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Expanding on Knowledge of Student Thinking through Mathematics Teachers' Engagement in Lesson Study

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Abstract

This study aims to expand on knowledge of student thinking by examining the practices of mathematics teachers engaged in a lesson study process designed to enhance their knowledge of student thinking. Conducted as a qualitative case study, the research involved three high school mathematics teachers as participants. The primary data sources for the study are the collaborative planning meetings conducted by mathematics teachers involved in the lesson study process and their classroom teaching practices. In this context, twenty-two hours of lesson observations served as the core dataset, supplemented by field notes documenting the entire process. Teachers' discourses and actions regarding student thinking were systematically analyzed using the constant comparison method. The main components of knowledge of student thinking were (a) building on students' mathematical ideas, (b) promoting students' thinking of mathematics, (c) triggering and considering divergent thoughts, (d) engaging students in mathematical learning, (e) evaluating students' understanding, (f) motivating students' learning, (g) considering students' misconceptions and errors, (h) considering students' difficulties, and (i) estimating students' possible ideas and approaches. The 47 subcodes identified within these categories elucidate the specific actions undertaken by teachers to consider and respond to student thinking within their classroom practices. This framework, encompassing components related to teachers' knowledge of student thinking, is proposed as a valuable tool for research on the professional development of both pre-service and in-service teachers.

Keywords: Knowledge Of Student Thinking, Lesson Study, Mathematics Teacher, Pedagogical Content Knowledge, Professional Development, Qualitative Case Study.

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Matematik Öğretmenlerinin Katılımıyla Gerçekleştirilen Ders İmecesi ile Öğrenci Düşüncesi Bilgisinin Detaylandırılması

Özet

Bu çalışmanın amacı, öğretmenlerin öğrenci düşüncesine ilişkin bilgilerini geliştirmek için tasarlanmış bir ders imecesi sürecinde matematik öğretmenlerinin uygulamalarına dayalı olarak öğrenci düşüncesi bilgisini detaylandırmaktır. Bu nitel durum çalışmasının katılımcıları üç lise matematik öğretmenidir. Ders imecesi sürecini gerçekleştiren matematik öğretmenlerinin işbirliğine dayalı yürüttükleri planlama toplantıları ve gerçekleştirdikleri sınıf içi öğretim uygulamaları araştırmanın temel veri kaynaklarıdır. Bu kapsamda yirmi iki saatlik ders gözlemlerinden elde edilen verilere odaklanılmış ve tüm sürece ilişkin alan notları ile bu veriler zenginleştirilmiştir. Sürekli karşılaştırmalar yoluyla öğretmenlerin öğrenci düşüncesi bilgisi ile ilişkili olan söylemleri ve eylemleri analiz edilmiştir. Bu analizlere dayalı olarak, öğrenci düşüncesi bilgisinin temel bileşenleri (a) öğrencilerin matematiksel fikirlerini dayanak alıp onları geliştirme, (b) öğrencilerin matematik düşünmeye teşvik etme, (c) farklı düşünceleri ortaya çıkarma ve dikkate alma, (d) öğrencilerin matematik öğrenmeye katılımlarını sağlama, (e) öğrenci anlayışlarını değerlendirme, (f) öğrencileri öğrenmeye motive etmek, (g) öğrencilerin kavram yanılgılarını ve hatalarını dikkate alma, (h) öğrencilerin zorluklarını dikkate alma ve (i) öğrencilerin olası fikirlerini ve yaklaşımlarını tahmin etme olarak kategorilendirilmiştir. Bu kategoriler altında ortaya çıkarılan 47 alt kod öğretmenlerin sınıf içi uygulamalarında öğrenci düşüncelerini dikkate alıp onlara yanıt verme eylemlerini detaylandırmaktadır. Öğrenci düşüncesi bilgisi ile ilgili bileşenleri içeren bu çerçevenin hem öğretmen adaylarının hem de öğretmenlerin mesleki gelişimlerine odaklanan araştırmalarda kullanılabileceği düşünülmektedir.

Anahtar Kelimeler: Alan Öğretimi Bilgisi, Ders İmecesi, Matematik Öğretmeni, Mesleki Gelişim, Nitel Durum Çalışması, Öğrenci Düşüncesi Bilgisi.

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1. Introduction

Several educational reform documents advocate for a transition from teacher-centered to student-centered paradigms in instructional practices (Wilson et al., 2013). The report "Principles and Standards for School Mathematics" by the National Council of Teachers of Mathematics (NCTM, 2000) underscores the importance of understanding students' existing knowledge, their learning needs, and the support required to enhance their conceptual understanding. It clearly asserts that an effective mathematics teaching approach necessitates a focus on students and their cognitive processes. Mathematics educators are expected to utilize their specialized mathematical knowledge, taking into account students' individual needs, to implement effective teaching strategies.

A teacher's understanding of their students' learning of a specific mathematical concept is closely linked to their knowledge of content and student. Hill et al. (2008) emphasized that "knowledge of content and students, or teachers' knowledge of students' mathematical thinking and learning, which is a subset of pedagogical content knowledge, widely believed to be an important component of teacher knowledge" (p. 373). Similarly, Rowland et al. (2005) conceptualized mathematical content knowledge as an integration of subject matter knowledge and pedagogical content knowledge. They empirically developed the Knowledge Quartet framework to capture the complexities of teacher knowledge in mathematics education. In this framework, the dimension of contingency, which encompasses the consideration of students' thinking, involves an approach to responding to students' ideas. According to Franke and Kazemi (2001), emphasizing students' mathematical thinking is a critical component for integrating knowledge of teaching, mathematics, and student understanding. Implementing teaching practices that prioritize student thinking involves considering multiple factors, such as students' existing knowledge, potential areas of misconception, and preferred learning methods. Such teachers' knowledge encompasses an awareness of students' thinking processes and their comprehension of the subject matter (Schilling et al., 2007). Within this framework, teachers' 'Knowledge of Student Thinking (KoST)' plays a crucial role.

