$K_n(y) = \frac{n(1+u_n)}{\sqrt{\pi}} e^{-n^2 y^2}$$

and T_n *defined by (2) is a convolution operator.*

Notice that Theorem 2.4 and Corollary 2.5 of [12] can not be applied for K_n since (u_n) is neither convergent nor statistically convergent. However, Theorem 2 can be applied to obtain that

$$st_P - \lim_n ||T_n(f) - f||_\delta = 0$$

for every $f \in C[a,b]$ and for a fixed $0 < \delta < \frac{b-a}{2}$.
The above example is standard to give but here, we construct another example which is extraordinary and is motivated by a result in [10].

Example 2. *Let* (u_n) *and the method* P *be constructed as below:*

$$u_n = \begin{cases} 0 & , \quad n = 2k \\ 1 & , \quad n = 2k+1 \end{cases} , \qquad p_n = \begin{cases} 0 & , \quad n = 2k \\ 1 & , \quad n = 2k+1 \end{cases} .$$

Notice that P *is regular and* (u_n) *is* P*-statistically convergent to 1. Then define* T_n *on* $C\left[\frac{-1}{2}, \frac{1}{2}\right]$ *as follows:*

$$T_n(f;x) = u_n \int_{\frac{-1}{2}}^{\frac{1}{2}} f(y) \lambda_n(y-x) dy = u_n L_n(f;x) \text{ with } \lambda_n(y) = c_n (1-y^2)^n,$$

$f \in C\left[\frac{-1}{2}, \frac{1}{2}\right]$ *and* (c_n) *chosen such that* $\int_{-1}^{1} \lambda_n(y) dy = 1.$
Since (u_n) *is neither convergent nor statistically convergent, one can not approximate* f *by the earlier theorems in the classical or statistical settings. But we still have the opportunity to approximate* f *since* (u_n) *is* P*-statistically convergent to 1 by using Theorem 2 and the uniform convergence of* $L_n(f;x)$ *to* f *on* $[\frac{-1}{2} + \delta, \frac{1}{2} + \delta]$ *for each* $0 < \delta < \frac{1}{2}$ *which is also known from [10].*

Furthermore, the rate of this approximation can be given as follows with the use of modulus of continuity and the concept of P-statistical convergence with the rate $o(a_n)$.
The other main tool of this study is P-statistical rate and it was introduced in [1] in light of [15] in 2023.

Definition 1. *[1] Let* (a_n) *be a non-increasing, positive real number sequence and* P *be regular. If*

$$\lim_{0<t\to R^-} \left[\frac{1}{p(t)} \sum_{n:|s_n - l| \geq \varepsilon a_n} p_n t^n \right] = 0$$

is true for every $\varepsilon > 0$ *then we say that* $s = (s_n)$ *is* P*-statistically convergent to* l *with the rate* $o(a_n)$ *and we denote by* $s_n - l = st_P - o(a_n)$, $(n \to \infty)$.

Here it is noteworthy to mention that the terms of the sequence (s_n) are controlling the rate.

Theorem 3. *Let* P *be regular and* (T_n) *be given by (1). Suppose also that* (a_n), (b_n) *are non-increasing sequences of positive numbers and* $0 < \delta < \frac{b-a}{2}$ *be fixed.*
If

$$||T_n(e_0) - e_0||_\delta = st_P - o(a_n), \quad (n \to \infty),$$

and

$$\omega(f, \lambda_n) = st_P - o(b_n), \quad (n \to \infty),$$

then

$$||T_n(f) - f||_\delta = st_P - o(\gamma_n), \quad (n \to \infty)$$

holds for all $f \in C[a,b]$. *Here* $\lambda_n := \sqrt{||T_n(\varphi)||_\delta}$ *and* $\gamma_n := \max\{a_n, b_n, a_n b_n\}$.

