Konuralp Journal of Mathematics, 12 (2) (2024) 124-129

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

On Schur Stability and Oscillation of Linear Difference Equation Systems with Constant Coefficients

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Abstract

In this study, the solutions of linear difference equation systems with constant coefficients were examined with respect to whether they were Schur stability and oscillatory or not. A new parameter $\gamma(A)$ which indicates the quality of Schur stability and oscillatory of the system has been defined. For Schur stable and oscillation linear difference equation system with constant coefficients, continuity theorems which show how much change is permissible without disturbing the Schur stability and oscillatory have been proved, and some examples illustrating the efficiency of the theorems have been given.

Keywords: *Condition number; Difference equation systems; Oscillation; Perturbation systems; Schur stability; Sensitivity*
2010 Mathematics Subject Classification: *39A06; 39A21; 39A30*

1. Introduction

Difference equation systems have been used in modelling the problems given in many fields such as the study of number of living population in biology, economics, medical science. A great number of studies have been conducted on the behavior of solutions of difference equation systems, focusing on Schur stability or oscillation characteristics. However, to our knowledge, there has not been sufficient studies conducted in which oscillation and Schur stability of difference equation systems are provided together.

Now, let us explain the conditions under which linear difference equation systems with constant coefficients are both Schur stable and oscillatory (SSO), depending on the spectral criterion [1]. Consider the linear difference equation system

$$x(n+1) = Ax(n),\ n \in \mathbb{Z} \tag{1.1}$$

where A is a matrix with $N \times N$ dimensions and $x(n)$ is an N dimensional vector. Let the eigenvalues of the coefficient matrix A be $\lambda_j(A) \in \mathbb{C}, (j = 1,2,...,N)$.

i. Difference equation system (1.1) (or the matrix A) is Schur stable if and only if $|\lambda_j(A)| < 1$, for all $j = 1,2,...,N$ (see, for example, [2]-[7]).
ii. Let $\lambda_j(A) = \alpha_j \pm i\beta_j, (j = 1,2,...,N)$. Difference equation system (1.1) (or the matrix A) is oscillatory if and only if
a) If $\beta_j = 0$, then $\lambda_j(A) < 0$, $j = 1,2,...,N$. The solution $x(n)$ oscillates.
b) Let $\beta_j \neq 0$,
− $r < 1 \Rightarrow \lambda_j(A), (j = 1,2,...,N)$ lie inside unit disk. The solution$x(n)$ oscillates and converges to zero;
− $r = 1 \Rightarrow \lambda_j(A), (j = 1,2,...,N)$ lie on the unit circle. The solution $x(n)$ oscillates and constant in magnitude;
− $r > 1 \Rightarrow \lambda_j(A), (j = 1,2,...,N)$ are outside the unit circle. $x(n)$ oscillates and increasing in magnitude

where $r = \sqrt{\alpha_j^2 + \beta_j^2}, (j = 1,2,...,N)$ ([2, 3]).
The conditions for the solutions of the difference equation system (1.1) to be SSO are given above. In addition, the conditions under which homogeneous second order difference equations with constant coefficients are SSO are examined and the behavior of solutions is investigated as a result of perturbating such conditions [1].

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Now, let's explain in which case the matrix A is Hurwitz stable. Then, let's explain the connection between system (1.1) being oscillatory and being Hurwitz stable. According to the spectral criterion, if the real parts of all eigenvalues of the matrix A are less than zero, i.e. $Re\lambda_j(A) < 0$ $(j = 1, 2, \ldots, N)$, then the matrix A is called to be Hurwitz stable (see, for example, [2]-[6]).
Note. *It is clearly seen that if matrix A is Hurwitz stable, the difference equation system (1.1), where we take matrix A as the coefficient matrix, is oscillatory.*

Remark 1.1. *Let $\lambda_j(A) \in \mathbb{C}, (j = 1, 2, ..., N)$ be eigenvalues of the matrix A.*

i. $|\lambda_j(A)| < 1$, $(j = 1, 2, ..., N) \Leftrightarrow$ *the matrix A (difference equation system (1.1)) is Schur stable,*
ii. $Re\lambda_j(A) < 0$ $(j = 1, 2, \ldots, N) \Leftrightarrow$ *the matrix A is Hurwitz stable. Also difference equation system (1.1) is oscillating.*

It is clear that the system (1.1) is SSO if and only if i) and ii) of Remark 1.1 are considered together [5].