The KoST enables teachers to interpret students' errors, misconceptions, and conceptual understandings, as well as to recognize interactions that can enhance students' cognitive processes and promote more effective learning (Empson & Junk, 2004). In their study, Brendefur et al. (2013) elucidate that KoST pertains to the pedagogical knowledge that enables teachers to anticipate potential student solution strategies, foresee common misconceptions, and interpret students' mathematical ideas. An et al. (2004) identify four key components of KoST, which include addressing students' misconceptions, building on students' mathematical ideas, engaging students in mathematical learning, and promoting students' thinking mathematically. Lee (2006) investigated teachers' understanding of students' mathematical thinking and further expanded this framework to include components such as questioning that triggers divergent thinking, motivating student learning, evaluating student understanding, and using prior knowledge. Researchers and policymakers in mathematics education emphasize the importance of teachers' awareness of students' existing knowledge and cognitive processes related to specific mathematical concepts, as this is crucial for fostering deeper student understanding (National Board for Professional Teaching Standards, 1997; NCTM, 2000). Therefore, it is essential that teachers develop an awareness of the significance of KoST and its reflection in teaching practices.

Numerous researchers have examined teachers' understanding of students' mathematical thinking (Corey et al., 2021; Moon, 2023; Van Zoest et al., 2010), while others have emphasized the development of teachers' knowledge and instructional practices informed by students' mathematical thinking (Gehrtz et al., 2022; Fernández et al., 2012; Liang, 2023). Corey et al. (2021) conducted an analysis of ten written instructional products to examine how these materials conveyed knowledge of student mathematical thinking. The findings revealed that the most effective instructional products were those that explicitly addressed specific tasks or mathematical topics, incorporated diverse explanations of multiple solution strategies or reasoning pathways, and provided detailed information that was practical and actionable for teachers. Moon (2023) investigated how pre-service teachers develop their understanding of students' thinking regarding big ideas in algebra. The study revealed that while pre-service teachers were able to design contextualized tasks to engage students, they struggled to articulate strategies for using these tasks to foster the development of big ideas by connecting multiple representations. Van Zoest et al. (2010) conducted their study with 16 qualified mathematics teachers who participated in a one-day focus group session designed to serve as a professional development opportunity. Subsequently, at least two lessons taught by six of these teachers were observed over three consecutive days. The study identified the primary objectives of the teachers in eliciting student thinking as fostering classroom engagement and enhancing students' mathematical understanding. These objectives were achieved by providing students with opportunities to compare diverse solutions and engage in questioning each other's reasoning.

Gehrtz et al. (2022) highlighted a significant gap in the literature regarding how instructors leverage student thinking in undergraduate STEM education and the factors that enable them to do so effectively. The study revealed that even if the courses identified as student-centered, they failed to genuinely embody student thinking centered practices. Similarly, Liang (2023) examined a teacher's KoST. She provided an in-depth analysis of the cognitive processes underpinning the teacher's learning from student thinking. Drawing on Piagetian learning theories, the study elucidated the mechanisms through which the teacher developed knowledge in response to student thinking.

Considering that this knowledge is most effectively discerned through processes in which teachers systematically reflect on their students' understanding and continuously enhance their capacity to notice and interpret students' thinking, it becomes imperative to articulate the components of this knowledge. Furthermore, it is essential to delineate the aspects and dimensions of teaching where this knowledge is manifested. Such detailing must be situated within the context of a dynamic and evolving teaching process, as this approach ensures a more precise and coherent framework for understanding and application. In this study we aimed to expand on KoST by examining the practices of mathematics teachers engaged in a lesson study process designed to enhance their KoST. We hypothesized that this process would facilitate a deeper examination of their approaches to KoST and allow for the extension of its content throughout the lesson study process.

Lesson Study

Teachers' continuous engagement in professional development plays a crucial role in shaping their teaching effectiveness. According to the National Council of Teachers of Mathematics (NCTM, 2000), professional development processes are essential for enhancing teachers' understanding of students' mathematical thinking and instructional strategies. Borko (2004) emphasized that the primary goal of these programs is to enable teachers to deepen their understanding and refine their instructional practices to improve the quality of teaching. The

National Comprehensive Center for Teacher Quality (2011) identifies five key purposes of high-quality professional development:

1. complying with the other professional learning activities including aims of school, state standards and assessments and formative teacher assessments,

2. focusing on the foundation of domain and on the model of teaching strategies related to the domain,