Proof. Since it is already shown that there exists $L > 0$ such that

$$||T_n(f) - f||_\delta \leq L\Big\{\omega(f, \lambda_n) + \omega(f, \lambda_n)||T_n(e_0) - e_0||_\delta + ||T_n(e_0) - e_0||_\delta\Big\}$$

holds for all $n \in \mathbb{N}$, we immediately obtain that

$$||T_n(f) - f||_\delta = st_P - o(\gamma_n)$$

where $\gamma_n := \max\{a_n, b_n, a_n b_n\}$. This gives the desired result. □

2. P-statistical approximation in C^*

Here, we present an approximation result of convolution type operators in the space of periodic functions of period 2π and continuous on $\mathbb{R}$, for short C^* by P-statistical convergence.
It is beneficial to recall $||f||_{C^*} = \sup_{x\in\mathbb{R}} |f(x)|$ is the standard norm on C^*.
Construct T_n for $f \in C^*$ and for each $n \in \mathbb{N}$ as follows:

$$T_n(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) K_n(y - x) dy \tag{3}$$

where $K_n \in C^*$, $K_n(y) \geq 0$ for any $y \in [-\pi, \pi]$. Hence K_n is nonnegative on $\mathbb{R}$. Following a similar way as in earlier results, we can also conclude the next theorem.

Theorem 4. *Let* P *be regular and* (T_n) *be defined by (3). If*

$$\delta_P\left(\{n \in \mathbb{N} : \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y)\, dy = 1\}\right) = 1$$

and

$$st_P - \lim_n \left(\sup_{|y|\geq\delta} K_n(y)\right) = 0$$

for any $\delta > 0$ *then we get*

$$st_P - \lim_n ||T_n(f) - f||_{C^*} = 0$$

for any $f \in C^*$.

3. P-Statistical Approximation Of Convolution Operators For Multivariables

Korovkin type approximation theorems have been investigated in real m-dimensional space by statistical convergence and summation process in [4], [13]. In this section, we examine the approximation properties of the below convolution operators for multivariables:

$$T_n(f; x, y) = \int_c^d \int_a^b f(u, v) K_n(u - x, v - y)\, du dv \tag{4}$$

where $(x, y) \in J := [a, b] \times [c, d]$, $f \in C(J)$ and $C(J) = \{f | f : J \longrightarrow \mathbb{R} \quad \textit{continuous}\}$ with the norm $||f|| := \sup_{(x,y)\in J} |f(x, y)|$.
For the positivity we suppose for all $n \in \mathbb{N}$ that $K_n(t, z)$ are continuous and $K_n(t, z) \geq 0$ on $[a - b, b - a] \times [c - d, d - c]$. Hence, $T_n : C(J) \longrightarrow C(J)$ given by (4) are positive and linear operators.
Unfortunately, one variable is insufficient to give a model of real world problems therefore considering multivariable cases in approximation theory has great importance.
First, let us prove the following lemmas which lead us to our main theorem.

Lemma 2. *Let P be regular and $T_n : C(J) \longrightarrow C(J)$ be positive and linear operators for every $n \in \mathbb{N}$. If*

$$st_P - \lim_n ||T_n(f_i) - f_i|| = 0 \quad for \quad i = 0, 1, 2, 3$$

where $f_0(u, v) = 1,\ f_1(u, v) = u,\ f_2(u, v) = v,\ f_3(u, v) = u^2 + v^2$ then

$$st_P - \lim_n ||T_n(f) - f|| = 0$$

holds for all $f \in C(J)$.