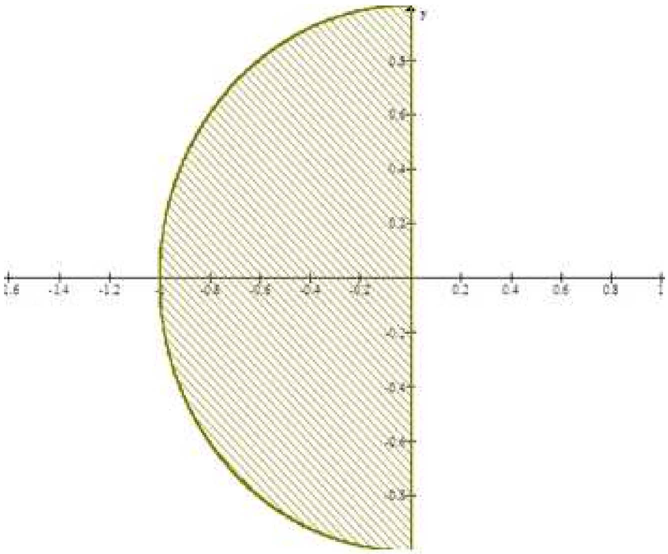

Figure 1.1: The regions of eigenvalues of the matrix A where difference equation system (1.1) is SSO.

With the help of the eigenvalues of the coefficient matrix A, the conditions under which the difference equation system (1.1) is Schur stable and oscillatory were investigated. However, since the eigenvalue problem is not well-posed for non-symmetric matrices, it is more practical to use parameters calculated with the help of the solution of a linear algebraic equation, rather than eigenvalues, to determine these conditions. Now, let us present the Lyapunov theorems to introduce the Schur stability parameter for difference equation systems with constant coefficient, followed by the Hurwitz stability parameter for differential equation systems with constant coefficient.

Theorem 1.2. *The matrix A (trivial solution of the system (1.1)) is Schur stable if and only if there is a solution $H = H^* > 0$ of the Lyapunov matrix equation $A^*HA - H = -I$ [4].*

Schur stability parameter of system (1.1) is represented by $\omega(A)$ and defined as:

$$\omega(A) = \|H\|; H = \sum_{k=0}^{\infty} (A^*)^k A^k,\ A^*HA - H = -I,\ H = H^* > 0,$$

where I is unit matrix, A^* is adjoint of the matrix A, $\|A\|$ is the spectral norm of the matrix A and $\|x\|$ is Euclidean norm for the vector $x = (x_1, x_2, ..., x_N)^T$. If $\omega(A) < \infty$, the matrix A (the system (1.1)) is Schur stable. Otherwise, we set $\omega(A) = \infty$ [4]-[8].
In conclusion, it is easily seen from above that

$$\left(|\lambda_j(A)| < 1, (j = 1, 2, ..., N)\right) \Leftrightarrow (\exists H = H^* > 0 \ni A^*HA - H = -I) \Leftrightarrow (\omega(A) < \infty).$$

Theorem 1.3. *The matrix A is Hurwitz stable if and only if there is a solution $F = F^* > 0$ of the Lyapunov matrix equation $A^*F - FA = -I$ [9, 10].*

Hurwitz stability parameter $\kappa(A)$ for the matrix A is defined as

$$\kappa(A) = 2\|A\|\,\|F\|;\ F = \int_0^{\infty} e^{tA^*} e^{tA} dt,\ A^*F - FA = -I,\ F = F^* > 0.$$

If $\kappa(A) < \infty$, then the matrix A is Hurwitz stable. Otherwise, we set $\kappa(A) = \infty$.
In conclusion, it is easily seen from above that

$$\left(Re\lambda_j(A) < 0, (j = 1, 2, ..., N)\right) \Leftrightarrow (\exists F = F^* > 0 \ni A^*F - FA = -I) \Leftrightarrow (\kappa(A) < \infty).$$

2. Both Schur Stability and Oscillation of Linear Difference Equation System with Constant Coefficients

In this section, a new parameter $\gamma(A)$ which indicates the quality of Schur stability and oscillatory of the system has been defined. In addition, equivalent conditions for SSO of the linear difference equation system have been obtained.