3. including the opportunities to actively learn current teaching strategies,

4. providing opportunities for collaboration with teachers and,

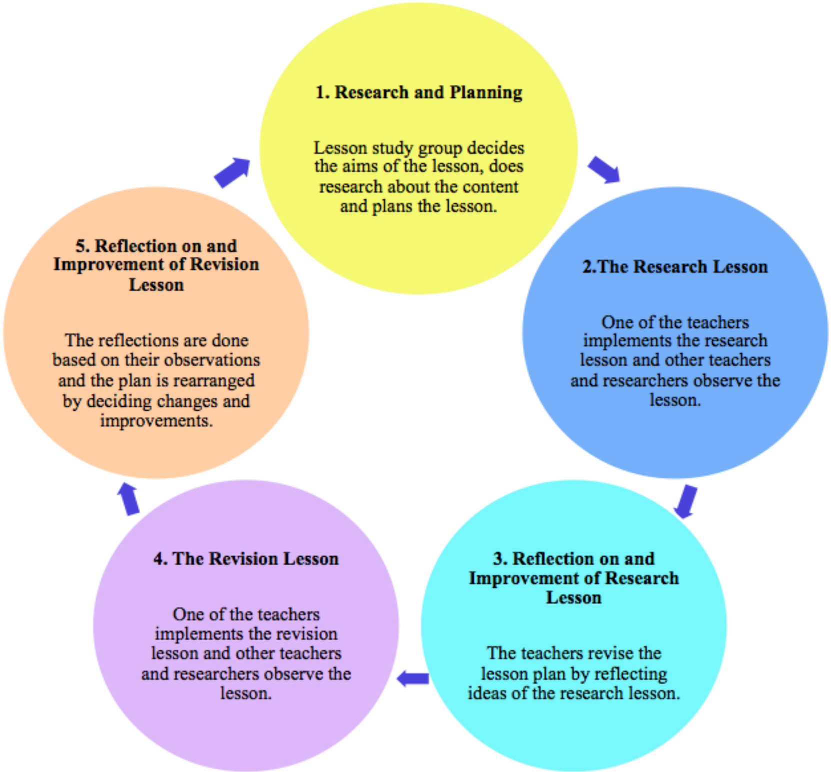
5. including continuous feedback.

Lesson study, recognized as an effective model for enhancing teachers' professional development, embodies many characteristics of high-quality professional development programs that emphasize teacher collaboration (Perry & Lewis, 2009). Hurd and Licciardo-Musso (2005) describe lesson study as a cyclical process that involves teachers in the stages of planning, observing, and revising a research lesson. Yoshida (1999) elaborates on this process, suggesting that it includes the development of lesson plans, their implementation in real classroom settings, careful observation, and subsequent reflection. This cyclical nature of lesson study can be conceptualized as a series of expert-led meetings aimed at refining teaching practices. Throughout this process, students' learning and cognitive processes are considered integral to all stages, including planning, observation, and revision. Furthermore, teachers have the opportunity to gain valuable insights into the effectiveness of their teaching by observing students within classroom settings. It is therefore considered essential that mathematics educators adopt a critical stance towards their own instructional practices as well as those of their peers. Huang and Shimizu (2016) argue that lesson study can facilitate the development of an inquiry-based approach and enhance teachers' critical reflection on their methodologies. Similarly, White and Lim (2008) suggest that lesson study supports the design of high-quality lessons and fosters a deeper understanding of student learning processes. Moreover, according to Lewis et al. (2012), by observing lessons from the students' perspective, teachers can also enhance their KoST and gain motivation to refine their instructional strategies. Given the characteristics of lesson study, we posit that participation in such a framework can aid mathematics teachers in the advancement of their KoST, which can be articulated through an examination of their pedagogical actions.

We adopted a five-stage lesson study cycle (see Figure 1) to provide the teachers with the opportunity to develop their KoST.

Figure 1.

The cycle of lesson study used in the study



The cycle consisted of (1) research and planning, (2) implementing the research lesson, (3) reflecting and improving the research lesson, (4) implementing the revision lesson, and (5) reflecting and improving the revision lesson. In the initial stage, teachers collaboratively examined the key concepts, deliberately considering potential student thinking during the lesson planning process. During planning and revision meetings, they engaged in discussions about how students might conceptualize and interact with the material, exploring various strategies to facilitate meaningful learning. This process culminated in the development of a research lesson. In the second stage, one teacher implements the collaboratively designed lesson plan, while the remaining team members observe with a particular focus on student responses and thinking during the lesson. In the third stage, the team reconvenes to analyze the research lesson, reflecting on student engagement and their observations to refine and further develop the lesson plan. Following these discussions, the revised lesson is implemented in the fourth stage, after which the team meets again in the fifth stage to evaluate and further improve the revised lesson. This iterative, collaborative cycle fosters a culture of mutual support among the teachers, thereby contributing to their professional growth and enhancing the overall effectiveness of the lesson study process.

2. Method

This study employed a qualitative case study methodology based on a nine-month lesson study. Throughout the duration, the focus was on analyzing teachers' actions and discourses in response to their students' thinking. The examination of these teacher actions during the lesson study facilitated the development of dynamically new approaches related to the KoST.

Participants

The participants consisted of three mathematics teachers from the same high school, selected through typical-case sampling. We assumed that engaging these teachers within a single school setting would facilitate effective meetings and encourage interaction among them, enabling them to observe each other's lessons. Additionally, the participants had a shared history, having graduated from the same university and worked together in the same school for an extended period, although they had not previously engaged in professional discussions with one another. They also had an established relationship with the researchers, stemming from years of collaboration through school-based mentoring of pre-service teachers from our university and participation in workshops and seminars organized by the research team. The genders, educational backgrounds and professional experiences were given in Table 1. Due to T3 being assigned to a different school, her participation was limited to the first lesson study cycle.