Proof. Since $f \in C(J)$, we have $\delta > 0$ for every $\varepsilon > 0$ satisfying that $|f(u, v) - f(x, y)| < \varepsilon$ for every $(u, v) \in J$ such that $|u - x| < \delta$ and $|v - y| < \delta$. Then we can write that

$$\begin{aligned} |f(u, v) - f(x, y)| =& |f(u, v) - f(x, y)|\chi_{J_\delta}(u, v) + |f(u, v) - f(x, y)|\chi_{J \setminus J_\delta}(u, v) \\ \leq& \varepsilon + 2H\chi_{J\setminus J_\delta}(u, v) \end{aligned}$$

where χ_J is the characteristic function of J, $J_\delta = [x - \delta, x + \delta] \times [y - \delta, y + \delta] \cap J$ and $H := ||f||$. Also

$$\chi_{J\setminus J_\delta}(u, v) \leq \frac{(u - x)^2}{\delta^2} + \frac{(v - y)^2}{\delta^2}$$

holds and by combining the above inequalities, we have that

$$|f(u, v) - f(x, y)| \leq \varepsilon + \frac{2H}{\delta^2}\Big\{(u - x)^2 + (v - y)^2\Big\}$$

for every u, v, x, y.

Since T_n are positive and linear for all $n \in \mathbb{N}$, we can also obtain that

$$\begin{aligned} |T_n(f; x, y) - f(x, y)| \leq& T_n\Big(|f(u, v) - f(x, y)|; x, y\Big) \\ &+ |f(x, y)||T_n(f_0; x, y) - f_0(x, y)| \end{aligned}$$

and

$$\begin{aligned} |T_n(f; x, y) - f(x, y)| \leq& \varepsilon + \Big(\varepsilon + H + \frac{(h_1^2 + h_2^2)2H}{\delta^2}|T_n(f_0; x, y) - f_0(x, y)|\Big) \\ &+ \frac{4h_1 H}{\delta^2}|T_n(f_1; x, y) - f_1(x, y)| \\ &+ \frac{4h_2 H}{\delta^2}|T_n(f_2; x, y) - f_2(x, y)| \\ &+ \frac{2H}{\delta^2}|T_n(f_3; x, y) - f_3(x, y)| \end{aligned}$$

where $h_1 = \max\{|a|, |b|\}$, $h_2 = \max\{|c|, |d|\}$.

Thus by taking supremum over J, we have that

$$||T_n(f) - f|| \leq \varepsilon + K\Big\{||T_n(f_0) - f_0|| + ||T_n(f_1) - f_1|| + ||T_n(f_2) - f_2|| + ||T_n(f_3) - f_3||\Big\} \tag{5}$$

where

$$K := \max\Big\{\varepsilon + H + \frac{(h_1^2 + h_2^2)2H}{\delta^2}, \frac{4h_1 H}{\delta^2}, \frac{4h_2 H}{\delta^2}, \frac{2h}{\delta^2}\Big\}.$$

For a given $r > 0$ pick $\varepsilon > 0$ such that $\varepsilon < r$ and define the followings:

$$F = \{n : ||T_n(f) - f|| \geq r\}$$

$$F_1 = \{n : ||T_n(f_0) - f_0|| \geq \frac{r - \varepsilon}{4K}\}$$

$$F_2 = \{n : ||T_n(f_1) - f_1|| \geq \frac{r - \varepsilon}{4K}\}$$

$$F_3 = \{n : ||T_n(f_2) - f_2|| \geq \frac{r - \varepsilon}{4K}\}$$

$$F_4 = \{n : ||T_n(f_3) - f_3|| \geq \frac{r - \varepsilon}{4K}\}.$$

It is easy to notice that $F \subseteq F_1 \cup F_2 \cup F_3 \cup F_4$ by (5).
Then

$$\frac{1}{p(t)}\sum_{n\in F} p_n t^{n-1} \leq \frac{1}{p(t)}\Big\{\sum_{n\in F_1} p_n t^{n-1} + \sum_{n\in F_2} p_n t^{n-1} + \sum_{n\in F_3} p_n t^{n-1} + \sum_{n\in F_4} p_n t^{n-1}\Big\}$$

holds and by taking limit in both sides we obtain that

$$\lim_{t\to R^-} \frac{1}{p(t)}\sum_{n\in F} p_n t^{n-1} = 0$$

then we complete the proof. □

By setting $\Gamma(u,v) = (u-x)^2 + (v-y)^2$, one can immediately get the following via a slight modification.