Definition 2.1. *For the coefficient matrix A, let us define the condition number as $\gamma(A) = \max\{\omega(A), \kappa(A)\}$. If $\gamma(A) < \infty$, then the difference equation system (1.1) is SSO. Otherwise we set $\gamma(A) = \infty$. The matrix functional $\gamma(A)$ defined in this way is referred to as the parameter that indicates the quality of both Schur stability and oscillatory of the system (1.1)*

Now, let's consider the system $x(n+1) = A_i x(n), i = 1,2$ where $A_1 = \begin{pmatrix} -0.1 & 0 \\ 0 & -0.7 \end{pmatrix}, A_2 = \begin{pmatrix} -0.1 & 9 \\ 0 & -0.7 \end{pmatrix}$ and illustrate that the parameter $\gamma(A)$ represents the quality of SSO. The eigenvalues of both matrices are -0.1 and -0.7. It can be easily seen that both systems are SSO. But knowledge of the eigenvalues does not give information about the quality of the SSO. However, since $\gamma(A_1) = 7 < \gamma(A_2) = 13149.8$, it is seen that the quality of SSO of the system $x(n+1) = A_1 x(n)$ is better than the quality of SSO of the system $x(n+1) = A_2 x(n)$. This means that the SSO of the system $x(n+1) = A_2 x(n)$ deteriorates than the SSO of the system $x(n+1) = A_1 x(n)$ with less perturbation. For example; when A_1 and A_2 matrices are perturbed with matrix $B = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$, the system $x(n+1) = A_1 x(n)$ is SSO, while the system $x(n+1) = A_2 x(n)$ is not SSO. As can be seen, while eigenvalues do not give an idea about the quality of SSO of a system, the parameter $\gamma(A)$ calculates the quality of SSO.

Remark 2.2. *The following conditions are equivalent;*

1. *The matrix A is Schur stable and Hurwitz stable,*
2. $|\lambda_j(A)| < 1, (j = 1,2,...,N)$ *and* $Re\lambda_j(A) < 0, (j = 1,2,...,N)$,
3. $A^*HA - H = -I, \exists H = H^* > 0$ *and* $A^*F - FA = -I, \exists F = F^* > 0$,
4. $\omega(A) < \infty$ *and* $\kappa(A) < \infty$,
5. *The difference system (1.1) is SSO, i.e.* $\gamma(A) < \infty$.

Let the matrix family $\mathscr{A}$, be defined as matrix family that are simultaneously Schur stable and Hurwitz stable matrices, i.e. $\mathscr{A} = \{A | \gamma(A) < \infty\}$.

Remark 2.3. *If $A \in \mathscr{A}$, then the difference equation system (1.1) is SSO.*

Let's provide a remark explaining the above results and giving equivalent conditions. Let us express this result with the following diagram.

