THE STRUCTURE OF THE UNIT GROUP OF $\mathbb{F}_{3^k}(C_3 \times D_{2n})$ AND FINITE GROUP ALGEBRAS OF GROUPS OF ORDER 42

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Dedicated to the memory of Professor Syed M. Tariq Rizvi

ABSTRACT. Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$ with $|\mathbb{F}_q| = q = p^k$ and $\mathcal{U}(\mathbb{F}_q G)$ be the unit group of the group algebra $\mathbb{F}_q G$ for some group G. There are 6 groups of order 42 up to isomorphism. In this paper, we provide a characterization of $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \times D_{2n}))$ and establish the structures of the unit groups of some finite group algebras of groups of order 42.

1. Introduction

Let $\mathbb{F}_q G$ be the group algebra of a group G over a finite field $\mathbb{F}_q$ of order $q = p^k$, for some prime p and a positive integer k. The group containing invertible elements of $\mathbb{F}_q G$ is denoted by $\mathcal{U}(\mathbb{F}_q G)$. Let $J(\mathbb{F}_q G)$ be the Jacobson radical of $\mathbb{F}_q G$ and $V = 1 + J(\mathbb{F}_q G)$. For $H \lhd G$, the canonical homomorphism $\omega : G \to G/H$ can be extended to form an epimorphism $\omega' : \mathbb{F}_q G \to \mathbb{F}_q(G/H)$ given by $\omega'(\sum_{g \in G} \alpha_g g) = \sum_{g \in G} \alpha_g \omega(g)$. Here, $Ker(\omega') = \Delta(G, H)$ is a two-sided ideal of $\mathbb{F}_q G$ generated by the set $\{h - 1 : h \in H\}$. For fundamental definitions and results utilized in this paper, see [16].

The structure of the unit groups of several finite group algebras have already been established in [1,4,5,11,14,19]. The dihedral group of order $2n$ is denoted by D_{2n}. Characterizations of $\mathcal{U}(\mathbb{Z}D_8)$ and $\mathcal{U}(\mathbb{Z}D_{12})$ are provided by the authors in [12]. Some general results describing $\mathcal{U}(\mathbb{F}_{3^k}(C_n \times D_6))$, $\mathcal{U}(\mathbb{F}_{2^k}D_{2p})$ for prime p and $\mathcal{U}(\mathbb{F}_{2^k}D_{2n})$ for odd integers n are given in [7,8,10]. In [2,3], the study of unitary subgroups of some group algebras have been undertaken.

The six non-isomorphic groups of order 42 are: D_{42}, C_{42}, $C_3 \times D_{14}$, $C_7 \times D_6$, $C_7 \rtimes C_6$ and $C_2 \times (C_7 \rtimes C_3)$. The structure of $\mathcal{U}(\mathbb{F}_q(D_{42}))$ is investigated by the

authors in [13]. Section 3 of this paper describes the unit group $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \times D_{2n}))$. Further, we characterize the structure of the unit groups of $\mathbb{F}_q C_{42}$, $\mathbb{F}_q(C_3 \times D_{14})$, $\mathbb{F}_q(C_7 \times D_6)$ in Section 4. Additionally, the semisimple case is discussed for the two semidirect products $C_7 \rtimes C_6$ and $C_2 \times (C_7 \rtimes C_3)$ by configuring the corresponding Wedderburn decomposition.

2. Preliminaries

Some helpful results to explore the structure of $\mathbb{F}_q G/J(\mathbb{F}_q G)$ were given by Ferraz [6]. Let G be a finite group and e be the l.c.m. of the orders of all the p-regular elements in G. Let η be the primitive e-th root of unity over $\mathbb{F}_q$ and $\mathcal{B}$ be the set of integers $t \bmod e$ for which $\eta \to \eta^t$ is an automorphism of $\mathbb{F}_q(\eta)$ over $\mathbb{F}_q$. If l is the multiplicative order of $q \bmod e$, then $\mathcal{B} = \{1, q, \ldots, q^{l-1}\} \bmod e$. For a p-regular element g, define β_g to be the sum of all conjugates of g. The cyclotomic $\mathbb{F}_q$-class of β_g is defined by

$$S(\beta_g) = \{\beta_{g^t} \mid t \in \mathcal{B}\}.$$

Lemma 2.1. [6, Proposition 1.2] *The number of cyclotomic $\mathbb{F}_q$-classes in G is equal to the number of simple components of $\mathbb{F}_q G/J(\mathbb{F}_q G)$.*

Lemma 2.2. [6, Theorem 1.3] *Let η be the same as defined above and d be the number of cyclotmic $\mathbb{F}_q$-classes in G. If $S_1, \ldots, S_d$ are the cyclotomic $\mathbb{F}_q$-classes in G and $P_1, \ldots, P_d$ are the simple components of the center of $\mathbb{F}_q G/J(\mathbb{F}_q G)$, then $|S_i| = [P_i : \mathbb{F}_q]$ for a suitable ordering of the indices.*

Let us recall a very useful result from [16, Proposition 3.6.11] which states that if $\mathbb{F}_q G$ is semisimple, then

$$\mathbb{F}_q G \cong \mathbb{F}_q(G/G') \bigoplus \Delta(G, G')$$

where $\mathbb{F}_q(G/G')$ is the sum of all the commutative simple components of $\mathbb{F}_q G$ and $\Delta(G, G')$ is the sum of all others.

Let I_p be the set of all p-elements including the identity element of G. Define a map $\theta : G \to \mathbb{F}_q$ such that $\theta(g) = 1$ if $g \in I_p$ and $\theta(g) = 0$ otherwise. We linearly extend θ from $\mathbb{F}_q G \mapsto \mathbb{F}_q$ such that $\theta(\alpha) = \sum_{g \in G} \alpha_g \theta(g) = \sum_{g \in I_p} \alpha_g$ for all $\alpha = \sum_{g \in G} \alpha_g g \in \mathbb{F}_q G$.

Lemma 2.3. [21, Lemma 2.2] *Let G be a finite group and θ be the map defined above. Then,*

(1) $J(\mathbb{F}_q G) \subseteq Ker(\theta)$.
(2) $Ker(\theta) = Ann(\widehat{I_p})$.
(3) $J(\mathbb{F}_q G) \subseteq Ann(\widehat{I_p})$.

3. The structure of $\mathcal{U}(\mathbb{F}_{3^k}(C_3 \times D_{2n}))$

The group $C_3 \times D_{2n}$ of order $6n$, for $n \geq 1$, is represented as:

$$C_3 \times D_{2n} = \langle r, s, t \mid r^n = s^2 = t^3 = 1, \ rs = sr^{-1}, \ rt = tr, \ st = ts \rangle.$$

Theorem 3.1. *Let $\mathbb{F}_q$ be a finite field of order $q = 3^k$. Then,*

$$\mathcal{U}(\mathbb{F}_q(C_3 \times D_{2n})) \cong ((\cdots(C_3^{3nk} \rtimes \underbrace{C_3^k) \rtimes C_3^k) \rtimes \cdots \rtimes C_3^k}_{n \ times}) \rtimes \mathcal{U}(\mathbb{F}_q D_{2n}).$$

Proof. Let $G = C_3 \times D_{2n}$, then $N = \langle t \rangle$ is a normal subgroup of G of order 3. Let $M = \langle r, s \rangle$, then the factor group $G/N \cong M \cong D_{2n}$. Now, define a map $\Psi : \mathbb{F}_q G \to \mathbb{F}_q M$ given by

$$\Psi\left(\sum_{j=0}^{n-1} \sum_{i=0}^{2} t^i r^j (a_{i+3j} + a_{i+3j+3n}s)\right) = \sum_{j=0}^{n-1} \sum_{i=0}^{2} r^j (a_{i+3j} + a_{i+3j+3n}s).$$

We obtain a group epimorphism $\Psi' : \mathcal{U}(\mathbb{F}_q G) \to \mathcal{U}(\mathbb{F}_q M)$ by restricting the ring epimorphism Ψ. Again, the restriction of the inclusion map from $\mathbb{F}_q M \to \mathbb{F}_q G$ results in a group monomorphism $\Phi : \mathcal{U}(\mathbb{F}_q M) \to \mathcal{U}(\mathbb{F}_q G)$ defined by

$$\Phi\left(\sum_{j=0}^{n-1} r^j (z_j + z_{j+n}s)\right) = \sum_{j=0}^{n-1} r^j (z_j + z_{j+n}s).$$

Let $\mathcal{K} = ker(\Psi')$. Since $\Psi' \circ \Phi = 1_{\mathcal{U}(\mathbb{F}_q M)}$, $\mathcal{U}(\mathbb{F}_q G) \cong \mathcal{K} \rtimes \mathcal{U}(\mathbb{F}_q M)$.

