DOI: 10.38061/idunas.1566513

IDUNAS	NATURAL & APPLIED SCIENCES JOURNAL	2024 Vol. 7 No. 2 (61-68)

The Heat Problem with Non-Local Boundary Conditions

Research Article

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DOI: 10.38061/idunas.1566513

Received: 13.10.2024; Accepted: 12.11.2024

Abstract

In this article, the two-dimensional inverse nonlinear parabolic problem is discussed. The most important feature of the problem is its solution with the Fourier approach. The solution was obtained by Fourier implicit and iteration methods [1]-[6].

Keywords: Heat problem, Fourier iterative method, nonlocal conditions, implicit finite-difference methods.

1. INTRODUCTION

The inverse two-dimensional parabolic problem finds applications in many fields, including diffusion applications, heat transfer, population, medicine, electrochemistry, engineering, chemistry, plasma physics.

2. MATERIALS AND METHODS

To solve this inverse problem, a combination of the Fourier method, Picard successive approximations, and the finite difference method was employed.

3. THE EXISTENCE OF SOLUTIONS

$$\frac{\partial \upsilon}{\partial \tau} = \frac{\partial^2 \upsilon}{\partial \xi^2} + \sigma(\tau)\frac{\partial^2 \upsilon}{\partial \eta^2} + \theta\left(\xi, \eta, \tau, \upsilon\right), \tag{1}$$

$$\upsilon(\xi, \eta, 0) = \varphi(\xi, \eta), \xi \in \left[0, \pi\right], \eta \in \left[0, \pi\right] \tag{2}$$

$$\upsilon(0,\eta,\tau)=\upsilon(\pi,\eta,\tau),\eta\in[0,\pi],\tau\in[0,T]$$
$$\upsilon(\xi,0,\tau)=\upsilon(\xi,\pi,\tau),\xi\in[0,\pi],\tau\in[0,T]$$
$$\upsilon_{\xi}(0,\eta,\tau)=\upsilon_{\xi}(\pi,\eta,\tau),\eta\in[0,\pi],\tau\in[0,T] \tag{3}$$
$$\upsilon_{\eta}(\xi,0,\tau)=\upsilon_{\eta}(\xi,\pi,\tau),\xi\in[0,\pi],\tau\in[0,T]$$
$$\varsigma(t)=\int_0^{\pi}\int_0^{\pi}\xi\eta v(\xi,\eta,\tau)d\xi d\eta\,,\tau\in[0,T], \tag{4}$$

Here, (1) represents the inverse coefficient problem, (2)-(3) define the initial and periodic boundary conditions of the problem [8] and (4) is the integral condition for the inverse coefficient [7].

The following solution is obtained with the Fourier Method:

$$\upsilon(\xi,\eta,\tau)=\frac{1}{4}\left(\varphi_0+\frac{4}{\pi^2}\int_0^t\theta_0(\tau,\upsilon)d\tau\right)$$
$$+\sum_{m,n=1}^{\infty}\left(\varphi_{cmn}+\frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[(2m)^2+\sigma(\iota)(2n)^2\right]d\iota}\theta_{cmn}(\tau,\upsilon)d\tau\right)$$
$$\cos(2m\xi)\cos(2n\eta)$$
$$+\sum_{m,n=1}^{\infty}\left(\varphi_{csmn}+\frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[(2m)^2+\sigma(\iota)(2n)^2\right]d\iota}\theta_{csmn}(\tau,\upsilon)d\tau\right)$$
$$\cos(2m\xi)\sin(2n\eta)$$
$$+\sum_{m,n=1}^{\infty}\left(\varphi_{scmn}+\frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[(2m)^2+\sigma(\iota)(2n)^2\right]d\iota}\theta_{scmn}(\tau,\upsilon)d\tau\right)$$
$$\sin(2m\xi)\cos(2n\upsilon)$$
$$+\sum_{m,n=1}^{\infty}\left(\varphi_{smn}+\frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[(2m)^2+\sigma(\iota)(2n)^2\right]d\iota}\theta_{smn}(\tau,\upsilon)d\tau\right)$$
$$\sin(2m\xi)\sin(2n\eta)$$