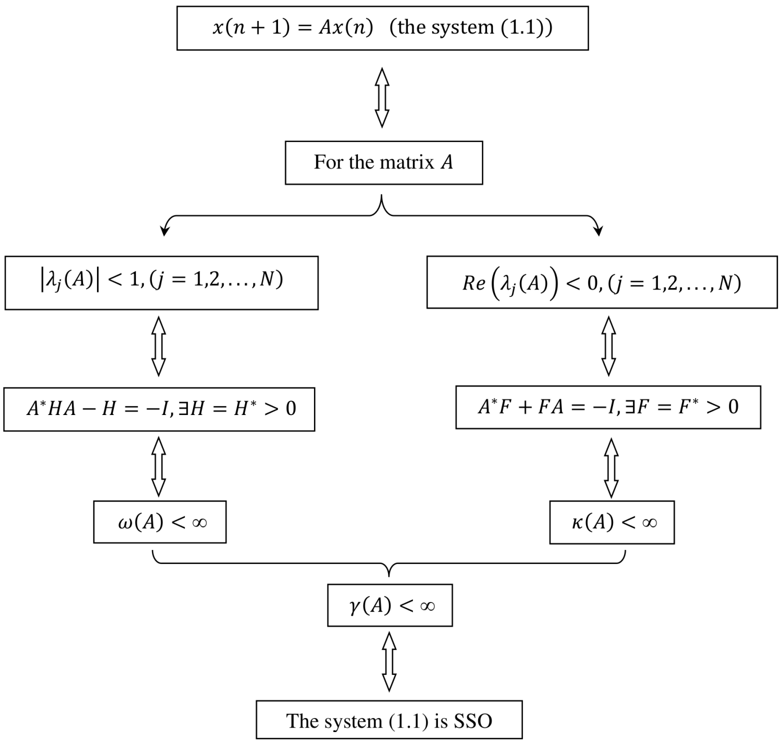

Figure 2.1: The equivalent results obtained for the difference equation system (1.1) to be SSO

Example 2.4. *Consider the linear difference equation system*

$$x(n+1) = \begin{pmatrix} -0.9 & 0 \\ 0 & -0.7 \end{pmatrix} x(n), \; x(0) = (-1000, 1000)^T.$$

$\gamma(A) = 5.26316$ *for the coefficient matrix A. Therefore the difference equation system $x(n+1) = Ax(n)$ is SSO.*

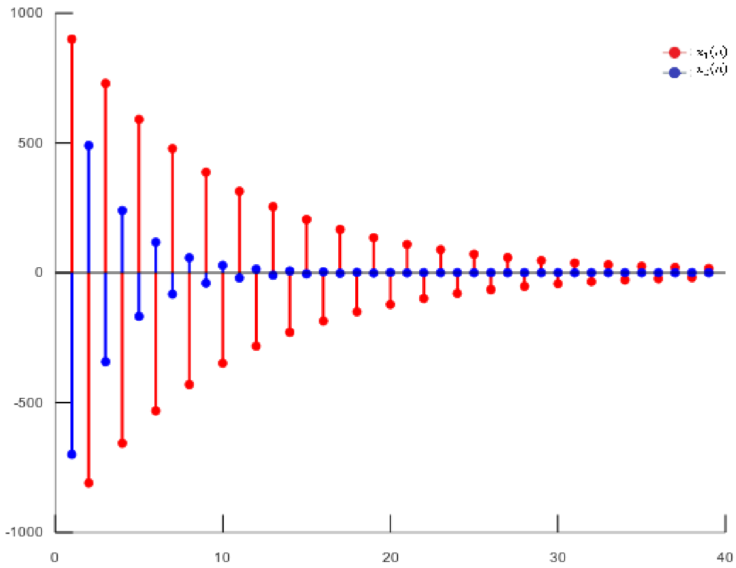

Figure 2.2: The behavior of the solutions of the difference equation system in Example 2.4

Example 2.5. *Consider the linear difference equation system*

$$x(n+1)=\begin{pmatrix} -0.8 & 0 & 0 \\ 0 & -0.5 & 1 \\ 1 & 0 & -0.9 \end{pmatrix} x(n),\; x(0)=(-1000,1000,1000)^T.$$

$\gamma(A)=456.494$ *for the coefficient matrix A. Therefore the difference equation system* $x(n+1)=Ax(n)$ *is SSO.*