Consider $v = \sum_{j=0}^{n-1} \sum_{i=0}^{2} t^i r^j (a_{i+3j} + a_{i+3j+3n}s) \in \mathcal{K}$, i.e., $\Psi'(v) = 1$ and this results in the following equations:

$$a_0 = 1 - a_1 - a_2, \ a_{3d} = -a_{3d+1} - a_{3d+2} \text{ for } d = 1, \ldots, 2n - 1.$$

Given this, an equivalent way of writing the set $\mathcal{K}$ is

$$\mathcal{K} = \{1 + \sum_{j=0}^{n-1} \sum_{i=1}^{2} (t^i - 1)r^j (b_{i+2j} + b_{i+2j+2n}s) \mid b_i \in \mathbb{F}_q\}.$$

It can be verified that $\mathcal{K}$ is a non-abelian group satisfying $\mathcal{K}^3 = 1$. The presumption $q = 3^k$ concludes $|\mathcal{K}| = 3^{4nk}$.

Consider some subgroups of $\mathcal{K}$ defined as:

$$S_d = \{1 + x_1 \hat{t} + x_2(t + 2t^2)r^d s \mid x_i \in \mathbb{F}_q\} \qquad \text{for } d = 0, 1, \ldots, n - 1,$$

and $U_n = \mathcal{K}$,

$$U_0 = \{1 + \sum_{j=0}^{n-1} \sum_{i=1}^{2} (t^i - 1)r^j y_{i+2j} + \hat{t} \sum_{i=0}^{n-1} r^i y_{i+2n+1}s \mid y_i \in \mathbb{F}_q\},$$

$$U_d = \{1 + \sum_{i=1}^{2}(t^i - 1)(\sum_{j=0}^{n-1} r^j y_{i+2j} + \sum_{j=0}^{d-1} r^j y_{i+2j+2n} s)$$

$$+ \hat{t} \sum_{i=d}^{n-1} r^i y_{i+d+2n+1} s \mid y_i \in \mathbb{F}_q\} \qquad \text{for } d = 1, \dots, n-1.$$

Here, S_d and U_d are subgroups of U_{d+1} and $I = S_d \cap U_d = \{1 + x_1 \hat{t} \mid x_1 \in \mathbb{F}_q\} \cong C_3^k$, for $d = 0, 1, \dots, n-1$. Moreover, S_d is an abelian group and therefore, for each $d = 0, 1, \dots, n-1$, there exists some subgroup R_d of S_d satisfying $S_d = I \times R_d$ with $R_d \cong C_3^k$. Considering some general elements

$$p_d = 1 + x_1 \hat{t} + x_2(t + 2t^2)r^d s \in S_d \qquad \text{for } d = 0, \dots, n-1,$$

$$q_0 = 1 + \sum_{j=0}^{n-1}\sum_{i=1}^{2}(t^i - 1)r^j y_{i+2j} + \hat{t}\sum_{i=0}^{n-1} r^i y_{i+2n+1} s \in U_0,$$

$$q_d = 1 + \sum_{i=1}^{2}(t^i - 1)(\sum_{j=0}^{n-1} r^j y_{i+2j} + \sum_{j=0}^{d-1} r^j y_{i+2j+2n} s)$$

$$+ \hat{t} \sum_{i=d}^{n-1} r^i y_{i+d+2n+1} s \in U_d \qquad \text{for } d = 1, \dots, n-1.$$

Let us define:
$E_1 = \sum_{j=0}^{n-1}\sum_{i=1}^{2}(t^i - 1)r^j y_{i+2j}$, $E_1' = \sum_{j=0}^{n-1}\sum_{i=1}^{2}(t^i - 1)r^{-j} y_{i+2j}$,

$E_{2,0} = 0 = E_{2,0}'$, $E_{2,d} = \sum_{j=0}^{d-1}\sum_{i=1}^{2}(t^i - 1)r^j y_{i+2j+2n}$ for $d = 1, \dots, n-1$,

$E_{2,d}' = \sum_{j=0}^{d-1}\sum_{i=1}^{2}(t^i - 1)r^{-j} y_{i+2j+2n}$ for $d = 1, \dots, n-1$ and

$E_{3,d} = \hat{t} \sum_{i=d}^{n-1} r^i y_{i+d+2n+1}$ for $d = 0, \dots, n-1$.

Equivalently, q_d can be written as:

$$q_d = 1 + E_1 + E_{2,d}s + E_{3,d}s \in U_d \qquad \text{for } d = 0, \dots, n-1.$$

The fact that $S_d \subseteq \mathcal{K}$, gives us $S_d^3 = 1$. Thus, for $p_d \in S_d$,

$$p_d^{-1} = p_d^2 = 1 + (2x_1 + x_2^2)\hat{t} + 2x_2(t + 2t^2)r^d s \qquad \text{for } d = 0, \dots, n-1.$$

The structure of $\mathcal{K}$ can be concluded by combining the information given so far along with the following steps.

Step 1: Let $q_0 \in U_0$ and $p_0 \in S_0$. Then,

$$q_0^{p_0} = p_0^{-1} q_0 p_0$$
$$= q_0 + x_2(t + 2t^2)(E_1 - E_1')s \in U_0.$$

By definition, S_0 normalizes U_0. Furthermore, U_0 is abelian and therefore $U_0 \cong C_3^{3nk}$. Clearly, $U_0 \cap R_0 = \{1\}$ and hence $U_1 \cong U_0 \rtimes R_0 \cong C_3^{3nk} \rtimes C_3^k$.

Step 2: Let $q_1 \in U_1$ and $p_1 \in S_1$. Then,

$$q_1^{p_1} = p_1^{-1} q_1 p_1$$
$$= q_1 + x_2(t + 2t^2)(E_1 - E_1')rs + x_2(t + 2t^2)(E_{2,1}r^{-1} - E_{2,1}'r) \in U_1.$$

Again, it follows that S_1 normalizes U_1 and as $U_1 \cap R_1 = \{1\}$, therefore $U_2 \cong U_1 \rtimes R_1 \cong (C_3^{3nk} \rtimes C_3^k) \rtimes C_3^k$.

In general,

$$q_d^{p_d} = q_d + x_2(t + 2t^2)(E_1 - E_1')r^d s + x_2(t + 2t^2)(E_{2,d}r^{-d} - E_{2,d}'r^d) \in U_d$$

for $d = 0, \ldots, n-1$.

Proceeding in a similar manner, it can be shown that S_d normalizes U_d and therefore $U_{d+1} \cong U_d \rtimes R_d$ for $d = 2, \ldots, n-1$. Consequently, $U_n \cong U_{n-1} \rtimes R_{n-1}$, that is

$$\mathcal{K} \cong ((\cdots (C_3^{3nk} \rtimes \underbrace{C_3^k) \rtimes C_3^k) \rtimes \ldots \rtimes C_3^k}_{n \text{ times}}).$$

Since $M \cong D_{2n}$, we get

$$\mathcal{U}(\mathbb{F}_q(C_3 \times D_{2n})) \cong ((\cdots (C_3^{3nk} \rtimes \underbrace{C_3^k) \rtimes C_3^k) \rtimes \ldots \rtimes C_3^k}_{n \text{ times}}) \rtimes \mathcal{U}(\mathbb{F}_q D_{2n}). \qquad \square$$

4. Some structures of $\mathcal{U}(\mathbb{F}_q G)$ for $|G| = 42$

The results pertaining to the structures of $\mathcal{U}(\mathbb{F}_q G)$ for the earlier mentioned groups G of order 42 are provided in this section.