Here,

$$\varphi_0 = \upsilon_0(0),$$

$$\varphi_{cmn} = \upsilon_{cmn}(0)e^{-\int_0^t \left[(2m)^2 + \sigma(\iota)(2n)^2\right]d\iota},$$

$$\varphi_{csmn} = \upsilon_{csmn}(0)e^{-\int_0^t \left[(2m)^2 + \sigma(\iota)(2n)^2\right]d\iota},$$

$$\varphi_{scmn} = \upsilon_{scmn}(0)e^{-\int_0^t \left[(2m)^2 + \sigma(\iota)(2n)^2\right]d\iota},$$

$$\varphi_{smn} = \upsilon_{smn}(0)e^{-\int_0^t \left[(2m)^2 + \sigma(\iota)(2n)^2\right]d\iota}.$$

Let the following rules apply:

(C1) $\varsigma(t) \in C^1[0,T]$
$\phi(\xi,\eta) \in C^{1,1}([0,\pi] \times [0,\pi])$,
(C2) $\phi(0,\eta) = \phi(\pi,\beta), \phi_\xi(0,\eta) = \phi_\xi(\pi,\eta), \int_0^\pi \int_0^\pi \xi\eta\phi(\xi,\eta)d\xi d\eta = \varsigma(0)$,
$\phi(\xi,0) = \phi(\xi,\pi), \phi_\eta(\xi,0) = \phi_\eta(\xi,\pi)$,
(C3) **(1)** Let $\theta(\xi,\eta,\tau,\upsilon)$ have the following properties:

$$\left|\frac{\partial\theta(\xi,\eta,\tau,\upsilon)}{\partial\xi} - \frac{\partial\theta(\xi,\eta,\tau,\bar{\upsilon})}{\partial\xi}\right| \le \chi(\xi,\eta,\tau)\left|\upsilon - \bar{\upsilon}\right|,$$

$$\left|\frac{\partial\theta(\xi,\eta,\tau,\upsilon)}{\partial\eta} - \frac{\partial\theta(\xi,\eta,\tau,\bar{\upsilon})}{\partial\eta}\right| \le \chi(\xi,\eta,\tau)\left|\upsilon - \bar{\upsilon}\right|,$$

$$\left|\frac{\partial\theta(\xi,\eta,\tau,\upsilon)}{\partial\eta\partial\xi} - \frac{\partial\theta(\xi,\eta,\tau,\bar{\upsilon})}{\partial\xi\partial\eta}\right| \le \chi(\xi,\eta,\tau)\left|\upsilon - \bar{\upsilon}\right|,$$

where $\chi(\xi,\eta,\tau) \in L_2(D), \chi(\xi,\eta,\tau) \ge 0$.

(2) $\theta(\xi,\eta,\tau,\upsilon) \in C^{2,2,0}[0,\pi],\ \tau \in 0,T]$,

$$\int_0^\pi\int_0^\pi \xi\eta\upsilon_t(\xi,\eta,\tau)d\xi d\eta = \varsigma^{'}(\tau), 0 \le t \le T.$$

$$\sigma(\tau) = \frac{\varsigma^{'}(\tau) - \int_0^\pi\int_0^\pi \xi\eta\theta(\xi,\eta,\tau,\upsilon)d\xi d\eta - \frac{\pi^3}{2}\upsilon_\eta(\pi,\tau)}{\frac{\pi^3}{2}\upsilon_\xi(\pi,t)}.$$

Definition 1.

$\{\upsilon(\tau)\} = \{\upsilon_0(\tau), \upsilon_{c\alpha\beta}(\tau), \upsilon_{cs\alpha\beta}(\tau), \upsilon_{sc\alpha\beta}(\tau), \upsilon_{s\alpha\beta}(\tau)\}$ of continuous functions on [0, T].

