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SILVER STRUCTURES ON THE RIEMANN EXTENSIONS

FILIZ OCAK AND ŞEMSI EKEN MERIÇ

0000-0003-4157-6404 and 0000-0003-2783-1149

Abstract. In the present paper we deal with an $n-$dimensional differentiable manifold M with a torsion-free linear connection ∇. Here we study some properties of a silver structure on the cotangent bundle T^*M equipped with the Riemannian extension ${}^R\nabla$ and obtain a necessary condition for which the silver semi-Riemannian manifold $\left(T^*M, {}^R\nabla, S\right)$ to be locally decomposable.

1. Introduction

The notion of metallic structure on Riemannian manifolds has been studied intesively recently. One of the most studied structure on Riemannian manifolds is silver structure. As a mathematical point of view, the positive solution of the equation

$$x^2 - px - q = 0,$$

for some positive integers p and q is called a $(p,q)-$ structure number which has the form

$$\mu_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}.$$

In particular case $p = 2$ and $q = 1$, we note that the last equality gives a silver ratio. In the recent years, the silver sturucture on the differentiable manifolds has been studied intensively in [4, 8, 9].

On the other hand, the cotangent bundle is the dual space of tangent bundle for a differentiable manifold which is very popular topic in Differential Geometry and Mathematical Physics. There are many different types of metrics on the cotangent bundle to study the geometric of such a bundle, for instance, Sasaki metric, Cheeger-Gromoll metric, general natural metrics, Oproius metrics, and etc. One of the most interesting metric is the Riemann extension which is defined by Patterson and Walker in [10]. Then, the notion of Riemann extension has been extensively studied by several authors on different smooth manifolds, for more [2, 3, 5-7, 12, 16].

In the present paper, we study some properties of a silver structure on the

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cotangent bundle equipped with the Riemannian extension. In Sect. 2, we recall some preliminaries on the details concerning the cotangent bundle. In Sect. 3, considering a silver structure on the cotangent bundle T^*M , we give some necessary conditions for which the triple $\left(T^*M, {}^R\nabla, S\right)$ is a locally decomposable silver semi-Riemannian manifold.

2. Preliminaries

In this section, we recall some basic notations about the cotangent bundle of [16].

Let (M, g) be a $n-$dimensional differentiable manifold whose cotangent bundle is denoted by T^*M. The bundle projection is given as $\pi : T^*M \rightarrow M$ and the local coordinates $\left(U, x^j\right),\ j = 1, ..., n$ on M induces a system of local coordinates $\left(\pi^{-1}(U),\ x^j,\ x^{\bar{j}} = p_j\right), \bar{j} = n + j = n + 1, ..., 2n$ on T^*M, where $x^{\bar{j}} = p_j$ are the components of the covector p in each cotangent space $T_x^*M, x \in U$ with respect to the natural coframe $\left\{dx^j\right\}$.

Also, the set $(r, s)-$type of all tensor fields is denoted by $\Im_s^r(M)$ and $\Im_s^r(T^*M)$ on M and T^*M, respectively. Suppose that the vector and a covector (1-form) field $X \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$ have the local expression $X = X^j \frac{\partial}{\partial x^j}$ and $\omega = \omega_j dx^j$ in $U \subset M$, respectively. Then, the horizontal lift ${}^HX \in \Im_0^1(T^*M)$ of $X \in \Im_0^1(M)$ and the vertical lift ${}^V\omega \in \Im_0^1(T^*M)$ of $\omega \in \Im_1^0(M)$ are given, respectively, by

$$\begin{aligned} &{}^HX = X^j \tfrac{\partial}{\partial x^j} + \sum_j p_h \Gamma_{ji}^h X^i \tfrac{\partial}{\partial x^{\bar{j}}}, \\ &{}^V\omega = \sum_j \omega_j \tfrac{\partial}{\partial x^{\bar{j}}} \end{aligned} \tag{2.1}$$

with respect to the natural frame $\left\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^{\bar{j}}}\right\}$, where Γ_{ji}^h are the components of the Levi-Civita connection ∇_g on M.