Theorem 4.1. *Let $\mathbb{F}_q$ be a finite field of order $q = p^k$ with characteristic p and let $G = C_7 \times D_6$.*

(1) *If **Char** $\mathbb{F}_q = 2$, then $\mathcal{U}(\mathbb{F}_q G) \cong C_2^{7k} \rtimes \mathcal{U}(\mathbb{F}_q G / J(\mathbb{F}_q G))$.*

(2) *If* **Char** $\mathbb{F}_q = 3$, *then*

$$\mathcal{U}(\mathbb{F}_q G) \cong \begin{cases} (C_3^{21k} \rtimes C_3^k) \rtimes C_{q-1}^{14}, & \textit{if } q \equiv 1 \textit{ mod } 14; \\ (C_3^{21k} \rtimes C_3^k) \rtimes (C_{q-1}^2 \times C_{q^2-1}^6), & \textit{if } q \equiv -1 \textit{ mod } 14; \\ (C_3^{21k} \rtimes C_3^k) \rtimes (C_{q-1}^2 \times C_{q^6-1}^2), & \textit{if } q \equiv 3,5 \textit{ mod } 14; \\ (C_3^{21k} \rtimes C_3^k) \rtimes (C_{q-1}^2 \times C_{q^3-1}^4), & \textit{if } q \equiv -3,-5 \textit{ mod } 14. \end{cases}$$

(3) *If* **Char** $\mathbb{F}_q = 7$, *then*

$$\mathcal{U}(\mathbb{F}_q G) \cong \mathcal{K} \rtimes (C_{q-1}^2 \times GL(2, \mathbb{F}_q)),$$

where $\mathcal{K}$ is a non-abelian group of order 7^{36k} satisfying $\mathcal{K}^7 = 1$.

(4) *If* **Char** $\mathbb{F}_q \neq 2, 3, 7$, *then $\mathcal{U}(\mathbb{F}_q G)$ is isomorphic to*

   (a) $C_{q-1}^{14} \times GL(2, \mathbb{F}_q)^7$,   *if $q \equiv 1, 29 \textit{ mod } 42$.*

   (b) $C_{q-1}^2 \times C_{q^6-1}^2 \times GL(2, \mathbb{F}_q) \times GL(2, \mathbb{F}_{q^6})$,   *if $q \equiv 5, 17, 19, 31 \textit{ mod } 42$.*

   (c) $C_{q-1}^2 \times C_{q^3-1}^4 \times GL(2, \mathbb{F}_q) \times GL(2, \mathbb{F}_{q^3})^2$,   *if $q \equiv 11, 23, 25, 37 \textit{ mod } 42$.*

   (d) $C_{q-1}^2 \times C_{q^2-1}^6 \times GL(2, \mathbb{F}_q) \times GL(2, \mathbb{F}_{q^2})^3$,   *if $q \equiv 13, 41 \textit{ mod } 42$.*

Proof. Let $G = \langle r, s, t \mid r^3 = s^2 = t^7 = 1, \; rs = sr^{-1}, \; rt = tr, \; st = ts \rangle$.

1. Char $\mathbb{F}_q = 2$: For $p = 2$, we have $I_2 = \{1, s, rs, r^2 s\}$. Then, $\widehat{I_2} = 1 + \hat{r}s$. Let us consider a general element $\gamma = \sum_{k=0}^{1} \sum_{j=0}^{6} \sum_{i=0}^{2} h_{i+3j+21k} t^j r^i s^k \in \mathbb{F}_q G$ such that

$$\gamma(1 + \hat{r}s) = 0 \text{ i.e., } \gamma + \gamma \hat{r}s = 0.$$

After simplifying, we get the equations:

$$h_{3i} = h_{3i+m} = h_{3i+m+21} \text{ for } i = 0, 1, \ldots, 6, \text{ and } m = 0, 1, 2.$$

Consequently, $\gamma = \sum_{i=0}^{6} q_i t^i \hat{r}(1 + s)$. Thus,

$$\mathrm{Ann}(\widehat{I_2}) = \{\sum_{i=0}^{6} q_i t^i \hat{r}(1 + s) \mid q_i \in \mathbb{F}_q\}.$$

Observe that $t, \hat{r} \in Z(\mathbb{F}_q G)$ and $(1 + s)^2 = 0$, therefore $\mathrm{Ann}(\widehat{I_2})^2 = (0)$. As a result, $\mathrm{Ann}(\widehat{I_2}) \subseteq J(\mathbb{F}_q G)$. By using Lemma 2.3, we obtain $\mathrm{Ann}(\widehat{I_2}) = J(\mathbb{F}_q G)$. It is trivial to see that $V^2 = 1$ and therefore $V \cong C_2^{7k}$. Hence, with the help of [9, Lemma 2.1], we conclude that

$$\mathcal{U}(\mathbb{F}_q G) \cong C_2^{7k} \rtimes \mathcal{U}(\mathbb{F}_q G / J(\mathbb{F}_q G)).$$

2. Char $\mathbb{F}_q = 3$: Using the result given by Gildea [7, Theorem 1.1] for the particular case of $n = 7$, we get

$$\mathcal{U}(\mathbb{F}_{3^k}(C_7 \times D_6)) \cong (C_3^{21k} \rtimes C_3^{7k}) \rtimes \mathcal{U}(\mathbb{F}_{3^k}(C_{14})).$$

Finally, we deduce our result by utilizing [18, Theorem 4.3].

3. Char $\mathbb{F}_q = 7$: Let $M = \langle r, s \rangle$ and $N = \langle t \rangle$. Since $N \lhd G$, $G/N \cong M \cong D_6$. Now, define a map $\Psi : \mathbb{F}_q G \to \mathbb{F}_q M$ given by

$$\Psi(\sum_{j=0}^{2} \sum_{i=0}^{6} t^i r^j (a_{i+7j} + a_{i+7j+21}s)) = \sum_{j=0}^{2} \sum_{i=0}^{6} r^j (a_{i+7j} + a_{i+7j+21}s).$$

We obtain a group epimorphism $\Psi' : \mathcal{U}(\mathbb{F}_q G) \to \mathcal{U}(\mathbb{F}_q M)$ by restricting the ring epimorphism Ψ. Again, the restriction of the inclusion map from $\mathbb{F}_q M \to \mathbb{F}_q G$ provides a group monomorphism $\Phi : \mathcal{U}(\mathbb{F}_q M) \to \mathcal{U}(\mathbb{F}_q G)$ defined by

$$\Phi(\sum_{j=0}^{2} r^j (z_j + z_{j+3}s)) = \sum_{j=0}^{2} r^j (z_j + z_{j+3}s).$$

Let $\mathcal{K} = ker(\Psi')$. Since $\Psi' \circ \Phi = 1_{\mathcal{U}(\mathbb{F}_q M)}$,

$$\mathcal{U}(\mathbb{F}_q G) \cong \mathcal{K} \rtimes \mathcal{U}(\mathbb{F}_q M) \cong \mathcal{K} \rtimes \mathcal{U}(\mathbb{F}_q D_6).$$

Consider $v = \sum_{j=0}^{2} \sum_{i=0}^{6} t^i r^j (a_{i+7j} + a_{i+7j+21}s) \in \mathcal{K}$ i.e., $\Psi'(v) = 1$, which leads to the following equations:

$$\sum_{i=0}^{6} a_i = 1, \ \sum_{i=0}^{6} a_{i+7d} = 0 \text{ for } d = 1, \ldots, 5.$$

Thus,

$$\mathcal{K} = \{1 + \sum_{j=0}^{2} \sum_{i=1}^{6} (t^i - 1) r^j (b_{i+6j} + b_{i+6j+18}s) \mid b_i \in \mathbb{F}_q\}.$$

Consequently, it can be verified that $\mathcal{K}$ is a non-abelian group satisfying $\mathcal{K}^7 = 1$ with $|\mathcal{K}| = 7^{36k}$. Also, by [20, Theorem 2.3], we get $\mathcal{U}(\mathbb{F}_q D_6) \cong C_{q-1}^2 \times GL(2, \mathbb{F}_q)$. Hence,

$$\mathcal{U}(\mathbb{F}_q G) \cong \mathcal{K} \rtimes (C_{q-1}^2 \times GL(2, \mathbb{F}_q)).$$

4. Char $\mathbb{F}_q \neq 2, 3, 7$: Since $p \nmid |G|$, by Maschke's theorem $\mathbb{F}_q G$ is a semisimple group algebra and $J(\mathbb{F}_q G) = (0)$. Also,

$$\mathbb{F}_q G \cong \mathbb{F}_q(G/G') \bigoplus \Delta(G, G').$$

The conjugacy classes of G are:

$$[t^i] = \{t^i\} \qquad\qquad \text{for } i = 0, 1, \ldots, 6;$$
$$[rt^i] = \{rt^i, r^2 t^i\} \qquad \text{for } i = 0, 1, \ldots, 6;$$
$$[st^i] = \{st^i, rst^i, r^2 st^i\} \quad \text{for } i = 0, 1, \ldots, 6.$$

Observe that $G/G' \cong C_{14}$, thus the Wedderburn decomposition is

$$\mathbb{F}_q G \cong \mathbb{F}_q C_{14} \bigoplus_{j=1}^{m} M(n_j, R_j)$$

where $n_j \geq 2$ and for each $j \in \{1, \ldots, m\}$, R_j represents a division algebra over $\mathbb{F}_q$. Since class sums form a basis for $Z(\mathbb{F}_q G)$, $dim(Z(\mathbb{F}_q G)) = 21$. This implies, $m \leq 7$. Also, for any choice of characteristic p, $e = 42$. The structure of $\mathbb{F}_q C_{14}$ has been given by [18, Theorem 4.3].