Lemma 3. *Let P be regular and $T_n : C(J) \longrightarrow C(J)$ be positive and linear operators for every $n \in \mathbb{N}$. If*

$$st_P - \lim_n ||T_n(f_0) - f_0|| = 0$$

and

$$st_P - \lim_n ||T_n(\Gamma)|| = 0$$

then we have

$$st_P - \lim_n ||T_n(f) - f|| = 0$$

for all $f \in C(J)$.

Let

$$||f||_\delta = \sup_{a+\delta\leq x\leq b-\delta,\, c+\delta\leq y\leq d-\delta} |f(x,y)|$$

where $0 < \delta < \min\{\frac{b-a}{2}, \frac{d-c}{2}\}$, $f \in C(J)$ and also let $B_\gamma := [a-b, b-a]\times[c-d, d-c]\backslash[-\gamma,\gamma]\times[-\gamma,\gamma]$ for any $\gamma > 0$ satisfying $\gamma < \min\{b-a, d-c\}$ along the paper.

Lemma 4. *Let P be regular, $\delta \in (0, \min\{\frac{b-a}{2}, \frac{d-c}{2}\})$ be fixed and consider the operators T_n given by (4). If*

$$st_P - \lim_n \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_n(u,v)\, dudv = 1$$

and

$$st_P - \lim_n \Big(\sup_{(u,v)\in B_\gamma} K_n(u,v)\Big) = 0$$

for any $\gamma > 0$ then

$$st_P - \lim_n ||T_n(f_0) - f_0||_\delta = 0$$

holds.

Proof. Let $\delta \in (0, \min\{\frac{b-a}{2}, \frac{d-c}{2}\})$ be fixed and $(x,y) \in [a+\delta, b-\delta]\times[c+\delta, d-\delta]$.
We have that

$$\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_n(u,v)\, dudv \leq T_n(f_0; x,y) \leq \int_{-(d-c)}^{d-c}\int_{-(b-a)}^{b-a} K_n(u,v)\, dudv$$

and

$$||T_n(f_0) - f_0||_\delta \leq v_n$$

where

$$v_n := \max\Big\{\Big|\int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_n(u,v)\, dudv - 1\Big|, \Big|\int_{-(d-c)}^{d-c}\int_{-(b-a)}^{b-a} K_n(u,v)\, dudv - 1\Big|\Big\}.$$

By the hypothesis we get $st_P - \lim_n v_n = 0$.
We also have that

$$F := \{n : ||T_n(f_0) - f_0||_\delta \geq \varepsilon\} \subseteq \{n : v_n \geq \varepsilon\} =: F'$$

for a given $\varepsilon > 0$.
Then

$$\frac{1}{p(t)}\sum_{n\in F} p_n t^{n-1} \le \frac{1}{p(t)}\sum_{n\in F'} p_n t^{n-1}$$

holds and by taking limit we obtain

$$\lim_{t\to R^-}\frac{1}{p(t)}\sum_{n\in F} p_n t^{n-1} = 0$$

which means that

$$st_P - \lim_n ||T_n(f_0) - f_0||_\delta = 0.$$

□

Lemma 5. *Let P be regular, $\delta \in (0, \min\{\frac{b-a}{2}, \frac{d-c}{2}\})$ be fixed and consider the operators T_n given by (4). If*

$$st_P - \lim_n \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_n(u,v)\, dudv = 1$$

and

$$st_P - \lim_n \Big(\sup_{(u,v)\in B_\gamma} K_n(u,v)\Big) = 0$$

for any $\gamma > 0$ then we have

$$st_P - \lim_n ||T_n(\Gamma)||_\delta = 0.$$

Proof. Let $\delta \in (0, \min\{\frac{b-a}{2}, \frac{d-c}{2}\})$ be fixed and $(x,y) \in [a+\delta, b-\delta] \times [c+\delta, d-\delta]$. Since $\Gamma(u,v) = (u-x)^2 + (v-y)^2 \in C(J)$ we have that