Table 1.

The information of participants

Participant	Gender	Educational background	Professional experience
T1	Male	Master	13
T2	Female	College	13
T3	Female	College	13

Data Collection

The data consisted of video recordings of the lessons and meetings. Before the lesson study, we observed two-hour lessons of each teacher. During the lesson study, twelve-hour lessons, including research and revision lessons in the three lesson study cycles, were observed. After the lesson study, we observed two-hour lessons of two teachers. During all the class observations, we took detailed field notes by considering the KoST and videotaped the lessons to capture the teachers' and students' discourses/actions/gestures by two cameras. While one of them focused on the board, the other focused on the students. We also videotaped the meetings and took field notes to support the results of the study. The purpose of the lesson observations and video recordings was to identify instances that could serve as evidence of the teachers' KoST. Video transcriptions of the meetings were utilized to enhance the field notes and strengthen the findings of the study. Throughout the process, the field notes taken during these meetings were consistently referenced to expand the framework with a focus on student thinking. We aimed to maintain the contextual integrity by integrating the transcripts of lesson observations and meetings with the field notes collected throughout the process The triangulation of data was crucial for ensuring validity in the process of refining and expanding the KOST categories (Creswell & Miller, 2000).

Procedure

We first conducted semi-structured interviews and observed the participants' lessons to establish a baseline understanding prior to the lesson study process. Notably, the teachers did not observe each other's lessons. One of the objectives of this initial phase was to assess the participants' KoST prior to engaging in the lesson study. Following these initial observations and interviews, we provided a seminar for the teachers. This seminar aimed to introduce the research purposes, the lesson study process, and the concept of KoST.

In the initial cycle, teachers planned instructional topics on radical expressions for 9th-grade students, and subsequently, in the following cycles, they focused on 'trigonometric ratios in a right-angled triangle' and 'coterminal angles and the unit circle' for 10th-grade students. During the implementation of the lesson study process, we engaged in discussions with the teachers, examining the sections related to the evidence of KoST that had been previously identified through the transcriptions. We asked some questions to the teachers with the aim of supporting their improvement. We guided teachers through reflective questions such as: "Which sections of the lesson were most effective according to your plan?", "What challenges did students have during the learning process?", "What changes would you make to this lesson if given the opportunity?", "Which practices do you consider most and least effective in terms of fostering KoST?", "What actions did you take, or could you have taken, to assess whether students learned the material?", "Why was the allocated time insufficient to implement all activities?", "What strategies would you employ to gain a deeper understanding of student thinking as a teacher?", and "Given the student's statement, what might their thoughts be?" Through these discussions, we aimed to more effectively and thoroughly explore instances related to KoST. Following the completion of all cycles, we observed teachers' lessons to draw inferences about their KoST.

Data Analysis

We attended all lessons and meetings to become thoroughly familiar with the data. Initially, we transcribed all video recordings verbatim. We then independently examined these transcriptions, segmenting them into contextual parts that included both teacher discourses and interactions between teachers and students, reflecting students' thinking. Following this initial analysis, we convened to reach a consensus by comparing and discussing the segmented parts. After this first meeting, we individually proceeded with a detailed analysis of the segments in three distinct stages. In the first stage, key terms were identified that described different aspects of students' thinking, such as prior knowledge, mistakes, varied representations, and different thought processes. In the second stage, we examined teachers' actions in relation to these keywords. For instance, teachers utilized different representations to address misconceptions and posed questions to identify students' errors. These actions were consistently compared to identify the main components. In connection with the teachers' objectives, a specific action related to the same keyword could be categorized under different main components. For example, the use of representations was not a standalone component; rather, teachers' implementation of this action varied—sometimes used to correct mistakes and at other times to assist with overcoming students' difficulties. To categorize the teachers' actions, we carefully considered students' thinking. This methodological approach facilitated the identification of key components in the data. An excerpt from the data analysis process is provided below.

Table 2.

An excerpt from the data analysis process

Excerpt		Keywords	Teacher actions	Main Component
Teacher:	$\sqrt[12]{2^{13}}$ is a response to a question. But this response was not included in the choices. How can you find the response? What can you do?		Challenging the students' expressions	
Student 1:	$2^{\frac{13}{12}}$			
Teacher:	This is not such an answer in the choices, also.	Different thoughts	Asking them to explain their thoughts	Triggering and considering divergent thoughts
Student 2:	$2^{\frac{26}{24}}$	Different responses		
Student 3:	I simplify.			
Teacher:	You cannot simplify $\frac{13}{12}$		Triggering students to give different responses	
Student 4:	$6\sqrt[12]{2}$			
Student 5:	Is it $\sqrt{4}$?			
Teacher:	I am listening, what else?			
Student 6:	$6\sqrt[12]{2}$			

We conducted this process by analyzing the first two lessons, forming an initial code list that included main components and sub-components. We then treated each lesson study cycle as a unit of analysis. By examining the transcriptions of lessons from subsequent cycles in a similar manner, we revised the code list through a process of constant comparison, adding new components and sub-components as needed. This iterative approach led to the development of a final code list. Through retrospective analysis, we re-examined all lessons comprehensively.