(a) If $q \equiv 1, 29 \bmod 42$, then $\mathcal{B} = \{1\} \bmod 42$ or $\mathcal{B} = \{1, 29\} \bmod 42$. This provides $|S(\beta_g)| = 1$ for all $g \in G$. Using Lemmas 2.1 and 2.2, we get

$$\mathbb{F}_q G \cong \mathbb{F}_q^{14} \bigoplus_{j=1}^{7} M(n_j, \mathbb{F}_q).$$

The equation generated by equating the dimensions of both sides is $\sum_{j=1}^{7} n_j^2 = 28$. The only possible solution is $n_j = 2$ for all $j \in \{1, \ldots, 7\}$. Hence,

$$\mathbb{F}_q G \cong \mathbb{F}_q^{14} \bigoplus M(2, \mathbb{F}_q)^7.$$

(b) If $q \equiv 5, 17, 19, 31 \bmod 42$, then $\mathcal{B} = \{1, 5, 17, 25, 37, 41\} \bmod 42$ or $\mathcal{B} = \{1, 13, 19, 25, 31, 37\} \bmod 42$. Thus, $|S(\beta_g)| = 1$ for $g = 1$, r, s, and $|S(\beta_g)| = 6$ for $g = t$, rt, st. Then,

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^6}^2 \bigoplus M(n_1, \mathbb{F}_q) \bigoplus M(n_2, \mathbb{F}_{q^6}),$$

by using Lemmas 2.1 and 2.2. The necessary condition $n_1^2 + 6n_2^2 = 28$ is true only when $n_1 = n_2 = 2$. Hence,

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^6}^2 \bigoplus M(2, \mathbb{F}_q) \bigoplus M(2, \mathbb{F}_{q^6}).$$

(c) If $q \equiv 11, 23, 25, 37 \bmod 42$, then $\mathcal{B} = \{1, 11, 23, 25, 29, 37\} \bmod 42$ or $\mathcal{B} = \{1, 25, 37\} \bmod 42$. This gives $|S(\beta_g)| = 1$ for $g = 1$, r, s and $|S(\beta_g)| = 3$ for $g = t$, t^3, rt, rt^3, st, st^3. Using Lemmas 2.1 and 2.2,

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^3}^4 \bigoplus M(n_1, \mathbb{F}_q) \bigoplus M(n_2, \mathbb{F}_{q^3}) \bigoplus M(n_3, \mathbb{F}_{q^3}),$$

with the restriction that $n_1^2 + 3n_2^2 + 3n_3^2 = 28$. The only possible solution of the equation is $n_1 = n_2 = n_3 = 2$. Hence,

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^3}^4 \bigoplus M(2, \mathbb{F}_q) \bigoplus M(2, \mathbb{F}_{q^3})^2.$$

(d) If $q \equiv 13, 41 \bmod 42$, then $\mathcal{B} = \{1, 13\} \bmod 42$ or $\mathcal{B} = \{1, 41\} \bmod 42$. Thus, $|S(\beta_g)| = 1$ for $g = 1$, r, s, and $|S(\beta_g)| = 2$ for $g = t$, t^2, t^3, rt, rt^2, rt^3, st, st^2, st^3. Lemmas 2.1 and 2.2 provide us

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^6 \bigoplus M(n_1, \mathbb{F}_q) \bigoplus_{j=2}^{4} M(n_j, \mathbb{F}_{q^2}),$$

under the condition that $n_1^2 + \sum_{j=2}^{4} 2n_j^2 = 28$, which has the only solution as $n_j = 2$ for all $j \in \{1, \dots, 4\}$. Hence,

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^6 \bigoplus M(2, \mathbb{F}_q) \bigoplus M(2, \mathbb{F}_{q^2})^3. \qquad \square$$

Theorem 4.2. *Let $\mathbb{F}_q$ be a finite field of order $q = p^k$ with characteristic p and let $G = C_3 \times D_{14}$.*

(1) *If* **Char** $\mathbb{F}_q = 2$, *then*
$$\mathcal{U}(\mathbb{F}_q G) \cong C_2^{3k} \rtimes \mathcal{U}(\mathbb{F}_q G / J(\mathbb{F}_q G)).$$

(2) *If* **Char** $\mathbb{F}_q = 3$, *then*
$$\mathcal{U}(\mathbb{F}_q G) \cong \begin{cases} \mathcal{K} \rtimes (C_{q-1}^2 \times GL(2, \mathbb{F}_q)^3), & \text{if } q \equiv \pm 1 \bmod 7; \\ \mathcal{K} \rtimes (C_{q-1}^2 \times GL(2, \mathbb{F}_{q^3})), & \text{if } q \equiv \pm 2, \pm 3 \bmod 7, \end{cases}$$
where $\mathcal{K} \cong (((((((C_3^{21k} \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k)$.

(3) *If* **Char** $\mathbb{F}_q = 7$, *then*
$$\mathcal{U}(\mathbb{F}_q G) \cong \mathcal{K} \rtimes C_{q-1}^6,$$
where $\mathcal{K}$ *is a non-abelian group of order* 7^{36k} *satisfying* $\mathcal{K}^7 = 1$.

(4) *If* **Char** $\mathbb{F}_q \neq 2, 3, 7$, *then* $\mathcal{U}(\mathbb{F}_q G)$ *is isomorphic to*
  (a) $C_{q-1}^6 \times GL(2, \mathbb{F}_q)^9$,   *if* $q \equiv 1, 13 \bmod 42$.
  (b) $C_{q-1}^2 \times C_{q^2-1}^2 \times GL(2, \mathbb{F}_{q^3}) \times GL(2, \mathbb{F}_{q^6})$,   *if* $q \equiv 5, 11, 17, 23 \bmod 42$.
  (c) $C_{q-1}^6 \times GL(2, \mathbb{F}_{q^3})^3$,   *if* $q \equiv 19, 25, 31, 37 \bmod 42$.
  (d) $C_{q-1}^2 \times C_{q^2-1}^2 \times GL(2, \mathbb{F}_q)^3 \times GL(2, \mathbb{F}_{q^2})^3$, *if* $q \equiv 29, 41 \bmod 42$.

Proof. Let $G = \langle r, s, t \mid r^7 = s^2 = t^3 = 1, \; rs = sr^{-1}, \; rt = tr, \; st = ts \rangle$.

1.   Char $\mathbb{F}_q = 2$: For $p = 2$, we have $I_2 = \{1, s, rs, r^2 s, \dots, r^6 s\}$. Then, $\widehat{I_2} = 1 + \hat{r}s$. Let us consider a general element $\gamma = \sum_{k=0}^{1} \sum_{j=0}^{2} \sum_{i=0}^{6} h_{i+7j+21k} t^j r^i s^k \in \mathbb{F}_q G$ such that

$$\gamma(1 + \hat{r}s) = 0 \text{ i.e., } \gamma + \gamma \hat{r}s = 0.$$

After simplifying, we get the equations:

$$h_{7i} = h_{7i+m} = h_{7i+m+21} \text{ for } i = 0, 1, 2, \text{ and } m = 0, 1, \dots, 6.$$

Consequently, $\gamma = \sum_{i=0}^{2} q_i t^i \hat{r}(1+s)$. Thus,

$$\text{Ann}(\widehat{I_2}) = \{\sum_{i=0}^{2} q_i t^i \hat{r}(1+s) \mid q_i \in \mathbb{F}_q\}.$$

Observe that $t, \hat{r} \in Z(\mathbb{F}_q G)$ and $(1+s)^2 = 0$, therefore $\text{Ann}(\widehat{I_2})^2 = (0)$. As a result, $\text{Ann}(\widehat{I_2}) \subseteq J(\mathbb{F}_q G)$. By using Lemma 2.3, we obtain $\text{Ann}(\widehat{I_2}) = J(\mathbb{F}_q G)$. It is trivial to show that $V^2 = 1$ and therefore $V \cong C_2^{3k}$. Hence, with the help of [9, Lemma 2.1], we conclude that

$$\mathcal{U}(\mathbb{F}_q G) \cong C_2^{3k} \rtimes \mathcal{U}(\mathbb{F}_q G / J(\mathbb{F}_q G)).$$