$$\begin{aligned} T_n(\Gamma; x, y) &= \int_c^d \int_a^b [(u-x)^2 + (v-y)^2] K_n(u-x, v-y)\, dudv \\ &= \int_{c-y}^{d-y}\int_{a-x}^{b-x} (u^2+v^2) K_n(u,v)\, dudv \\ &\le \int_{-(d-c)}^{d-c}\int_{-(b-a)}^{b-a} (u^2+v^2) K_n(u,v)\, dudv \end{aligned}$$

for every $n \in \mathbb{N}$.
Since Γ is continuous at (0,0), for sufficiently small $\varepsilon > 0$ $(0 < \sqrt{\varepsilon} < \delta)$, $\Gamma(u,v) < 2\varepsilon$ holds whenever $|u| < \sqrt{\varepsilon}$, $|v| < \sqrt{\varepsilon}$.
Hence, we obtain that

$$\begin{aligned} T_n(\Gamma; x, y) =& 2\varepsilon \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}}\int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} K_n(u,v)\, dudv + \iint_{B_{\sqrt{\varepsilon}}} (u^2+v^2) K_n(u,v)\, du\, dv \\ \le& 2\varepsilon \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_n(u,v)\, dudv + \iint_{B_{\sqrt{\varepsilon}}} (u^2+v^2) K_n(u,v)\, du\, dv \\ \le& 2\varepsilon \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_n(u,v)\, dudv + \mathcal{R} \sup_{(u,v)\in B_{\sqrt{\varepsilon}}} K_n(u,v) \end{aligned}$$

where

$$\mathcal{R} = \int_{c-d}^{d-c}\int_{a-b}^{b-a} (u^2+v^2)\, dudv.$$

Following the similar ways in the earlier results and with the use of hypothesis, we conclude that

$$st_P - \lim_n ||T_n(\Gamma)||_\delta = 0.$$

□

Combining the above results, we can present the following approximation theorem for convolution operators in multivariable case.

Theorem 5. *Let P be regular, $\delta \in (0, \min\{\frac{b-a}{2}, \frac{d-c}{2}\})$ be fixed and the operators T_n given by (4). If*

$$st_P - \lim_n \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} K_n(u,v)\, dudv = 1$$

and

$$st_P - \lim_n \Big(\sup_{(u,v)\in B_\gamma} K_n(u,v) \Big) = 0$$

for any $\gamma > 0$ then

$$st_P - \lim_n ||T_n(f) - f||_\delta = 0$$

holds for every $f \in C(J)$.

Now we can reorganize our Example 1 for multivariable case:

Example 3. *Let $T_n : C(J) \longrightarrow C(J)$ be constructed by*

$$T_n(f;x,y) = n^2 \frac{(1+u_n)}{\pi} \int_c^d \int_a^b f(u,v) e^{-n^2(u-x)^2} e^{-n^2(v-y)^2}\, dudv$$

where (u_n) and (p_n) defined as in Example 1,

$$K_n(u,v) = n^2 \frac{(1+u_n)}{\pi} e^{-n^2u^2} e^{-n^2v^2}.$$

For every $\delta \in (0, \min\{\frac{b-a}{2}, \frac{d-c}{2}\})$, one can have that

$$\int_{-\delta}^{\delta} \int_{-\delta}^{\delta} K_n(u,v)\, dudv = n^2 \frac{(1+u_n)}{\pi} \Big\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-n^2u^2} e^{-n^2v^2}\, dudv - \iint_{(u,v)\in \mathcal{B}_\delta} e^{-n^2u^2} e^{-n^2v^2}\, du\, dv \Big\}.$$