Banach norm:

$$\max_{0\le t\le T}\frac{|\upsilon_0(\tau)|}{4} + \sum_{\alpha,\beta=1}^{\infty}\begin{pmatrix}\max\limits_{0\le t\le T}|\upsilon_{c\alpha\beta}(\tau)| + \max\limits_{0\le t\le T}|\upsilon_{cs\alpha\beta}(\tau)| \\ +\max\limits_{0\le t\le T}|\upsilon_{sc\alpha\beta}(\tau)| + \max\limits_{0\le t\le T}|\upsilon_{s\alpha\beta}(\tau)|\end{pmatrix} < \infty .$$

$$\|\upsilon(\tau)\| = \max_{0\le t\le T}\frac{|\upsilon_0(\tau)|}{4} + \sum_{\alpha,\beta=1}^{\infty}\begin{pmatrix}\max\limits_{0\le t\le T}|\upsilon_{c\alpha\beta}(\tau)| + \max\limits_{0\le t\le T}|\upsilon_{cs\alpha\beta}(\tau)| \\ +\max\limits_{0\le t\le T}|\upsilon_{sc\alpha\beta}(\tau)| + \max\limits_{0\le t\le T}|\upsilon_{s\alpha\beta}(\tau)|\end{pmatrix}.$$

Theorem 1. The problem has a solution under conditions (C1) -(C3).

Proof. Let's iterate the coefficients:

$$\upsilon_0^{(N+1)}(\tau) = \varphi_0 + \frac{4}{\pi^2}\int_0^t \theta_0(\tau,\upsilon)d\tau$$

$$\upsilon_{cmn}^{(N+1)}(\tau) = \varphi_{cmn} + \frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[(2m)^2+\sigma(\iota)(2n)^2\right]d\iota}\theta_{cmn}(\tau,\upsilon)d\tau$$

$$\upsilon_{csmn}^{(N+1)}(\tau) = \varphi_{csmn} + \frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[(2m)^2+\sigma(\iota)(2n)^2\right]d\iota}\theta_{csmn}(\tau,\upsilon)d\tau$$

$$\upsilon_{scmn}^{(N+1)} = \varphi_{scmn} + \frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[(2m)^2+\sigma(\iota)(2n)^2\right]d\iota}\theta_{scmn}(\tau,\upsilon)d\tau$$

$$\upsilon_{smn}^{(N+1)} = \varphi_{smn} + \frac{4}{\pi^2}\int_0^t e^{-\int_\tau^t\left[(2m)^2+\sigma(\iota)(2n)^2\right]d\iota}\theta_{smn}(\tau,\upsilon)d\tau$$

According to the assumptions, we get $\upsilon_0^{(0)}(\tau) \in B$, $t\in [0, T]$. To find the approximation principle from 1 to N, Cauchy, Hölder, Bessel inequalities and Lipschitzs condition were employed, respectively. These tools are utilized to demonstrate the existence of the solution. Finally, we arrive at:

$$\left\|\upsilon^{(N+1)}(\tau)\right\| = \max_{0\le\tau\le T}\frac{\left\|\upsilon_{0}^{(N)}(\tau)\right\|}{4} +$$

$$\sum_{m,n=1}^{\infty}\max_{0\le\tau\le T}\left\|\upsilon_{cmn}^{(N)}(\tau)\right\| + \max_{0\le\tau\le T}\left\|\upsilon_{csmn}^{(N)}(\tau)\right\| + \max_{0\le\tau\le T}\left\|\upsilon_{scmn}^{(N)}(\tau)\right\| + \max_{0\le\tau\le T}\left\|\upsilon_{smn}^{(N)}(\tau)\right\|$$

$$\le \frac{\sqrt{9T\pi}+16\sqrt{T}}{3\pi}\left\|\gamma\right\|\left\|\upsilon^{(0)}(\tau)\right\| + \frac{\left(\sqrt{9T\pi}+16\sqrt{T}\right)M}{3\pi}$$

we get $\upsilon^{(N)}(\tau) \in B$.

$$\{\upsilon(\tau)\} = \left\{\upsilon_{0}^{(N)}(\tau), \upsilon_{cmn}^{(N)}(\tau), \upsilon_{csmn}^{(N)}(\tau), \upsilon_{scmn}^{(N)}(\tau), \upsilon_{smn}^{(N)}(\tau), m, n = 1, 2, \ldots\right\} \in B.$$

By using the same operations, we obtain:

$$\left\|\sigma^{(N+1)}(\tau)\right\| \le \frac{\pi}{2} + \frac{1}{\left\|\upsilon^{(N)}(\tau)\right\|}$$

we get $\sigma^{(N)}(\tau) \in B$.