2. Char $\mathbb{F}_q = 3$: Specifically, for $n = 7$, applying Theorem 3.1 provides

$$\mathcal{U}(\mathbb{F}_q(C_3 \times D_{14})) \cong \mathcal{K} \rtimes \mathcal{U}(\mathbb{F}_q D_{14})$$

where

$$\mathcal{K} \cong (((((((C_3^{21k} \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k) \rtimes C_3^k).$$

Furthermore, based on [17, Theorem 4.1], we find

$$\mathcal{U}(\mathbb{F}_q D_{14}) \cong \begin{cases} C_{q-1}^2 \times GL(2, \mathbb{F}_q)^3, & \text{if } q \equiv \pm 1 \bmod 7; \\ C_{q-1}^2 \times GL(2, \mathbb{F}_{q^3}), & \text{if } q \equiv \pm 2, \pm 3 \bmod 7. \end{cases}$$

3. Char $\mathbb{F}_q = 7$: Let $M = \langle s, t \rangle$ and $N = \langle r \rangle$. Since $N \lhd G$, $G/N \cong M \cong C_6$. Now, define a map $\Psi : \mathbb{F}_q G \to \mathbb{F}_q M$ given by

$$\Psi\left(\sum_{j=0}^{2} \sum_{i=0}^{6} t^j r^i (a_{i+7j} + a_{i+7j+21}s) \right) = \sum_{j=0}^{2} \sum_{i=0}^{6} t^j (a_{i+7j} + a_{i+7j+21}s).$$

We obtain a group epimorphism $\Psi' : \mathcal{U}(\mathbb{F}_q G) \to \mathcal{U}(\mathbb{F}_q M)$ by restricting the ring epimorphism Ψ. Again, the restriction of the inclusion map from $\mathbb{F}_q M \to \mathbb{F}_q G$ provides a group monomorphism $\Phi : \mathcal{U}(\mathbb{F}_q M) \to \mathcal{U}(\mathbb{F}_q G)$ defined by

$$\Phi\left(\sum_{j=0}^{2} t^j (z_j + z_{j+3}s) \right) = \sum_{j=0}^{2} t^j (z_j + z_{j+3}s).$$

Let $\mathcal{K} = ker(\Psi')$. Since $\Psi' \circ \Phi = 1_{\mathcal{U}(\mathbb{F}_q M)}$,

$$\mathcal{U}(\mathbb{F}_q G) \cong \mathcal{K} \rtimes \mathcal{U}(\mathbb{F}_q M) \cong \mathcal{K} \rtimes \mathcal{U}(\mathbb{F}_q C_6).$$

Consider $v = \sum_{j=0}^{2} \sum_{i=0}^{6} t^j r^i (a_{i+7j} + a_{i+7j+21}s) \in \mathcal{K}$ i.e., $\Psi'(v) = 1$, which leads to the following equations:

$$\sum_{i=0}^{6} a_i = 1, \ \sum_{i=0}^{6} a_{i+7d} = 0 \text{ for } d = 1, \ldots, 5.$$

Thus,

$$\mathcal{K} = \left\{ 1 + \sum_{j=0}^{2} \sum_{i=1}^{6} (r^i - 1)t^j (b_{i+6j} + b_{i+6j+18}s) \mid b_i \in \mathbb{F}_q \right\}.$$

Consequently, it can be verified that $\mathcal{K}$ is a non-abelian group satisfying $\mathcal{K}^7 = 1$ with $|\mathcal{K}| = 7^{36k}$. Also, by [18, Theorem 4.1], we get $\mathcal{U}(\mathbb{F}_q C_6) \cong C_{q-1}^6$. Hence,

$$\mathcal{U}(\mathbb{F}_q G) \cong \mathcal{K} \rtimes C_{q-1}^6.$$

4. Char $\mathbb{F}_q \neq 2, 3, 7$: Since $p \nmid |G|$, $\mathbb{F}_q G$ is a semisimple group algebra and $J(\mathbb{F}_q G) = (0)$. Also,

$$\mathbb{F}_q G \cong \mathbb{F}_q(G/G') \bigoplus \Delta(G, G').$$

The conjugacy classes of G are:

$$
\begin{aligned}
[t^i] &= \{t^i\} & &\text{for } i = 0, 1, 2; \\
[rt^i] &= \{rt^i, r^6 t^i\} & &\text{for } i = 0, 1, 2; \\
[r^2 t^i] &= \{r^2 t^i, r^5 t^i\} & &\text{for } i = 0, 1, 2; \\
[r^3 t^i] &= \{r^3 t^i, r^4 t^i\} & &\text{for } i = 0, 1, 2; \\
[st^i] &= \{st^i, rst^i, r^2 st^i, \ldots, r^6 st^i\} & &\text{for } i = 0, 1, 2.
\end{aligned}
$$

Observe that $G/G' \cong C_6$, thus

$$\mathbb{F}_q G \cong \mathbb{F}_q C_6 \bigoplus_{j=1}^m M(n_j, R_j)$$

where $n_j \geq 2$ and for each $j \in \{1, \ldots, m\}$, R_j represents a division algebra over $\mathbb{F}_q$. Since $dim(Z(\mathbb{F}_q G)) = 15$, $m \leq 9$. Also, for any value of characteristic p, $e = 42$. By [18, Theorem 4.1], the structure of $\mathbb{F}_q C_6$ has been determined.

(**a**) If $q \equiv 1, 13 \mod 42$, then $\mathcal{B} = \{1\} \mod 42$ or $\mathcal{B} = \{1, 13\} \mod 42$. This gives $|S(\beta_g)| = 1$ for all $g \in G$. Using Lemmas 2.1 and 2.2, we get

$$\mathbb{F}_q G \cong \mathbb{F}_q^6 \bigoplus_{j=1}^9 M(n_j, \mathbb{F}_q).$$

The equation $\sum_{j=1}^9 n_j^2 = 36$ is obtained by equating the dimensions of both sides. The only possible solution is when $n_j = 2$ for all $j \in \{1, \ldots, 9\}$. Hence,

$$\mathbb{F}_q G \cong \mathbb{F}_q^6 \bigoplus M(2, \mathbb{F}_q)^9.$$

(**b**) If $q \equiv 5, 11, 17, 23 \mod 42$, then $\mathcal{B} = \{1, 5, 17, 25, 37, 41\} \mod 42$ or $\mathcal{B} = \{1, 11, 23, 25, 29, 37\} \mod 42$. Thus, $|S(\beta_g)| = 1$ for $g = 1$, s, $|S(\beta_g)| = 2$ for $g = t$, st, $|S(\beta_g)| = 3$ for $g = r$, and $|S(\beta_g)| = 6$ for $g = rt$. Then,

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus M(n_1, \mathbb{F}_{q^3}) \bigoplus M(n_2, \mathbb{F}_{q^6}),$$

by using Lemmas 2.1 and 2.2. The necessary condition $3n_1^2 + 6n_2^2 = 36$ is true only when $n_1 = n_2 = 2$. Hence,

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus M(2, \mathbb{F}_{q^3}) \oplus M(2, \mathbb{F}_{q^6}).$$

(c) If $q \equiv 19, 25, 31, 37 \ mod \ 42$, then $\mathcal{B} = \{1, 13, 19, 25, 31, 37\} \ mod \ 42$ or $\mathcal{B} = \{1, 25, 37\} \ mod \ 42$. This provides $|S(\beta_g)| = 1$ for $g = 1, \ s, \ t, \ t^2, \ st, \ st^2$, and $|S(\beta_g)| = 3$ for $g = r, \ rt, \ rt^2$. Using Lemmas 2.1 and 2.2,

$$\mathbb{F}_q G \cong \mathbb{F}_q^6 \bigoplus_{j=1}^{3} M(n_j, \mathbb{F}_{q^3}),$$

with the restriction that $3(n_1^2 + n_2^2 + n_3^2) = 36$. The only possible solution of this equation is $n_1 = n_2 = n_3 = 2$. Hence,

$$\mathbb{F}_q G \cong \mathbb{F}_q^6 \oplus M(2, \mathbb{F}_{q^3})^3.$$

(d) If $q \equiv 29, 41 \ mod \ 42$, then $\mathcal{B} = \{1, 29\} \ mod \ 42$ or $\mathcal{B} = \{1, 41\} \ mod \ 42$. Thus, $|S(\beta_g)| = 1$ for $g = 1, \ s, \ r, \ r^2, \ r^3$, and $|S(\beta_g)| = 2$ for $g = t, \ st, \ rt, \ r^2t, \ r^3t$. Lemmas 2.1 and 2.2 provide us