3. Result

During the lesson study, nine primary components and forty-eight sub-components related to KoST were identified (see Appendix 1). The main components were (1) building on students' mathematical ideas, (2) promoting students thinking mathematics, (3) triggering and considering divergent thoughts, (4) engaging students in mathematical learning, (5) evaluating students' understanding, (6) motivating students learning, (7) considering students' misconceptions and errors, (8) considering students' difficulties, (9) estimating students' possible ideas and approaches. These components were integrated throughout the teaching processes. The results present the content within these components as reflected in the teachers' actions, followed by evidence drawn from lesson excerpts during the lesson study.

KoST Components

Building on students' mathematical ideas involved an in-depth understanding of their prior knowledge, current understanding, interests, and deficiencies, as well as a consideration of mathematical concepts, their interconnections, representations, and the associated rules and procedures. This approach integrated all these elements into the teaching process to enhance students' understanding and conceptual development. Teachers' actions related to this component emerged through their efforts to build upon students' mathematical thinking and to design instruction that takes into account the students' thought processes. When teachers acknowledge students' prior knowledge and address their deficiencies, they facilitate a more conceptual learning experience for students. In the process of conceptual development, teachers encouraged students to relate new concepts to their existing cognitive frameworks. This approach demonstrated an acknowledgment of students' prior knowledge and engaged their mental processes. By integrating students' existing knowledge into the teaching process, it directed their attention to the new concepts, fostering a more active mental participation. The students' motivation and interest in learning mathematical concepts were heightened, which in turn increased their engagement and participation in lessons. When learning tasks captured students' interest, they were more likely to articulate their ideas and thought processes, thereby enhancing their understanding and identifying the critical aspects of the concepts being taught.

Promoting students' thinking mathematics involved several strategies, including the use of questioning, engaging learning tasks and activities, and employing diverse representations. This approach encouraged students to relate mathematical concepts to real-life scenarios and provides sufficient time for them to process and respond to questions. Observations indicated that certain teachers' actions were effective in fostering students' thinking processes. Teachers prompted students to engage with mathematical ideas by having them work through problems and make estimations. These interactions were instrumental in guiding teachers in formulating and utilizing questions effectively. Furthermore, the purposeful use of various representations—such as algebraic, figural, tabular, and graphical—helped students develop a more holistic understanding of mathematical ideas. By integrating different types of representations, teachers were able to tap into different cognitive processes among students. This approach highlights the importance of using multiple representations to support and enhance students' mathematical thinking in diverse contexts.

Triggering and considering divergent thoughts involved various methods such as exploring different solution strategies, presenting contradictory examples, comparing students' ideas, and encouraging students to question their peers and teachers. This component emerged from the teacher's actions at moments when diverse student thinking became apparent. Given the varying mental processes among students in the classroom, teachers acknowledged these differences to support student learning. Throughout the study, participants showed interest in students' diverse ideas and approaches in multiple ways. As lesson study cycles progressed, the teachers exhibited rich actions related to this component. They recognized that diverse thoughts were essential to the learning process and sought to bring them to light through discussions with one another and with researchers. Questioning was a fundamental action utilized by teachers, particularly in the context of triggering and considering divergent thoughts. This action included encouraging students to think differently, explore various solutions, provide different explanations, and engage with opposing viewpoints. By prompting students to reconsider and evaluate their own ideas, teachers were able to stimulate new and divergent thinking.

Engaging students in mathematical learning involved designing tasks, using different representations, connecting students' prior knowledge, giving examples of mathematical ideas, providing students to understand their difficulties. Focusing on the moments in which the students actively participated in the lessons revealed this component. Observations indicated that students were actively engaged in the lessons when working on tasks, with teachers using concrete examples to bridge abstract concepts and prevent passive learning. Additionally, students were more engaged when prompted to reflect on their difficulties and supported in overcoming them. The use of different representations was found to be particularly effective in fostering student focus and learning.

Evaluating students' understanding involved assessing their approaches, including how they comprehend, learn, and execute instructions. Teachers assessed students' interpretations while they were engaged in task-related activities and answering questions. During the observation of student work, teachers posed targeted questions to gauge their understanding. These observations were conducted both after individual work at their desks and during students' presentations and problem-solving on the board.

Motivating students' learning involved providing positive reinforcement when appropriate thoughts were expressed, offering guidance and advice, connecting concepts to real-world scenarios, presenting the historical development of mathematical ideas, and emphasizing the importance and relevance of the concepts. Comparing different solution approaches also served to enhance motivation. Furthermore, relating concepts to everyday life experiences was shown to improve students' reasoning and critical thinking skills.

Considering students' misconceptions and errors involved identifying, estimating, diagnosing, and preventing these issues. Teachers observed students' challenges, discussed their underlying reasons, and explored strategies for overcoming these difficulties during the lesson study process. During lesson planning, they also took into account students' potential ideas, solution strategies, and approaches. Teachers employed questioning techniques, provided hints, clarified problems step by step, and emphasized procedural understanding within the lessons.

The Teachers' Actions regarding KoST during the Lesson Study

Before the lesson study, one of the teachers taught on "finding the greatest common divisor (GCD) and least common multiple (LCM) of two or more polynomial functions". He first asked the students to find GCD and LCM of two natural numbers and after then to express GCD and LCM of two polynomials.

Teacher: Now we find GCD and LCM of two or more polynomials. I am writing these polynomials: $P(x)=x^2-2x-3$ and $Q(x)= x^2-9$.
How do you find GCD and LCM of these polynomials? [As the students did not respond, he explained by referring to the method related to GCD and LCM of two real numbers].