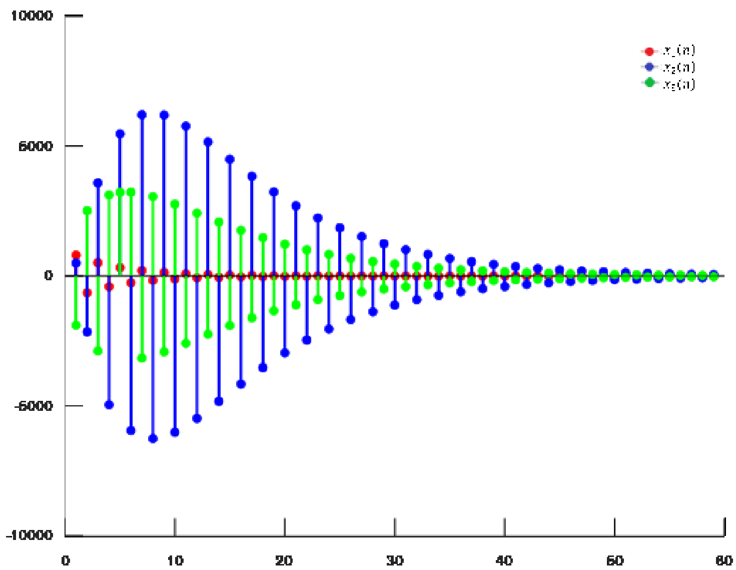

Figure 2.3: The behavior of the solutions of the difference equation system in Example 2.5

3. Sensitivity of SSO of Linear Difference Equation System with Constant Coefficients

Let $A,B\in M_N(\mathbb{R})$ and consider the following difference equation system

$$y(n+1)=(A+B)\,y(n),\; n\in\mathbb{Z}. \tag{3.1}$$

The difference equation system (3.1) is the perturbed system of the difference equation system (1.1).
Now let us give the following continuity theorem which shows how much perturbation is permissible for the difference equation system (1.1) in literature.

Theorem 3.1. *Let the matrix A (the system (1.1)) be Schur stable, i.e.,* $\omega(A)<\infty$. *If B satisfies the inequalities*

i. $\|B\|<\frac{\|A\|}{10\omega(A)}$; [11], *ii.* $\|B\| < \frac{1}{20(\omega(A))^{\frac{3}{2}}}$; [11],

iii. $\|B\| < \frac{1}{6\pi\omega(A)}$; [11],

iv. $\|B\| < \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} - \|A\|$; [7],

then the matrix $A+B$ (the difference equations system (3.1)) is Schur stable.

The continuity theorem that provides the largest perturbation bound for a Schur stable difference equation system, which does not disrupt its Schur stability, is the theorem that gives the best bound according to the given problem.

Result 3.1. Let the matrix A (the difference equations system (1.1)) be Schur stable, i.e., $\omega(A) < \infty$. If B satisfies the inequalities

$$\|B\| < \delta = \max\{\delta_1, \delta_2, \delta_3, \delta_4\}$$

then the matrix $A+B$ (the difference equations system (3.1)) is Schur stable where $\delta_1 = \frac{\|A\|}{10\omega(A)}$, $\delta_2 = \frac{1}{20(\omega(A))^{\frac{3}{2}}}$, $\delta_3 = \frac{1}{6\pi\omega(A)}$ and $\delta_4 = \sqrt{\|A\|^2 + \frac{1}{\omega(A)}} - \|A\|$.

We give a theorem in the literature on the sensitivity of the Hurwitz stability of the matrix A.

Theorem 3.2. *Let A be a Hurwitz stable matrix. If $\|B\| < \frac{\|A\|}{\kappa(A)}$ then the matrix $A+B$ is Hurwitz stable [12].*

Now, some new results on the sensitivity of the SSO are stated.

Theorem 3.3. *Let the system (1.1) is SSO i.e., $A \in \mathscr{A}$. If $\|B\| < \min\left\{\delta, \frac{\|A\|}{\kappa(A)}\right\}$ then the system (3.1) is SSO, where δ is as given in the Result 3.1.*

Proof. When the matrix A is Schur stable, from Result 3.1, the matrix $A+B$ is Schur stable for $\|B\| < \delta$. Furthermore, the Hurwitz stability of the coefficient matrix A implies that system (1.1) is oscillatory. From Theorem 3.2, when the system (1.1) is oscillatory, the pertubed system (3.1) is oscillatory for $\|B\| < \frac{\|A\|}{\kappa(A)}$. Therefore let the system (1.1) is SSO i.e., $A \in \mathscr{A}$. If $\|B\| < \min\left\{\delta, \frac{\|A\|}{\kappa(A)}\right\}$ then the system (3.1) is SSO. □

Theorem 3.4. *Let the system (1.1) is SSO ($A \in \mathscr{A}$). If B satisfies the inequality*

i. $\|B\| < \frac{\kappa(A)}{10\omega(A)}$ *for* $10\omega(A) > \kappa(A)$

ii. $\|B\| < \frac{10\omega(A)}{\kappa(A)}$ *for* $10\omega(A) < \kappa(A)$

then the difference equations system (3.1) is SSO.