Here $\mathcal{B}_\delta := \{(u,v) : |u| \geq \delta \quad or \quad |v| \geq \delta\}$. Since $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-n^2u^2} e^{-n^2v^2}\, dudv = \frac{\pi}{n^2} < \infty$, we have

$$\lim_n \iint_{(u,v)\in \mathcal{B}_\delta} e^{-n^2u^2} e^{-n^2v^2}\, du\, dv = 0$$

which implies that

$$st_P - \lim_n \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} K_n(u,v)\, dudv = 1.$$

One can also obtain that for any $\gamma > 0$

$$\sup_{(u,v)\in B_\gamma} K_n(u,v) \leq n^2 \frac{(1+u_n)}{\pi} \frac{1}{e^{n^2\gamma^2}}$$

and

$$\lim_n \frac{n^2}{e^{n^2\gamma^2}} = 0$$

which implies

$$st_P - \lim_n \Big(\sup_{(u,v)\in B_\gamma} K_n(u,v) \Big) = 0.$$

Therefore our theorem is satisfied for this example but the earlier results can not be applied since (u_n) is neither convergent nor statistically convergent.

In order to give the rate of this approximation we should recall full continuity modulus. Let $f : J \longrightarrow \mathbb{R}$ be continuous and $\lambda > 0$. The full continuity modulus of $f(x,y)$ is defined by

$$\omega(f,\lambda) = \max_{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \leq \lambda} |f(x_1,y_1) - f(x_2,y_2)|.$$

It is known that $\lim_{\lambda\to 0} \omega(f,\lambda) = 0$ and for any $\lambda > 0$, $\omega(f,\lambda\Upsilon) \leq ([\Upsilon]+1)\omega(f,\lambda)$ [24].

Theorem 6. *Let P be regular and T_n given by (4). Assume also that $(a_n), (b_n)$ are non-increasing sequences of positive real numbers and $0 < \delta < \min\{\frac{b-a}{2}, \frac{d-c}{2}\}$.*
If

$$||T_n(e_0) - e_0||_\delta = st_P - o(a_n), \quad (n \to \infty),$$

and

$$\omega(f, \lambda_n) = st_P - o(b_n), \quad (n \to \infty),$$

then we have that

$$||T_n(f) - f||_\delta = st_P - o(\gamma_n), \quad (n \to \infty)$$

for every $f \in C(J)$. Here $\lambda_n := \sqrt{||T_n((u-x)^2 + (v-y)^2; x, y)||_\delta}$ and $\gamma_n = \max\{a_n, b_n, a_n b_n\}$.

Proof. Let $0 < \delta < \min\{\frac{b-a}{2}, \frac{d-c}{2}\}$, $f \in C(J)$ and $(x, y) \in [a+\delta, b-\delta] \times [c+\delta, d-\delta]$.
For any $\lambda > 0$ we have that

$$\begin{aligned} |T_n(f; x, y) - f(x, y)| \leq & T_n(|f(u, v) - f(x, y)|; x, y) + |f(x, y)||T_n(f_0) - f_0| \\ \leq & \omega(f, \lambda) T_n\left(1 + \frac{(u-x)^2 + (v-y)^2}{\lambda^2}; x, y\right) + |f(x, y)||T_n(f_0) - f_0| \\ \leq & \omega(f, \lambda)\left\{T_n(f_0) + \frac{1}{\lambda^2} T_n((u-x)^2 + (v-y)^2; x, y)\right\} + |f(x, y)||T_n(f_0) - f_0| \end{aligned}$$

since T_n are positive and linear operators.
This implies for all $n \in \mathbb{N}$, that

$$||T_n(f) - f||_\delta \leq \omega(f, \lambda)\left\{||T_n(f_0)||_\delta + \frac{1}{\lambda^2}||T_n((u-x)^2 + (v-y)^2; x, y)||_\delta\right\} + H_1||T_n(f_0) - f_0||_\delta$$