	24	18	2
$A=2^3.3$	12	9	2
$B=3^2.2$	6	9	2
	3	9	3
	1	3	3
		1	

$GCD\ (A, B)=2^1.3^1$
$LCM\ (A, B)=3^2.2^3$

$P(x)=x^2-2x-3$	(x-3)(x+1)	(x-3)(x+3)	(x-3)
$Q(x)= x^2-9$	(x+1)	(x+3)	(x+3)

	(x+1)	1	(x+1)
P(x)=(x-3)(x+1)	1		
Q(x)=(x-3)(x+3)			

LCM(P(x),Q(x))=(x-3)(x+1)(x+3)
GCD(P(x),Q(x))=(x-3)

The teacher considered students' prior knowledge and used an example related to their prior knowledge to support them to construct new concepts (1a and 4d, see in Appendix 1). He thought that the students would understand the new concept by connecting the concepts in their minds. Thus, he supported the students to connect by using a representation which the students could relate to prior knowledge (1f). This example provided the students to remember how GCD and LCM of two expressions would be found. As he used this representation, finding GCD and LCM of two polynomials were meaningful for the students' cognition. Beyond connecting to a concrete model, using different representations (1f-4b) or analogies were significant for building on students' ideas and engaging students in mathematical learning. There was a necessity to identify how the models, representations or analogies were used in the teaching processes. The excerpt also showed the evidence related to the sub-component of "using concepts or definitions to provide understanding" (1c). The teacher explained the definitions of GCD and LCM of two natural numbers; he supported the students to understand the meaning of GCD and LCM of two polynomials.

In the research lesson of the first lesson study cycle, the teacher asked questions related to exponential and radical expressions decided to ask in the planning meetings and he tried to understand the students' prior knowledge whether their prior knowledge was lacking.

Teacher: [He wrote radical expressions on the board.] What do you know about radical expressions?

Student 1: A radical expression is a number. The result is the number which multiplied with itself.

Teacher: Is there anyone else who has another idea?

Student 2: When we square a number and write it in a root, it becomes the same number again.

Teacher: For instance, what are the numbers of which the square is 4?

Students: 2 and -2

Teacher: Today, we will use the numbers which are squared. You worked on this concept at the level of secondary school. Right?

Students: Yes.

Teacher: For example, do you know $\sqrt{4}$?

Students: Yes, 2.

Teacher: 16=?

Students: 4

Teacher: Ok, very well.

Based on the question what radical expressions mean, the students remembered their knowledge regarding radical expressions, and they related a radical expression to squaring a number or multiplying a number with itself. "Considering the deficiencies of their prior knowledge" (1b) was important in building on students' mathematical ideas.

In the revision meeting, the teachers decided that students were bored and were not actively engaged in the lesson because the content of the lesson was intensive. Thus, they revised the plan by extending with information about the historical development of the symbol of root to motivate the students. So, the teacher gave an example regarding historical information of the radical expressions and asked the students to interpret it.

Teacher: If we write the degree of roots as ∛ √ ∜, beforehand, while square root was √ in the expression of these, they made three the number of this line (√) as v√

Students: Wow..

Teacher: They noticed that the number of these zigzags increased as the degree was getting greater.

Students: And then?

Teacher: They couldn't pull off and considered it appropriate to write just 2, 3, 4, 5 above them.

When the teacher presented this information to the students, the students' attention was attracted to the lesson. In this process, discussing with the students on the historical development of the concept motivated them and supported the students to think about the meaning of the concept (6d).

In the research lesson of the second cycle, the teacher first focused on students' prior knowledge on trigonometric ratios of a right triangle. And then, the teacher asked the students to think whether the trigonometric ratios would change when the right triangle was bigger or smaller without changing any angles as seen in the following excerpt.

Teacher: What if we make the triangle bigger or smaller? Will the trigonometric ratios change?

Student 1: No.

Student 2: Yes.

Student 3: Will the ratio change?

Teacher: If we decrease the size of the triangle some more, how will the ratio change?

Student 4: It will not change.

Teacher: It will not change, because?

Student 4: It's a ratio.

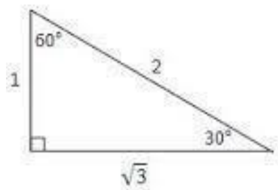
Student 5: Similar triangles.

Teacher: In fact, yes, it is the best sentence. These are similar triangles.

The teacher asked the students to explain the reasons for their responses (3c) and triggered them to think about the related concept. Encouraging the students to explain the reasons for ideas and creating a discussion environment became effective actions in revealing different students' thinking. However, he immediately confirmed a student who gave a response in accordance with his expectation. And ignored the student who related the question to the concept of ratio. Thus, he prevented the students from thinking about the question more and funnelled them in the direction of his own thinking. This excerpt presented the evidence in the context of triggering different responses and questioning the students even if it was interpreted that the teacher had actions in accordance with his own thoughts.

In the planning meeting, the teachers thought that the students could easily find the ratios in a triangle with angles of 30º-60º-90º but that they did not question the underlying reasons before. They decided to ask students what the ratios in a triangle with angles of 30º-60º-90º were with the aim of determining the students' prior knowledge and to improve their students' ideas with this question. As seen in the following excerpt, the teacher asked the student in the board to explain the response.

Board:



Teacher: Well, where did you find 1,2,√3?

Student 1: After his own heart.

Teacher: Why is 1,2,√3?

Student 1: Isn't this a rule? It results from the hypotenuse.