Let us show that the convergence of solution:

$$\left\|\upsilon^{(N+1)}(\tau) - \upsilon^{(N+1)}(\tau)\right\|$$

$$\le \left(\frac{\sqrt{9T\pi}+16\sqrt{T}}{3\pi\sqrt{N!}}\right)^{(N)}\left\|\gamma\right\|^{(N)}\left\|\upsilon^{(N)}(\tau)\right\|$$

$$\left\|\sigma^{(N+1)}(\tau) - \sigma^{(N)}(\tau)\right\| \le \frac{\pi}{2} + \left(\frac{\sqrt{9T\pi}+16\sqrt{T}}{3\pi\sqrt{N!}}\right)^{(N)}\left\|\gamma\right\|^{(N)}\left\|\upsilon^{(N)}(\tau)\right\|$$

Then $\upsilon^{(N+1)}(\tau) \to \upsilon^{(N)}(\tau), \sigma^{(N+1)}(\tau) \to \sigma^{(N)}(\tau), N \to \infty.$

$$\left\|\upsilon^{(N+1)}(\tau) - \upsilon(\tau)\right\|$$

$$\le \left(\frac{\sqrt{9T\pi}+16\sqrt{T}}{3\pi\sqrt{N!}}\right)^{(N)}\left\|\gamma\right\|^{(N)} x \exp\left(\frac{\sqrt{9T\pi}+16\sqrt{T}}{3\pi}\right)^{2}\left\|\gamma\right\|^{2}$$

$$\upsilon^{(N+1)}(\tau) \to \upsilon(\tau), \sigma^{(N+1)}(\tau) \to \sigma(\tau), N \to \infty.$$

Applying Cauchy inequality, Hölder Inequality, Lipschitzs condition, and Bessel inequality to the difference, we obtain

$$\left\|\upsilon(\tau) - \nu(\tau)\right\| \le 0x \exp\left(\frac{\sqrt{9T\pi}+16\sqrt{T}}{3\pi}\right)^{2}\left\|\gamma\right\|^{2}$$

$\upsilon(\tau) = \nu(\tau), \sigma(\tau) = \rho(\tau)$. Thus, we proved the uniqueness of the solution.

4. THE EXAMPLE FOR NUMERICAL METHOD

The linearize of the problem:

$$\upsilon_t^{(n)} = \upsilon_{xx}^{(n)} + \sigma(\tau)\upsilon_{yy}^{(n)} + \theta(\xi,\eta,\tau,\upsilon^{(n-1)}),$$

$$\upsilon^{(n)}(\xi,\eta,0) = \varphi(\xi,\eta), \xi \in [0,\pi], \eta \in [0,\pi]$$

$$\upsilon^{(n)}(0,\eta,\tau) = \upsilon^{(n)}(\pi,\eta,\tau), \eta \in [0,\pi], \tau \in [0,T]$$
$$\upsilon^{(n)}(\xi,0,\tau) = \upsilon^{(n)}(\xi,\pi,\tau), \xi \in [0,\pi], \tau \in [0,T]$$
$$\upsilon_\xi^{(n)}(0,\eta,\tau) = \upsilon_\xi^{(n)}(\pi,\eta,\tau), \eta \in [0,\pi], \tau \in [0,T]$$
$$\upsilon_\eta^{(n)}(\xi,0,\tau) = \upsilon_\eta^{(n)}(\xi,\pi,\tau), \xi \in [0,\pi], \tau \in [0,T]$$

Let us $\upsilon^{(n)}(\xi,\eta,\tau) = \psi(\xi,\eta,\tau)$.

$$\psi_t = \psi_{\xi\xi} + \sigma(\tau)\psi_{\eta\eta} + \theta(\xi,\eta,\tau,\psi),$$

$$\upsilon^{(n)}(\xi,\eta,0) = \varphi(\xi,\eta), \xi \in [0,\pi], \eta \in [0,\pi]$$

$$\upsilon^{(n)}(0,\eta,\tau) = \upsilon^{(n)}(\pi,\eta,\tau), \eta \in [0,\pi], \tau \in [0,T]$$
$$\upsilon^{(n)}(\xi,0,\tau) = \upsilon^{(n)}(\xi,\pi,\tau), \xi \in [0,\pi], \tau \in [0,T]$$
$$\upsilon_\xi^{(n)}(0,\eta,\tau) = \upsilon_\xi^{(n)}(\pi,\eta,\tau), \eta \in [0,\pi], \tau \in [0,T]$$
$$\upsilon_\eta^{(n)}(\xi,0,\tau) = \upsilon_\eta^{(n)}(\xi,\pi,\tau), \xi \in [0,\pi], \tau \in [0,T]$$

$[0,\pi]^2 \times [0,T]$ is divided into a $M^2 \times N$ mesh with the step size $h = \pi/M, \ \ \tau = T/N$.