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \bigoplus_{j=1}^{3} M(n_j, \mathbb{F}_q) \bigoplus_{j=4}^{6} M(n_j, \mathbb{F}_{q^2}),$$

with the condition that $\sum_{j=1}^{3} n_j^2 + \sum_{j=4}^{6} 2n_j^2 = 36$, which only has the solution as $n_j = 2$ for all $j \in \{1, \ldots, 6\}$. Hence,

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \oplus \mathbb{F}_{q^2}^2 \oplus M(2, \mathbb{F}_q)^3 \oplus M(2, \mathbb{F}_{q^2})^3. \qquad \square$$

Theorem 4.3. *Let* $\mathbb{F}_q$ *be a finite field of order* $q = p^k$ *with characteristic* $p \neq 2, 3, 7$ *and let* $G = C_7 \rtimes C_6$. *Then,*

$$\mathcal{U}(\mathbb{F}_q G) \cong \begin{cases} C_{q-1}^6 \times GL(6, \mathbb{F}_q), & \text{if } q \equiv 1, 13, 19, 25, 31, 37 \ mod \ 42; \\ C_{q-1}^2 \times C_{q^2-1}^2 \times GL(6, \mathbb{F}_q), & \text{if } q \equiv 5, 11, 17, 23, 29, 41 \ mod \ 42. \end{cases}$$

Proof. Let $G = \langle a, b \mid a^7 = b^6 = 1, \ bab^{-1} = a^3 \rangle$.

Since $p \nmid |G|$, $\mathbb{F}_q G$ is a semisimple group algebra and $J(\mathbb{F}_q G) = (0)$. Also,

$$\mathbb{F}_q G \cong \mathbb{F}_q(G/G') \oplus \Delta(G, G').$$

The conjugacy classes of G are:

$$[1] = \{1\},$$
$$[a] = \{a, a^2, \ldots, a^6\},$$
$$[b^i] = \{b^i, ab^i, a^2b^i, \ldots, a^6b^i\} \quad \text{for } i = 1, \ldots, 5.$$

Observe that $G/G' \cong C_6$, thus

$$\mathbb{F}_q G \cong \mathbb{F}_q C_6 \bigoplus_{j=1}^{m} M(n_j, R_j)$$

where $n_j \geq 2$ and for each $j \in \{1, \ldots, m\}$, R_j represents a division algebra over $\mathbb{F}_q$. Since $dim(Z(\mathbb{F}_q G)) = 7$, $m = 1$. Also, for any choice of characteristic p, $e = 42$. The structure of $\mathbb{F}_q C_6$ has been given by ([18], Theorem 4.1).

(**a**) If $q \equiv 1, 13, 19, 25, 31, 37 \ mod \ 42$, then $\mathcal{B} = \{1\} \ mod \ 42$ or $\mathcal{B} = \{1, 13\} \ mod \ 42$ or $\mathcal{B} = \{1, 13, 19, 25, 31, 37\} \ mod \ 42$ or $\mathcal{B} = \{1, 25, 37\} \ mod \ 42$. This gives $|S(\beta_g)| = 1$ for all $g \in G$. Using Lemmas 2.1 and 2.2, we get

$$\mathbb{F}_q G \cong \mathbb{F}_q^6 \bigoplus M(n_1, \mathbb{F}_q)$$

with the condition that $n_1^2 = 36$. Hence, $n_1 = 6$ and consequently

$$\mathbb{F}_q G \cong \mathbb{F}_q^6 \bigoplus M(6, \mathbb{F}_q).$$

(**b**) If $q \equiv 5, 11, 17, 23, 29, 41 \ mod \ 42$, then $\mathcal{B} = \{1, 5, 17, 25, 37, 41\} \ mod \ 42$ or $\mathcal{B} = \{1, 11, 23, 25, 29, 37\} \ mod \ 42$ or $\mathcal{B} = \{1, 29\} \ mod \ 42$ or $\mathcal{B} = \{1, 41\} \ mod \ 42$. This gives $|S(\beta_g)| = 1$ for $g = 1$, a, b^3, and $|S(\beta_g)| = 2$ for $g = b$, b^2. Using Lemmas 2.1 and 2.2, we get

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus M(n_1, \mathbb{F}_q)$$

with the restriction that $n_1^2 = 36$. Hence, $n_1 = 6$ and as a result

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus M(6, \mathbb{F}_q). \qquad \square$$

Theorem 4.4. *Let $\mathbb{F}_q$ be a finite field of order $q = p^k$ with characteristic $p \neq 2, 3, 7$ and let $G = C_2 \times (C_7 \rtimes C_3)$. Then,*

$$\mathcal{U}(\mathbb{F}_q G) \cong \begin{cases} C_{q-1}^6 \times GL(3, \mathbb{F}_q)^4, & if \ q \equiv 1, 25, 37 \ mod \ 42; \\ C_{q-1}^2 \times C_{q^2-1}^2 \times GL(3, \mathbb{F}_{q^2})^2, & if \ q \equiv 5, 17, 41 \ mod \ 42; \\ C_{q-1}^6 \times GL(3, \mathbb{F}_{q^2})^2, & if \ q \equiv 13, 19, 31 \ mod \ 42; \\ C_{q-1}^2 \times C_{q^2-1}^2 \times GL(3, \mathbb{F}_q)^4, & if \ q \equiv 11, 23, 29 \ mod \ 42. \end{cases}$$

Proof. Let $G = \langle a, b, c \mid a^7 = b^3 = c^2 = 1, \ bab^{-1} = a^2, \ ac = ca, \ bc = cb \rangle$. Since $p \nmid |G|$, by Maschke's theorem $\mathbb{F}_q G$ is a semisimple group algebra and $J(\mathbb{F}_q G) = (0)$. Also,

$$\mathbb{F}_q G \cong \mathbb{F}_q(G/G') \bigoplus \Delta(G, G').$$

The conjugacy classes of G are:

$$[1] = \{1\},$$
$$[c] = \{c\},$$
$$[ac^i] = \{ac^i, a^2 c^i, a^4 c^i\} \qquad \text{for } i = 0, 1;$$
$$[a^3 c^i] = \{a^3 c^i, a^5 c^i, a^6 c^i\} \qquad \text{for } i = 0, 1;$$
$$[bc^i] = \{bc^i, abc^i, a^2 bc^i, \ldots, a^6 bc^i\} \qquad \text{for } i = 0, 1;$$
$$[b^2 c^i] = \{b^2 c^i, ab^2 c^i, a^2 b^2 c^i, \ldots, a^6 b^2 c^i\} \qquad \text{for } i = 0, 1.$$

Observe that $G/G' \cong C_6$, thus

$$\mathbb{F}_q G \cong \mathbb{F}_q C_6 \bigoplus_{j=1}^{m} M(n_j, R_j)$$

where $n_j \geq 2$ and for each $j \in \{1, \ldots, m\}$, R_j represents a division algebra over $\mathbb{F}_q$. Since $dim(Z(\mathbb{F}_q G)) = 10$, $m \leq 4$. Here, for any value of characteristic p, $e = 42$. By [18, Theorem 4.1], the structure of $\mathbb{F}_q C_6$ has been determined.

(a) If $q \equiv 1, 25, 37 \mod 42$, then $\mathcal{B} = \{1\} \mod 42$ or $\mathcal{B} = \{1, 25, 37\} \mod 42$. This gives $|S(\beta_g)| = 1$ for all $g \in G$. Using Lemmas 2.1 and 2.2, we get

$$\mathbb{F}_q G \cong \mathbb{F}_q^6 \bigoplus_{j=1}^{4} M(n_j, \mathbb{F}_q).$$

The equation $\sum_{j=1}^{4} n_j^2 = 36$ is obtained by equating the dimensions of both the sides. The only possible solution is $n_j = 3$ for all $j \in \{1, \ldots, 4\}$. Hence,

$$\mathbb{F}_q G \cong \mathbb{F}_q^6 \bigoplus M(3, \mathbb{F}_q)^4.$$

(b) If $q \equiv 5, 17, 41 \mod 42$, then $\mathcal{B} = \{1, 5, 17, 25, 37, 41\} \mod 42$ or $\mathcal{B} = \{1, 41\} \mod 42$. Thus, $|S(\beta_g)| = 1$ for $g = 1$, c, and $|S(\beta_g)| = 2$ for $g = a$, ac, b, bc. Then,

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus M(n_1, \mathbb{F}_{q^2}) \bigoplus M(n_2, \mathbb{F}_{q^2}),$$

by using Lemmas 2.1 and 2.2. The necessary condition of $2(n_1^2 + n_2^2) = 36$ is satisfied only when $n_1 = n_2 = 3$. Hence,