where $H_1 := ||f||_\delta$. Now letting $\lambda = \lambda_n = \sqrt{||T_n((u-x)^2 + (v-y)^2; x, y)||_\delta}$,

$$\begin{aligned} ||T_n(f) - f||_\delta \leq & \omega(f, \lambda_n)\left\{||T_n(f_0)||_\delta + 1\right\} + H_1||T_n(f_0) - f_0||_\delta \\ \leq & 2\omega(f, \lambda_n) + \omega(f, \lambda_n)||T_n(f_0) - f_0||_\delta + H_1||T_n(f_0) - f_0||_\delta \end{aligned}$$

holds and also by letting $H := \max\{2, H_1\}$

$$||T_n(f) - f||_\delta \leq H\left\{\omega(f, \lambda_n) + \omega(f, \lambda_n)||T_n(f_0) - f_0||_\delta + ||T_n(f_0) - f_0||_\delta\right\}$$

for all $n \in \mathbb{N}$.
This implies

$$||T_n(f) - f||_\delta = st_P - o(\gamma_n)$$

where $\gamma_n = \max\{a_n, b_n, a_n b_n\}$. □

4. Concluding Remarks

The theory of approximation deals with the problem of expressing a given function by other functions which are good and simple. This problem goes back to Weierstrass and many mathematicians have studied on it after the well-known Weierstrass approximation theorem. Also after the simplest alternative proof of Bernstein to this theorem, Bohman, Korovkin and Popoviciu have extended this for positive and linear operators, independently. One of the important classes of such operators are convolution type operators and studying these operators as well as positive linear operators, relaxing positivity and linearity, assigning a limit when the classical limit fails have great importance since these type of results have applications in image processing, computer engineering, physics, statistics, computer aided geometric design, deep learning and 3D-modelling.

Here, we present approximation properties of convolution operators for multivariables via a special method called P-statistical convergence. It worths for mentioning that this method is not included in any other methods given before and unfortunately one variable is insufficient to give a model for real world problems. We also obtain the rate of this approximation and provide examples to support our results. Furthermore, an approximation result in the space of periodic functions of period 2π is presented by using similar techniques.

Author Contribution Statements The authors jointly worked on the results. Also they read and approved the final manuscript.

Declaration of Competing Interests The authors declare that there are no competing interests.

References

[1] Atlıhan, Ö. G., Yurdakadim, T., Taş, E., A new approach to approximation by positive linear operators in weighted spaces, *Ukrainian Mathematical Journal*, 74(11) (2023), 1447-1453. https://doi.org/10.1007/s11253-023-02161-2

[2] Atlıhan, Ö. G., Yurdakadim, T., Taş, E., Summation process of convolution operators for multivariables, *Sarajevo Journal of Mathematics*, 26(2) (2017), 207-216.

[3] Atlıhan, Ö. G., Yurdakadim, T., Taş, E., Korovkin type approximation theorems in weighted spaces via power series method,*Oper. Matrices*, 12(2) (2018), 529-535. https://doi.org/10.7153/oam-2018-12-32

[4] Atlıhan, Ö. G., Yurdakadim, T., Taş, E., Statistical approximation properties of convolution operators for multivariables, *AIP Conference Proceedings*, (2013), 1156-1159. https://doi.org/10.1063/1.4825713

[5] Bernstein, S. N., Demonstration du theoreme de Weierstrass fondee sur le calcul des probablites, *Communications of the Kharkov Mathematical Society*, 13 (1912), 1-2.