Using implicit finite-difference method for the problem:

$$\frac{1}{\tau}\left(\psi_{i,j}^{k+1} - \psi_{i,j}^{k}\right)$$
$$= \frac{1}{h^2}\begin{bmatrix} \left(\psi_{i-1,j}^{k+1} - 2\psi_{i,j}^{k+1} + \psi_{i+1,j}^{k+1}\right) \\ +\sigma^k\left(\psi_{i,j-1}^{k+1} - 2\psi_{i,j}^{k+1} + \psi_{i,j+1}^{k+1}\right) \end{bmatrix}$$
$$+\theta_{i,j}^{k+1},$$

$$\psi_{i,j}^0 = \varphi_i,$$

$$\psi_{0,j}^k = \psi_{M+1,j}^k, \psi_{M+1,j}^k = \frac{\psi_{1,j}^k - \psi_{M,j}^k}{2}$$

$$\psi_{i,0}^{k}=\psi_{i,M+1}^{k},\psi_{i,M+1}^{k}=\frac{\psi_{i,1}^{k}-\psi_{i,M}^{k}}{2}$$

$$\sigma(\tau)=\frac{\varsigma^{'}(\tau)-\int_0^\pi\int_0^\pi \xi\eta\,\theta(\xi,\eta,\tau)d\xi d\eta-\frac{\pi^3}{2}\psi_\eta(\pi,\tau)}{\frac{\pi^3}{2}\psi_\xi(\pi,\tau)}.$$

$$\sigma^{k+1}=-\frac{\left(\left(k^{k+2}-k^{k}\right)/\tau\right)}{\left(\frac{\pi^3}{2}\psi_\xi(\pi,\tau)\right)^{k}}$$

$$\frac{-\left(\int_0^\pi\int_0^\pi \xi\eta\theta(\alpha,\beta,\tau)d\xi d\eta\right)^{k}}{\left(\frac{\pi^3}{2}\psi_\xi(\pi,\tau)\right)^{k}}$$

$$-(\frac{\pi^3}{2}\psi_\xi(\pi,\tau))^{k}$$

where $k=0,1,...,N.$

From Simpson's scheme,

$$\frac{1}{\tau}\left(\psi_{i,j}^{k+1(s+1)}-\psi_{i,j}^{k+1(s)}\right)$$

$$=\frac{1}{h^2}\begin{bmatrix}\left(\psi_{i-1,j}^{k+1(s+1)}-2\psi_{i,j}^{k+1(s+1)}+\psi_{i+1,j}^{k+1(s+1)}\right)\\ +\sigma^{k(s+1)}\left(\psi_{i,j-1}^{k+1(s+1)}-2\psi_{i,j}^{k+1(s+1)}+\psi_{i,j+1}^{k+1(s+1)}\right)\end{bmatrix}$$

$$+\theta_{i,j}^{k+1},$$

$$\psi_{i,j}^{0}=\varphi_i,$$

$$\psi_{0,j}^{k+1(s)}=\psi_{M+1,j}^{k+1(s)},\psi_{M+1,j}^{k+1(s)}=\frac{\psi_{1,j}^{k+1(s)}-\psi_{M,j}^{k+1(s)}}{2}$$

$$\psi_{i,0}^{k+1(s)}=\psi_{i,M+1}^{k+1(s)},\psi_{i,M+1}^{k+1(s)}=\frac{\psi_{i,1}^{k+1(s)}-\psi_{i,M}^{k+1(s)}}{2}$$

$\psi_{i,j}^{k+1(s+1)}$ is found.

5. CONCLUSION

The inverse problem for the periodically bounded nonlinear two-dimensional parabolic equation was investigated. Two important investigations were carried out. These are determined by both theoretical and numerical examination. The Fourier method and finite difference method were used as methods.

6. ACKNOWLEDGMENTS

We would like to thank the reviewers.

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