Moreover, on the cotangent bundle T^*M, the Lie bracket satisfies the following relations:

$$\begin{aligned} &\text{i) } \left[{}^HX, {}^HY\right] = {}^H[X, Y] + \gamma R(X, Y) = {}^H[X, Y] + {}^V(pR(X, Y)), \\ &\text{ii) } \left[{}^HX, {}^V\omega\right] = {}^V(\nabla_X\omega), \qquad iii)\left[{}^V\omega, {}^V\theta\right] = 0, \\ &\text{iv) } {}^V\omega {}^Vf = 0, \qquad v){}^HX {}^Vf = {}^V(Xf) \end{aligned} \tag{2.2}$$

for any $X, Y \in \Im_0^1(M)$, $\omega, \theta \in \Im_1^0(M)$, R denoted the curvature tensor of ∇.

On the other hand, the Riemann extension ${}^R\nabla$ as a semi-Riemannian metric is defined by

$$\begin{aligned} &{}^R\nabla\left({}^V\omega, {}^V\theta\right) = {}^R\nabla\left({}^HX, {}^HY\right) = 0, \\ &{}^R\nabla\left({}^V\omega, {}^HY\right) = {}^V(\omega(X)) = \omega(X) \circ \pi \end{aligned} \tag{2.3}$$

for any $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$ on T^*M [2, 16].

3. Silver Structure

Let $P \in \Im_1^1(M)$ be an almost product structure on M and g be a (semi-) Riemannian metric such as

$$P^2 = I,\ g(PX, Y) = g(X, PY) \tag{3.1}$$

for any $X, Y \in \Im_0^1(M)$. Then, we call that the pair (M, g, P) is a (semi-)Riemannian almost product manifold [1, 11, 17]. Such metrics in the second equation of (3.1) are said to be pure with respect to P [12, 14].

A necessary and sufficient condition for the almost product structure P to be integrable is that $\nabla P = 0$, where ∇ is Levi-Civita connection of g. An almost product manifold with an integrable product structure P is called locally product Riemannian manifold. We know that the locally product Riemannian manifold with structure tensor P is locally decomposable if and only if P is covariantly constant with respect to the Levi–Civita connection ∇. Note that the condition $\nabla P = 0$ is equivalent to $\phi_P g = 0$ where ϕ is the Tachibana operator and

$$\begin{aligned}(\phi_P g)(X,Y,Z) &= (PX)(g(Y,Z)) - X(g(PY,Z)) + g((L_Y P)X, Z) \\ &\quad + g(Y,(L_Z P)X)\end{aligned} \tag{3.2}$$

for all $X, Y, Z \in \Im_0^1(M)$ [12, 15].

Definition 3.1. (see [8]) Let M be a C^∞ differentiable manifold. A (1, 1)-type tensor field S on M is called a silver structure on M if

$$S^2 = 2S + I \tag{3.3}$$

is satisfied, where I is the identity map on M.

A Riemnnian manifold (M, g) with a silver structure S is said to be Silver Riemannian manifold if the Riemannian metric g is pure with respect to S.

The next theorem gives the relationship between the Riemannian silver and almost product structures as follows:

Theorem 3.2. *(see[8]) Let M be a Riemannian manifold. If S is a silver structure on M, then*

$$P = \frac{1}{\sqrt{2}}(S - I)$$

is an almost product structure on M. Conversely, any almost product structure P on M yields a silver structure on M as follows:

$$S = I + \sqrt{2}P.$$

Theorem 3.3. *(see [4]). Let (M, g, S) be a silver Riemannian manifold, where S is the silver structure and g is the Riemannian metric. Then the followings are satisified:*
a) S is integrable if $\phi_S g = 0$,
b) The condition $\phi_S g = 0$ is equivalent to $\nabla S = 0$, where ∇ is the Riemannian connection of g,
where ϕ_S denotes the Tacibana operator and ∇ is the Riemannian connection of g.