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus M(3, \mathbb{F}_{q^2})^2.$$

(c) If $q \equiv 13, 19, 31 \mod 42$, then $\mathcal{B} = \{1, 13, 19, 25, 31, 37\} \mod 42$ or $\mathcal{B} = \{1, 13\} \mod 42$. It follows from here that $|S(\beta_g)| = 1$ for $g = 1$, c, b, b^2, bc, $b^2 c$, and $|S(\beta_g)| = 2$ for $g = a$, ac. Using Lemmas 2.1 and 2.2,

$$\mathbb{F}_q G \cong \mathbb{F}_q^6 \bigoplus M(n_1, \mathbb{F}_{q^2}) \bigoplus M(n_2, \mathbb{F}_{q^2}),$$

with the restriction that $2(n_1^2 + n_2^2) = 36$. The only possible solution of the equation is $n_1 = n_2 = 3$. Hence,

$$\mathbb{F}_q G \cong \mathbb{F}_q^6 \bigoplus M(3, \mathbb{F}_{q^2})^2.$$

(**d**) If $q \equiv 11, 23, 29 \ mod \ 42$, then $\mathcal{B} = \{1, 11, 23, 25, 29, 37\} \ mod \ 42$ or $\mathcal{B} = \{1, 29\} \ mod \ 42$. Thus, $|S(\beta_g)| = 1$ for $g = 1$, c, a, a^3, ac, $a^3 c$, and $|S(\beta_g)| = 2$ for $g = b$, bc. Lemmas 2.1 and 2.2 guide us

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus_{j=1}^{4} M(n_j, \mathbb{F}_q),$$

with the condition that $\sum\limits_{j=1}^{4} n_j^2 = 36$, which is only possible when $n_j = 3$ for all $j \in \{1, \ldots, 4\}$. Hence,

$$\mathbb{F}_q G \cong \mathbb{F}_q^2 \bigoplus \mathbb{F}_{q^2}^2 \bigoplus M(3, \mathbb{F}_q)^4. \qquad \square$$

Theorem 4.5. *Let $\mathbb{F}_q$ be a finite field of order $q = p^k$ with characteristic p and let $G = C_{42}$.*

(1) *If* **Char** $\mathbb{F}_q \neq 2, 3, 7$, *then*

$$\mathcal{U}(\mathbb{F}_q G) \cong \begin{cases} C_{q-1}^{42}, & \text{if } q \equiv 1 \ mod \ 21; \\ C_{q-1}^2 \times C_{q^2-1}^2 \times C_{q^3-1}^4 \times C_{q^6-1}^4, & \text{if } q \equiv 2, 11 \ mod \ 21; \\ C_{q-1}^6 \times C_{q^3-1}^{12}, & \text{if } q \equiv 4, 16 \ mod \ 21; \\ C_{q-1}^2 \times C_{q^2-1}^2 \times C_{q^6-1}^6, & \text{if } q \equiv 5, 17 \ mod \ 21; \\ C_{q-1}^{14} \times C_{q^2-1}^{14}, & \text{if } q \equiv 8 \ mod \ 21; \\ C_{q-1}^6 \times C_{q^6-1}^6, & \text{if } q \equiv 10, 19 \ mod \ 21; \\ C_{q-1}^6 \times C_{q^2-1}^{18}, & \text{if } q \equiv 13 \ mod \ 21; \\ C_{q-1}^2 \times C_{q^2-1}^{20}, & \text{if } q \equiv 20 \ mod \ 21. \end{cases}$$

(2) *If* **Char** $\mathbb{F}_q = 2$, *then*

$$\mathcal{U}(\mathbb{F}_q G) \cong \begin{cases} C_2^{21k} \times C_{q-1}^{21}, & \text{if } q \equiv 1 \ mod \ 21; \\ C_2^{21k} \times C_{q-1} \times C_{q^2-1} \times C_{q^3-1}^2 \times C_{q^6-1}^2, & \text{if } q \equiv 2, 11 \ mod \ 21; \\ C_2^{21k} \times C_{q-1}^3 \times C_{q^3-1}^6, & \text{if } q \equiv 4, 16 \ mod \ 21; \\ C_2^{21k} \times C_{q-1}^7 \times C_{q^2-1}^7, & \text{if } q \equiv 8 \ mod \ 21. \end{cases}$$

(3) *If* **Char** $\mathbb{F}_q = 3$, *then*

$$\mathcal{U}(\mathbb{F}_q G) \cong \begin{cases} C_3^{28k} \times C_{q-1}^{14}, & \text{if } q \equiv 1 \bmod 14; \\ C_3^{28k} \times C_{q-1}^2 \times C_{q^6-1}^2, & \text{if } q \equiv 3, 5 \bmod 14; \\ C_3^{28k} \times C_{q-1}^2 \times C_{q^3-1}^4, & \text{if } q \equiv 9, 11 \bmod 14; \\ C_3^{28k} \times C_{q-1}^2 \times C_{q^2-1}^6, & \text{if } q \equiv 13 \bmod 14. \end{cases}$$

(4) *If* **Char** $\mathbb{F}_q = 7$, *then* $\mathcal{U}(\mathbb{F}_q G) \cong C_7^{36k} \times C_{q-1}^6$.

Proof. Let $C_{42} = \langle a \mid a^{42} = 1 \rangle$.

1. Char $\mathbb{F}_q \neq 2, 3, 7$: By Maschke's theorem $\mathbb{F}_q G$ is a semisimple group algebra. Consider,

$$\mathbb{F}_q G \cong \mathbb{F}_q(C_2 \times C_{21}) \cong (\mathbb{F}_q C_2)C_{21} \cong (\mathbb{F}_q \bigoplus \mathbb{F}_q)C_{21} \cong (\mathbb{F}_q C_{21})^2.$$

Let $P = C_{21} = \langle b \mid b^{21} = 1 \rangle$. Now, our goal is to determine the structure of $\mathbb{F}_q P$. In view of this, all the conjugacy classes of P are p-regular and $e = 21$.
If $q \equiv 1 \bmod 21$, then $\mathcal{B} = \{1\} \bmod 21$ and thus $|S(\beta_g)| = 1$ for all $g \in P$. By Lemmas 2.1 and 2.2, we get

$$\mathbb{F}_q P \cong \mathbb{F}_q^{21}.$$

If $q \equiv 2, 11 \bmod 21$, then $\mathcal{B} = \{1, 2, 4, 8, 11, 16\} \bmod 21$, which implies $|S(\beta_g)| = 1$ for $g = 1$, $|S(\beta_g)| = 2$ for $g = b^7$, $|S(\beta_g)| = 3$ for $g = b^3$, b^9, and $|S(\beta_g)| = 6$ for $g = b$, b^5. By Lemmas 2.1 and 2.2,

$$\mathbb{F}_q P \cong \mathbb{F}_q \bigoplus \mathbb{F}_{q^2} \bigoplus \mathbb{F}_{q^3}^2 \bigoplus \mathbb{F}_{q^6}^2.$$

If $q \equiv 4, 16 \bmod 21$, then $\mathcal{B} = \{1, 4, 16\} \bmod 21$. Thus, $|S(\beta_g)| = 1$ for $g = 1$, b^7, b^{14}, and $|S(\beta_g)| = 3$ for $g = b$, b^2, b^3, b^5, b^9, b^{10}. Using Lemmas 2.1 and 2.2,

$$\mathbb{F}_q P \cong \mathbb{F}_q^3 \bigoplus \mathbb{F}_{q^3}^6.$$

If $q \equiv 5, 17 \bmod 21$, then $\mathcal{B} = \{1, 4, 5, 16, 17, 20\} \bmod 21$. Thus, $|S(\beta_g)| = 1$ for $g = 1$, $|S(\beta_g)| = 2$ for $g = b^7$, and $|S(\beta_g)| = 6$ for $g = b$, b^2, b^3. Using Lemmas 2.1 and 2.2,

$$\mathbb{F}_q P \cong \mathbb{F}_q \bigoplus \mathbb{F}_{q^2} \bigoplus \mathbb{F}_{q^6}^3.$$