[6] Bohman, H., On approximation of continuous and of analytic functions, *Ark.Mat.*, 2 (1952), 43-56. https://doi.org/10.1007/BF02591381

[7] Boos, J., Classical and Modern Methods in Summability, *Oxford University Press*, UK., 2000. https://doi.org/10.1093/oso/9780198501657.001.0001

[8] Çınar, S., Yıldız, S., P-statistical summation process of sequences of convolution operators, *Indian Journal of Pure and Applied Mathematics*, 53(3) (2022), 648-659. https://doi.org/10.1007/s13226-021-00156-y

[9] Demirkale, S., Taş, E., Statistical convergence of spliced sequences in terms of power series on topological spaces, *Mathematical Sciences and Applications E-Notes*, 11(2) (2023), 104-111. https://doi.org/10.36753/mathenot.1212331

[10] DeVore, R. A., Lorentz, G. G., Constructive Approximation, *Springer-Verlag*, 1993.

[11] Duman, O., Khan, M. K., Orhan, C., A-statistical convergence of approximating operators,*Math. Inequal. Appl.*, 6 (2003), 689-699. https://doi.org/10.7153/mia-06-62

[12] Duman, O., A-statistical convergence of sequences of convolution operators, *Taiwanese Journal of Mathematics*, 12(2) (2008), 523-536. https://doi.org/10.11650/twjm/1500574172

[13] Erkus, E., Duman, O., A Korovkin type approximation theorem in statistical sense, *Studia Sci. Math. Hungar.*, 43(3) (2006), 285-294. https://doi.org/10.1556/sscmath.43.2006.3.2

[14] Fast, H., Sur la convergence statistique, *Colloq. Math.*, 2 (1951), 241-244.

[15] Fridy, J. A., Miller, H. I., Orhan, C., Statistical rates of convergence, *Acta Sci. Math.*, 69 (2003), 147-157. (2003).

[16] Fridy, J. A., On statistical convergence, *Analysis*, 5 (1985), 301-313. https://doi.org/10.1524/anly.1985.5.4.301

[17] Gadjiev, A. D., Orhan, C., Some approximation theorems via statistical convergence,*Rocky Mountain Journal of Math.*, 32 (2002), 129-137. https://doi.org/10.1216/rmjm/1030539612

[18] Kratz, W., Stadtmüller, U., Tauberian theorems for J_p-summability, *J. Math. Anal. Appl.*, 139 (1989), 362-371.

[19] Korovkin, P. P., On convergence of linear positive operators in the space of continuous functions,*Doklady Akad. Nauk SSR.*, 90 (1953), 961-964.

[20] Popoviciu, T., Asupra demonstratiei teoremei lui Weierstrass cu ajutorul polinoamelor de interpolare,*Lucrarile Ses. Gen. Şt. Acad. Române din.*, (1950), 1664-1657.

[21] Šalát, T., On statistically convergent sequences of real numbers, *Mat. Slovaca.*, 30 (1980), 139-150.

[22] Soylemez, D., Ünver, M., Rates of power series statistical convergence of positive linear operators and power series statistical convergence of q-Meyer–Konig and Zeller operators, *Lobachevskii Journal of Mathematics*, 42(2) (2021), 426-434. https://doi.org/10.1134/S1995080221020189

[23] Srivastava, H. M., Gupta, V., A certain family of summation-integral type operators, *Math Comput. Modelling*, 37 (2003), 1307-1315.

[24] Tasdelen, F., Olgun, A., Tunca Başcanbaz, G., Approximation of functions of two variables by certain linear positive operators, *Proc. Indian Acad. Sci.*, 117(3) (2007), 387-399. https://doi.org/10.1007/s12044-007-0033-x

[25] Ünver, M., Orhan, C., Statistical convergence with respect to power series methods and applications to approximation theory, *Numer. Func. Anal. Opt.*, 40 (2019), 535-547. https://doi.org/10.1080/01630563.2018.1561467

[26] Yurdakadim, T., Taş, E., Effects of fuzzy settings in Korovkin theory via P p-statistical convergence, *Romanian Journal of Mathematics and Computer Science*, 2(12) (2022), 1-8.

[27] Weierstrass, K. G., Über die Analytische Dorstellbarkeit Sogenannter Willkürlicher Funktionen Einer Reellen, Veranderlichen. Sitzungsber, Akad. Berlin, 1885.