Teacher: Let us calculate it in a different way. Why aren't the sides 1,√5,√6, but 1,2,√3? That's what I am asking.

Student 2: That's a rule.

Student 3: Is it related to the unit circle?

ooo

Teacher: Why are the sides 1,2,√3? at the 30°-60°-90° triangle, 30°-60°-90° not 1,√5,√6.

Student 5: 1 is opposite to the angle, 30°.

Student 6: Equilateral triangle.

Teacher: Who said equilateral triangle?

Student 6: Me.

Teacher: Come and draw us an equilateral triangle.

Board: (Drawing of the student on the board)

When the student wrote the numbers of 1, 2, 3 for the sides of the right triangle, the teacher pushed the student to explain the reason (1b-3a-3d). All students in the classroom thought about this question to justify these values. They focused on the equilateral triangle in the direction of a student's response during the classroom discussion (3b) and the teacher provided the students to understand the reason why the side lengths are 1, 2, 3 (2e) and to improve their existing thoughts. The teacher also asked the students to estimate this question (2b). This process which was unexpected for the students provided them to reason about the relationship among 1, 2 and 3 and prompted them to think mathematically.

In the revision lesson of the second cycle, there were several questions to encourage the students to think. The teacher asked students to find the points whose coordinates on the unit circle were integers.

Teacher: Are there points whose coordinates on the unit circle are integers?

Student 1: Why not? There might be.

Teacher: Well done. For example, which ones?

Student 2: (1,0), (-1, 0)

Teacher: Are there points whose coordinates on the unit circle are integers?

Student 1: Why not? There might be.

Teacher: Well done. For example, which ones?

Student 2: (1,0), (-1, 0)

Teacher: Where is the point whose coordinate is (1,0)?

Student 2: (she showed each one on a unit circle one by one)

Teacher: Yes, these points are on the circle, are there other points which both coordinates are integers?

How many elements does the solution of the equation $x^2+y^2=1$ in the set of integers?

Student 1: 4 points. The points of (0,1) and (0,-1) are also possible.

The teacher asked a follow-up question to improve students' understanding and encourage them to think (2c). Additionally, the teacher promoted students to use graphical representations and to connect graphical and algebraic representations while they were determining these points (2d). This was a promoting-thinking action because the question included the relation among the concepts of equation, trigonometric equation, and sets of numbers and unit circle. In other words, by this question, the students related different concepts to each other and developed their understanding by thinking about their existing understanding.

The teachers' discussions about the students' possible misconceptions and mistakes in the meetings affected the teacher's actions during the teaching process. In the revision lesson of the second cycle, the teacher gave a value of sine and asked students to express what the value of cosine would be equal to.

Student 1: sinα=310

Teacher: What's ?

Student 1: Since it is exact opposite cosα=103

Teacher: This is one of the most common mistakes... if tangent is reciprocal of cotangent, why not cosine would be the reciprocal of sine just like cotangent and tangent. You can evaluate this response by examining the right triangle.

Student 1: Cosine is found on the adjacent side over the hypotenuse.

Teacher: Aren't cosine and sine the reciprocal of each other, are they?

Student 2: No, different sides affect this ratio.

When the teacher expressed the value of sinus and asked the value of cosine, a student gave the response of 10/3 which was reciprocal. The teacher stated that this response might be related to students' common mistakes and asked the students to think about the meaning of ratios on the right triangle. The teachers estimated this inappropriate thinking in the planning meetings (9a), and knowing the mistake (7a) and relating the concept with the triangle to eliminate the mistake (7e-7f) were significant for improving the students' learning and thinking.

4. Discussion and Conclusion

In this study, teachers collaboratively engaged in the teaching processes by planning and reflecting on these plans after their implementation. They were exposed to a variety of students, different from those in their own classrooms, and actively shared their ideas regarding content and student thinking with one another. Our interactions with them during meetings, as knowledgeable facilitators (Pehlivan & Bukova Güzel, 2020; Takahashi, 2014), contributed to the enhancement of their mathematical knowledge for teaching. Specifically, the study focused on improving their KoST.

Through the lesson study process, teachers had the opportunity to observe and critically analyze different lessons, thereby individually supporting the development of their teaching actions related to KoST. For instance, discussions on questions such as how to prompt students to think about concepts, how to uncover their incomplete understandings, or how to encourage them to think differently fostered the teachers' perspectives on teaching and learning. As they considered student thinking and discussed lesson plans across three lesson study cycles, their approaches in the classroom were increasingly influenced by these evolving perspectives.

The lesson study process facilitated the extension of actions regarding KoST. Through careful planning that centered on understanding students' thought processes, and through post-lesson evaluation meetings, the teachers were able to refine their instructional strategies. These discussions allowed them to anticipate student thinking, thereby enabling them to employ more effective methods to enhance students' conceptual understanding. It is essential for teachers to have a clear understanding of students' pre-existing knowledge prior to a lesson, as well as the intended learning outcomes (Kelting-Gibson, 2013). During the planning phase, teachers engaged in discussions about potential student ideas and adjusted the lesson content accordingly based on their predictions of student thinking.