Proof. Let the system (1.1) is SSO. The system (3.1) is Schur stable for the perturbation matrix B satisfying the condition $\|B\| < \frac{\|A\|}{10\omega(A)}$ given by Theorem 3.1 (i) and the system (3.1) is oscillatory for the perturbation matrix B satisfying the condition $\|B\| < \frac{\|A\|}{\kappa(A)}$ given by Theorem 3.2. Therefore

$$\frac{\|A\|}{10\omega(A)} < \frac{\|A\|}{\kappa(A)} \text{ then } \kappa(A) < 10\omega(A) \tag{3.2}$$

$$\frac{\|A\|}{\kappa(A)} < \frac{\|A\|}{10\omega(A)} \text{ then } 10\omega(A) < \kappa(A) \tag{3.3}$$

It can be seen from (3.2) and (3.3)

$$\|B\| < \frac{\kappa(A)}{10\omega(A)} < \frac{\|A\|}{\kappa(A)} \text{ then } \|B\| < \frac{\kappa(A)}{10\omega(A)} \text{ for } 10\omega(A) > \kappa(A)$$

and

$$\|B\| < \frac{\|A\|}{\kappa(A)} < \frac{\|A\|}{10\omega(A)} \text{ then } \|B\| < \frac{10\omega(A)}{\kappa(A)} \text{ for } 10\omega(A) < \kappa(A).$$

Thus, the proof is complete. □

Result 3.2. Let the system (1.1) is SSO i.e., $A \in \mathscr{A}$. If B satisfies the inequality

$$\|B\| < \Theta,$$

then the system (3.1) is SSO, where $\Theta = \min\left\{\frac{\|A\|}{10\omega(A)}, \frac{\|A\|}{\kappa(A)}\right\}$.

Theorem 3.5. *Let the system (1.1) is SSO i.e., $A \in \mathscr{A}$. If B satisfies the inequality*

$$\|B\| < \frac{\|A\|}{10\gamma(A)},$$

then the system (3.1) is SSO.

Proof. Let the system (1.1) is SSO. The system (1.1) is Schur stable for the perturbation matrix B satisfying the condition $\|B\| < \frac{\|A\|}{10\omega(A)}$ given by Theorem 3.1 (i) and the system (1.1) is oscillatory for the perturbation matrix B satisfying the condition $\|B\| < \frac{\|A\|}{\kappa(A)}$ given by Theorem 3.2. Therefore, it is seen from the definition of the parameter $\gamma(A)$

$$\omega(A) \leq \gamma(A) \Rightarrow \frac{\|A\|}{10\gamma(A)} \leq \frac{\|A\|}{10\omega(A)}$$

and

$$\kappa(A) \leq \gamma(A) \Rightarrow \frac{\|A\|}{10\gamma(A)} \leq \frac{\|A\|}{\kappa(A)}$$

then

$$\frac{\|A\|}{10\gamma(A)} \leq \min\left\{\frac{\|A\|}{10\omega(A)}, \frac{\|A\|}{\kappa(A)}\right\}$$

holds. Thus, the proof of the theorem is easily seen from Result 3.2. □

Example 3.6. *Let us consider the system in Example 2.4. The perturbation upper bounds that make this system SSO are as follows.*

- $\|B\| < 0.1$ *Theorem 3.3*
- $\|B\| < 0.0244$ *Theorem 3.4*
- $\|B\| < 0.0171$ *Result 3.2*
- $\|B\| < 0.0171$ *Theorem 3.5*

Let us perturb the system with the coefficient matrix A according to Theorem 3.3, which provides the maximum perturbation. Using matrix $B = \begin{pmatrix} -0.09 & 0 \\ 0 & -0.09 \end{pmatrix}$, $\|B\| = 0.09 < 0.1$. *While the system is perturbed by B we have* $\gamma(A+B) = 50.2513$ *hence the perturbed system* $y(n+1) = (A+B)y(n)$ *is SSO.*

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable

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