In [13], Salimov and Agca presented an almost product structure on T^*M by

$$\begin{aligned}P^H X &= {}^V\tilde{X}, \\ P^V\omega &= {}^H\tilde{\omega}.\end{aligned} \tag{3.4}$$

for any $X \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$, where $\tilde{X} = g \circ X \in \Im_1^0(M)$, $\tilde{\omega} = g^{-1} \circ \omega \in \Im_0^1(M)$ and $P^2 = I$. Applying Theorem 3.2 and (3.4), we find the following silver structure S:

$$\begin{aligned}S^H X &= {}^H X + \sqrt{2}{}^V\tilde{X}, \\ S^V\omega &= {}^V\omega + \sqrt{2}{}^H\tilde{\omega}.\end{aligned} \tag{3.5}$$

This silver structure defined by (3.5) is used for Sasaki metric on T^*M in [4].

Now we consider the Riemannian extension ${}^R\nabla$ and the silver structure S on the cotangent bundle T^*M. Then, using the Eqs. (3.1) and (3.5), we have the following theorem:

Theorem 3.4. *Let M be semi-Riemannian manifold and T^*M be a cotangent bundle of M. If T^*M is endowed with a Riemann extension ${}^R\nabla$ and silver structure S, then the triple $\left(T^*M, {}^R\nabla, S\right)$ is a silver semi-Riemannian manifold.*

Proof. Using (3.1), we write

$$Q\left(\tilde{X}, \tilde{Y}\right) = {}^R\nabla\left(S\tilde{X}, \tilde{Y}\right) - {}^R\nabla\left(\tilde{X}, S\tilde{Y}\right)$$

for any $\tilde{X}, \tilde{Y} \in \Im_0^1\left(T^*M\right)$. From (2.1), (2.3) and (3.5), we have

$$\begin{aligned}
Q\left({}^HX, {}^HY\right) &= {}^R\nabla\left(S^HX, {}^HY\right) - {}^R\nabla\left({}^HX, S^HY\right)\\
&= {}^R\nabla\left({}^HX + \sqrt{2}^V\tilde{X}, {}^HY\right) - {}^R\nabla\left({}^HX, {}^HY + \sqrt{2}^V\tilde{Y}\right)\\
&= \left({}^V\left(\tilde{X}\left(Y\right)\right) - \left(\tilde{Y}\left(X\right)\right)\right) = \sqrt{2}\left(\tilde{X}_iY^i - \tilde{Y}_iX^i\right)\\
&= \sqrt{2}\left(g_{ki}X^kY^i - g_{ki}Y^kX^i\right) = 0,\\
Q\left({}^HX, {}^V\omega\right) &= -Q\left({}^HY, {}^V\omega\right) = {}^R\nabla\left(S^HX, {}^V\omega\right) - {}^R\nabla\left({}^HX, S^V\omega\right)\\
&= {}^R\nabla\left({}^HX + \sqrt{2}^V\tilde{X}, {}^V\omega\right) - {}^R\nabla\left({}^HX, {}^V\omega + \sqrt{2}^H\tilde{\omega}\right)\\
&= {}^V\left(\omega\left(X\right) - \omega\left(X\right)\right) = 0\ ,\\
Q\left({}^V\omega, {}^V\theta\right) &= {}^R\nabla\left(S^V\omega, {}^V\theta\right) - {}^R\nabla\left({}^V\omega, S^V\theta\right)\\
&= {}^R\nabla\left({}^V\omega + \sqrt{2}^H\tilde{\omega}, {}^V\theta\right) - {}^R\nabla\left({}^V\omega, {}^V\theta + \sqrt{2}^H\tilde{\theta}\right)\\
&= \sqrt{2}^V\left(\theta\left(\tilde{\omega}\right) - \omega\left(\tilde{\theta}\right)\right) = 0\ ,
\end{aligned}$$

i.e. ${}^R\nabla$ is pure with respect to S, which completes the proof. □

Using the Eqs.(2.2), (2.3), (3.2) and (3.5), we obtain the following:

Lemma 3.5. *Let $\left(T^*M, {}^R\nabla, S\right)$ be a silver semi-Riemannian manifold. Then, the following component for the Tachibana operator with respect to the silver structure S defined by (3.5) is given by*

$$\begin{aligned}
\left(\phi_S{}^R\nabla\right)\left({}^HX, {}^HY, {}^V\omega\right) &= \left(S^HX\right)\left({}^R\nabla\left({}^HY, {}^V\omega\right)\right) - {}^HX\left({}^R\nabla\left(S^HY, {}^V\omega\right)\right)\\
&\quad + {}^R\nabla\left(\left(L_{{}^HY}S\right){}^HX, {}^V\omega\right) + {}^R\nabla\left({}^HY, \left(L_{{}^V\omega}S\right){}^HX\right)\\
&= -\left({}^R\nabla\left({}^V\omega, \sqrt{2}^H\left(g^{-1}\circ pR\left(Y,X\right)\right)\right)\right)\\
&= -\sqrt{2}^V\left(\omega\left(g^{-1}\circ pR\left(Y,X\right)\right)\right) = -\sqrt{2}^V\left(g^{-1}\left(pR\left(Y,X\right), \omega\right)\right)\\
&= \sqrt{2}^V\left(pR\left(X,Y\right)\tilde{\omega}\right),\\
\left(\phi_F{}^R\nabla\right)\left({}^V\omega, {}^HY, {}^HZ\right) &= \sqrt{2}\left({}^V\left(pR\left(Y,\tilde{\omega}\right)Z + pR\left(Z,\tilde{\omega}\right)Y\right)\right),\\
\left(\phi_F{}^R\nabla\right)\left({}^HX, {}^V\omega, {}^HY\right) &= \sqrt{2}^V\left(pR\left(X,Y\right)\tilde{\omega}\right)
\end{aligned}$$

Here, we note that the other components are zero.

Using above Lemma 3.5, we have the following:

Theorem 3.6. *The silver semi-Riemannian manifold $\left(T^*M, {}^R\nabla, S\right)$ is a locally decomposable if and only if M is flat.*

Example 3.7. Consider the n-dimensional Euclidean space $\mathbb{E}^n$ with the Riemannian metric $g_{ij} = \delta_j^i$. It is clear that the Christoffel symbols induced by the Levi-Civita connection ∇ on $\mathbb{E}^n$ are zero.

Let P be an almost product structure on $T^*\mathbb{E}^n$ is given by

$$P = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix},$$

such that $P^2 = I_n$, where I_n denotes the identity matrix of order n. Using Theorem 3.2, the almost product structure P on $T^*\mathbb{E}^n$ gives

$$S^H X = {}^H X + \sqrt{2}^H X, \quad S^V \omega = {}^V \omega + \sqrt{2}^V \omega, \tag{3.6}$$

such that the equalities (3.6) are silver structure. Then, one can see that ${}^R\nabla$ is pure with respect to S and the triple $\left(T^*\mathbb{E}^n, {}^R\nabla, S\right)$ becomes a silver semi-Riemannian manifold.

On the other hand, using the Eq.(3.2) and the Tachibana operator with respect to the silver structure defined by (3.6), one has

$$\left(\phi_S {}^R\nabla\right)(X, Y, Z) = 0$$

for any $X, Y, Z \in \Im_0^1(M)$. Then, we obtain that the silver semi-Riemannian manifold $\left(T^*\mathbb{E}^n, {}^R\nabla, S\right)$ is a locally decomposable.

4. Conclusion

In this study, a semi-Riemannian manifold M and its cotangent bundle T^*M is considered. Then, by considering the Riemann extension ${}^R\nabla$ and silver structure S on T^*M, the components of Tachibana operators are calculated and using them, this characterization is obtained: M is flat if and only if the silver semi-Riemannian manifold $\left(T^*M, {}^R\nabla, S\right)$ is a locally decomposable.

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(Filiz OCAK) Department of Mathematics, Karadeniz Technical University, 61080 Trabzon, TURKIYE.

Email address: filiz.ocak@ktu.edu.tr

(Şemsi Eken MERIÇ) Department of Mathematics, Faculty of Science and Arts, Mersin University, 33343, Mersin, TURKIYE

Email address: semsieken@hotmail.com, semsieken@mersin.edu.tr