During the lesson implementation phases, teachers encouraged students to engage in reasoning about the concepts and guided their learning processes by following their own thought processes. As the lesson study process advanced, teachers took more effective actions to support students' learning. These included activities such as estimation, relating ideas, sharing insights, considering diverse perspectives, and posing questions. These actions reflected an improvement in their knowledge of students' thinking (KoST), as evidenced by their alignment with established frameworks (An, et al., 2004; Cengiz, 2007; Lee, 2006). Furthermore, when teachers employed questions and activities designed to prompt student estimation, they were able to observe students' thinking more clearly and create a learning environment that centered on student-centered thinking. Teachers also questioned the concepts and underlying ideas during the planning stages, thereby encouraging students to engage in similar questioning in the classroom, even when they had only a procedural understanding. The effect of the lesson study cycle on teachers' actions was evident, with improvements in KoST becoming apparent as the lesson study process progressed across cycles. This process provided a productive framework for examining and conceptualizing teachers' KoST components.

The KoST components serve as an analytical framework for mathematics teachers to enhance their instructional practices. These components were developed by analyzing real classroom interactions, providing a detailed description of contexts in which teachers can consider students' thinking. They offer a range of possible instructional strategies to support the development of students' mathematical thinking (Corey et al., 2021; Van Zoest et al., 2010). By utilizing evidence from categorized components, discussions can be initiated with teachers. Even if they are not directly involved in the professional development program, these findings can be used to evaluate and reflect on real classroom practices, thereby enhancing their development and awareness. The findings provide specific examples that help teachers understand the extent of their own knowledge of student thinking within their practices. A teacher who has engaged with this framework, can personally reflect on its categories in her/his own teaching practices. Introducing this framework to teachers serves as a valuable guide in this regard. Additionally, mathematics teacher education programs could incorporate the KoST framework to better prepare student teachers.

Limitations

This study involved three mathematics teachers working with a large group of students. Observing the teachers' KoST in crowded classrooms posed certain challenges; however, it also allowed for the identification of diverse instructional strategies. Additionally, the participation of only three teachers within a single school could be considered a limiting factor in the lesson study process. Conducting a lesson study cycle with a larger number of teachers from different schools could enable researchers to implement a broader professional development program, offering insights into a wider range of teacher practices related to KoST.

We encountered some limitations in coordinating in-person meetings with teachers. In future studies, an online environment for the lesson study program could be explored to examine how this approach supports teachers' professional development. Additionally, this study did not focus on student learning. Given that teachers with a strong KoST are better equipped to address students' needs and create opportunities to enhance their understanding (Asquith et al., 2007), subsequent research could investigate how students' understanding evolves as teachers' KoST actions improve."

5. References

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6. Appendix The main components and sub-components of KoST.

Components	Sub-components
1-Building on students' mathematical ideas	(1a) Knowing prior knowledge related to concept and connect them to new knowledge (1b) Determining students' prior knowledge and consider their deficiencies (1c) Using concepts or definitions to provide understanding (1d) Focusing on rules and procedures to support/reinforce/improve the mathematical knowledge (1e) Attracting students' interests to subject/concept (1f) Using representations/analogies/concrete models defining concepts explicitly
2-Promoting students thinking mathematics	(2a) Asking questions and design tasks/examples for students to think (2b) Having students estimate about questions/problems (2c) Asking questions and design tasks to develop students' existing understanding (2d) Asking students to product mathematical thoughts by representations such as figural/tabular/graphical (2e) Providing students opportunities to think and respond questions (2f) Relating examples/questions/problems to real life
3-Triggering and considering divergent thoughts	(3a) Asking questions to elicit students' ideas (3b) Creating class discussion about a student's idea/solution/question or any thoughts (3c) Asking students to produce thoughts or to explain about teacher's expressions (3d) Asking students to explain/expand/interpret about ideas proposed by them (3e) Asking students to express each other's explanations in different ways (3f) Asking students to give contradictory examples (3g) Encouraging students to produce different solutions (3h) Explaining/expanding students' ideas
4-Engaging students in mathematical learning	(4a) Arranging activities to activate students (4b) Using different representations of concepts (4c) Giving example of mathematical ideas (4d) Knowing prior knowledge related to concept and connect them to new knowledge (4e) Allowing students to understand their difficulties/obstacles/failures while reflecting on instructions and strategies
5-Evaluating students' understanding	(5a) Evaluating how students understand the instructions, how they learn and how they perform during teaching
6-Motivating students learning	(6a) Praising students when they provide appropriate thoughts (6b) Giving students motivational advice when they struggle or fail (6c) Relating examples/questions/problems to real life (6d) Giving the historical development of concept

	(6e) Addressing the importance and necessity of concept
7-Considering students' misconceptions and errors	(7a) Knowing students' misconceptions and errors (7b) Determining students' misconceptions and errors (7c) Focusing on concepts/rules/procedures to prevent misconceptions and errors (7d) Using different representations to prevent misconceptions and errors (7e) Focusing on concepts/rules/procedures to remove misconceptions and errors (7f) Using different representations to remove misconceptions and errors (7g) Giving students clues to realize their misconceptions/errors (7h) Ensuring students' understanding of the problems/questions
8-Considering students' difficulties	(8a) Estimating students' difficulties (8b) Simplifying/Explaining step by step what students have difficulties (8c) Recognizing students' difficulties (8d) Asking questions to determine the reasons of students' difficulties (8e) Giving students clues to overcome difficulties (8f) Focusing on concepts/rules/procedures to overcome difficulties (8g) Using different representations to overcome difficulties
9-Estimating students' possible ideas and approaches	(9a) Estimating possible thoughts to be produced by students (9b) Estimating students' solutions related to questions/problems