If $q \equiv 8 \bmod 21$, then $\mathcal{B} = \{1, 8\} \bmod 21$. This leads us to $|S(\beta_g)| = 1$ for $g = 1$, b^3, b^6, b^9, b^{12}, b^{15}, b^{18}, and $|S(\beta_g)| = 2$ for $g = b$, b^2, b^4, b^5, b^7, b^{10}, b^{13}. Lemmas 2.1 and 2.2 imply

$$\mathbb{F}_q P \cong \mathbb{F}_q^7 \bigoplus \mathbb{F}_{q^2}^7.$$

If $q \equiv 10, 19 \bmod 21$, then $\mathcal{B} = \{1, 4, 10, 13, 16, 19\} \bmod 21$. Thus $|S(\beta_g)| = 1$ for $g = 1$, b^7, b^{14}, and $|S(\beta_g)| = 6$ for $g = b$, b^2, b^3. By Lemmas 2.1 and 2.2,

$$\mathbb{F}_q P \cong \mathbb{F}_q^3 \bigoplus \mathbb{F}_{q^6}^3.$$

If $q \equiv 13 \bmod 21$, then $\mathcal{B} = \{1, 13\} \bmod 21$. Hence $|S(\beta_g)| = 1$ for $g = 1$, b^7, b^{14}, and $|S(\beta_g)| = 2$ for $g = b$, b^2, b^3, b^4, b^6, b^8, b^9, b^{11}, b^{16}. By Lemmas 2.1 and 2.2,

$$\mathbb{F}_q P \cong \mathbb{F}_q^3 \bigoplus \mathbb{F}_{q^2}^9.$$

If $q \equiv 20 \bmod 21$, then $\mathcal{B} = \{1, 20\} \bmod 21$. Thus, $|S(\beta_g)| = 1$ for $g = 1$, and $|S(\beta_g)| = 2$ for $g = b^m$ for $m = 1, 2, \ldots, 10$. Lemmas 2.1 and 2.2 imply that

$$\mathbb{F}_q P \cong \mathbb{F}_q \bigoplus \mathbb{F}_{q^2}^{10}.$$

2. Char $\mathbb{F}_q = 2$: Let $H = \langle a^{21} \rangle$. Then, $[G : H] = 21 \neq 0$ in $\mathbb{F}_q$. By [15, Theorem 7.2.7 and Lemma 8.1.17], $J(\mathbb{F}_q G) = \Delta(G, H)$ and therefore $\mathbb{F}_q G / J(\mathbb{F}_q G) \cong \mathbb{F}_q(G/H) \cong \mathbb{F}_q C_{21}$. Since $dim(J(\mathbb{F}_q G)) = 21$ and $J(\mathbb{F}_q G)^2 = (0)$, $V^2 = 1$ and $V \cong C_2^{21k}$. Thus,

$$\mathcal{U}(\mathbb{F}_q G) \cong C_2^{21k} \times \mathcal{U}(\mathbb{F}_q C_{21}).$$

The structure of $\mathcal{U}(\mathbb{F}_q C_{21})$ has been derived in part 1.

3. Char $\mathbb{F}_q = 3$: Let $N = \langle a^{14} \rangle$. Then, $[G : N] = 14 \neq 0$ in $\mathbb{F}_q$. By [15, Theorem 7.2.7 and Lemma 8.1.17], $J(\mathbb{F}_q G) = \Delta(G, N)$ and $\mathbb{F}_q G / J(\mathbb{F}_q G) \cong \mathbb{F}_q C_{14}$. Since $dim(J(\mathbb{F}_q G)) = 28$ and $J(\mathbb{F}_q G)^3 = (0)$, $V^3 = 1$ and $V \cong C_3^{28k}$. Thus,

$$\mathcal{U}(\mathbb{F}_q G) \cong C_3^{28k} \times \mathcal{U}(\mathbb{F}_q C_{14}).$$

The rest follows by [18, Theorem 4.3].

4. Char $\mathbb{F}_q = 7$: Let $K = \langle a^6 \rangle$. Then, $[G : K] = 6 \neq 0$ in $\mathbb{F}_q$. Thus, by [15, Theorem 7.2.7 and Lemma 8.1.17], $J(\mathbb{F}_q G) = \Delta(G, K)$ and $\mathbb{F}_q G / J(\mathbb{F}_q G) \cong \mathbb{F}_q C_6$. Since $dim(J(\mathbb{F}_q G)) = 36$ and $J(\mathbb{F}_q G)^7 = (0)$, $V^7 = 1$ and $V \cong C_7^{36k}$. Hence,

$$\mathcal{U}(\mathbb{F}_q G) \cong C_7^{36k} \times \mathcal{U}(\mathbb{F}_q C_6).$$

By [18, Theorem 4.1],

$$\mathcal{U}(\mathbb{F}_q G) \cong C_7^{36k} \times C_{q-1}^6. \qquad \square$$

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References

[1] S. F. Ansari and M. Sahai, *Unit groups of group algebras of groups of order 20*, Quaest. Math., 44(4) (2021), 503-511.

[2] A. Bovdi and L. Erdei, *Unitary units in modular group algebras of 2-groups*, Comm. Algebra, 28(2) (2000), 625-630.

[3] V. A. Bovdi and A. N. Grishkov, *Unitary and symmetric units of a commutative group algebra*, Proc. Edinb. Math. Soc. (2), 62(3) (2019), 641-654.

[4] L. Creedon and J. Gildea, *The structure of the unit group of the group algebra* $\mathbb{F}_{3^k} D_6$, Int. J. Pure Appl. Math., 45(2) (2008), 315-320.

[5] L. Creedon and J. Gildea, *The structure of the unit group of the group algebra* $\mathbb{F}_{2^k} D_8$, Canad. Math. Bull., 54(2) (2011), 237-243.

[6] R. A. Ferraz, *Simple components of the center of* $FG/J(FG)$, Comm. Algebra, 36(9) (2008), 3191-3199.

[7] J. Gildea and R. Taylor, *Units of the group algebra of the group* $C_n \times D_6$ *over any finite field of characteristic* 3, Int. Electron. J. Algebra, 24 (2018), 62-67.

[8] K. Kaur and M. Khan, *Units in* $F_2 D_{2p}$, J. Algebra Appl., 13(2) (2014), 1350090 (9 pp).

[9] N. Makhijani, R. K. Sharma and J. B. Srivastava, *A note on units in* $\mathbb{F}_{p^m} D_{2p^n}$, Acta Math. Acad. Paedagog. Nyházi. (N.S.), 30(1) (2014), 17-25.

[10] N. Makhijani, R. K. Sharma and J. B. Srivastava, *Units in* $\mathbb{F}_{2^k} D_{2n}$, Int. J. Group Theory, 3(3) (2014), 25-34.

[11] N. Makhijani, R. K. Sharma and J. B. Srivastava, *The unit group of* $\mathbb{F}_q[D_{30}]$, Serdica Math. J., 41(2-3) (2015), 185-198.

[12] S. Malik and R. K. Sharma, *Describing the group of units of integral grouprings* $\mathbb{Z}D_8$ *and* $\mathbb{Z}D_{12}$, Asian-Eur. J. Math., 17(1) (2024), 2350236 (13 pp).

[13] S. Malik, R. K. Sharma and M. Sahai, *The structure of the unit group of the group algebras* $\mathbb{F}_{3^k} D_{6n}$ *and* $\mathbb{F}_q D_{42}$, Ann. Math. Inform., DOI: 10.33039/ami.2024.08.001.

[14] G. Mittal and R. K. Sharma, *Wedderburn decomposition of a semisimple group algebra* $\mathbb{F}_q G$ *from a subalgebra of factor group of* G, Int. Electron. J. Algebra, 32 (2022), 91-100.

[15] D. S. Passman, The Algebraic Structure of Group Rings, Wiley Interscience, New York, 1977.

[16] C. Polcino Milies and S. K. Sehgal, An Introduction to Group Rings, Kluwer Academic Publishers, Dordrecht, 2002.

[17] M. Sahai and S. F. Ansari, *Unit groups of group algebras of certain dihedral groups-II*, Asian-Eur. J. Math., 12(4) (2019), 1950066 (12 pp).

[18] M. Sahai and S. F. Ansari, *Unit groups of finite group algebras of abelian groups of order at most* 16, Asian-Eur. J. Math., 14(3) (2021), 2150030 (17 pp).

[19] M. Sahai and S. F. Ansari, *Unit groups of group algebras of groups of order* 18, Comm. Algebra, 49(8) (2021), 3273-3282.

[20] R. K. Sharma, J. B. Srivastava and M. Khan, *The unit group of FS_3*, Acta Math. Acad. Paedagog. Nyházi. (N.S.), 23(2) (2007), 129-142.

[21] G. Tang, Y. Wei and Y. Li, *Unit groups of group algebras of some small groups*, Czechoslovak Math. J., 64(139)(1) (2014), 149-157.

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