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GENERALIZED TENSORIAL SIMPSON TYPE INEQUALITIES FOR CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACE

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Abstract. Several generalized Simpson tensorial type inequalities for selfadjoint operators have been obtained with variation depending on the conditions imposed on the function $\mathfrak{f}$

$$\Bigg\| \frac{1}{6}\mathfrak{f}(\lambda\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B})+\frac{1-\lambda}{3}\mathfrak{f}\left(\frac{(1+\lambda)\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B}}{2}\right)$$
$$+\frac{\lambda}{3}\mathfrak{f}\left(\frac{\lambda\mathfrak{A}\otimes 1+(2-\lambda)1\otimes\mathfrak{B}}{2}\right)-\frac{1}{2}\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\Bigg\|$$
$$\leq\frac{5\left\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\right\|(2\lambda-1)^2+1)}{144}\left\|\mathfrak{f}'\right\|_{I,+\infty}.$$

1. Introduction and preliminaries

The concept we now call a "tensor" wasn't originally named that way. When Josiah Willard Gibbs first described the idea in the late 19th century, he used the term "dyadic." Today, mathematicians define a tensor as the mathematical embodiment of Gibbs' initial concept. Tensors and inequalities are natural partners, thanks to the widespread use of inequalities in mathematics. These mathematical statements about comparisons have a profound impact on various scientific disciplines. While many types of inequalities exist, some of the most significant ones include Jensen's, Ostrowski's, Hermite-Hadamard's, and Minkowski's inequalities. For those interested in delving deeper, references [21] and [23] provide more details about inequalities and their fascinating history. Regarding the generalizations of the aforementioned inequalities, numerous studies have been published; for additional information, check the following and the references therein [8, 24, 25, 22, 17, 16, 15, 28, 29, 30, 1, 2, 3, 4, 5, 7, 9, 10].
Classical inequalities of Simpson type have been given by Hezenci et al. [19] and Sarikaya et al. [26]. To enhance the presentation of this work, we will demonstrate

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new developments in the theory of inequalities in Hilbert spaces. One such development is the Dragomir's inequality for normal operators given by the following [11]:

Theorem 1.1. *Let $(\mathscr{H};\langle .,.\rangle)$ be a Hilbert space and $\mathfrak{T}:\mathscr{H}\to\mathscr{H}$ a normal linear operator on $\mathscr{H}$. Then*

$$\|\mathfrak{T}x\|^2 \geq \frac{1}{2}\left(\|\mathfrak{T}x\|^2 + |\langle \mathfrak{T}^2 x, x\rangle|\right) \geq |\langle \mathfrak{T}x, x\rangle|^2,$$

for any $x \in H$, $\|x\| = 1$. The constant $\frac{1}{2}$ is the best possible.

The Hermite-Hadamard inequality in the selfadjoint operator sense, as provided by Dragomir [12], is another intriguing conclusion.

Theorem 1.2. *Let $\mathfrak{f}: I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators $\mathfrak{A}$ and $\mathfrak{B}$ with spectra in I we have the inequality*

$$\begin{aligned}\mathfrak{f}\left(\frac{\mathfrak{A}+\mathfrak{B}}{2}\right) &\leq \mathfrak{f}\left(\frac{3\mathfrak{A}+\mathfrak{B}}{4}\right) + \mathfrak{f}\left(\frac{\mathfrak{A}+3\mathfrak{B}}{4}\right)\\ &\leq \int_0^1 \mathfrak{f}((1-t)\mathfrak{A}+t\mathfrak{B})dt\\ &\leq \frac{1}{2}\left[\mathfrak{f}\left(\frac{\mathfrak{A}+\mathfrak{B}}{2}\right) + \frac{\mathfrak{f}(\mathfrak{A})+\mathfrak{f}(\mathfrak{B})}{2}\right] \leq \frac{\mathfrak{f}(\mathfrak{A})+\mathfrak{f}(\mathfrak{B})}{2}.\end{aligned}$$

The first paper related to tensorial inequalities in Hilbert space was written by Dragomir [14]. In the paper, he proved the tensorial version of the Ostrowski type inequality given by the following.

Theorem 1.3. *Assume that $\mathfrak{f}$ is continuously differentiable on I with $\|\mathfrak{f}'\|_{I,+\infty} := \sup_{t\in I}|\mathfrak{f}'(t)| < +\infty$ and $\mathfrak{A},\mathfrak{B}$ are selfadjoint operators with $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I$. Then the following inequality holds:*

$$\begin{aligned}&\left\|\mathfrak{f}((1-\lambda)\mathfrak{A}\otimes 1 + \lambda 1\otimes\mathfrak{B}) - \int_0^1 \mathfrak{f}((1-u)\mathfrak{A}\otimes 1 + u1\otimes\mathfrak{B})du\right\| \qquad (1.1)\\ &\leq \|\mathfrak{f}'\|_{I,+\infty}\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|1\otimes\mathfrak{B} - \mathfrak{A}\otimes 1\|\end{aligned}$$

for $\lambda \in [0,1]$.

Recently, various inequalities in the same tensorial surrounding have been obtained. The following result of Simpson type was obtained by Stojiljković [31].

Theorem 1.4. *Assume that $\mathfrak{f}$ is continuously differentiable on I and $|\mathfrak{f}''|$ is convex and $\mathfrak{A},\mathfrak{B}$ are selfadjoint operators with $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I$. Then the following inequality holds:*

$$\begin{aligned}&\left\|\frac{1}{6}\left(\mathfrak{f}(\mathfrak{A})\otimes 1 + 4\mathfrak{f}\left(\frac{\mathfrak{A}\otimes 1 + 1\otimes\mathfrak{B}}{2}\right) + 1\otimes\mathfrak{f}(\mathfrak{B})\right)\right.\\ &-\frac{1}{2}\alpha\Bigg(\int_0^1 \mathfrak{f}\left(\left(\frac{1-k}{2}\right)\mathfrak{A}\otimes 1 + \left(\frac{1+k}{2}\right)1\otimes\mathfrak{B}\right)k^{\alpha-1}dk\end{aligned}$$

$$+\int_0^1 \mathfrak{f}\left(\left(1-\frac{k}{2}\right)\mathfrak{A}\otimes 1+\frac{k}{2}1\otimes\mathfrak{B}\right)(1-k)^{\alpha-1}dk\Big)\Big\|$$

$$\leq \|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|^2\frac{\left(\|\mathfrak{f}''(\mathfrak{A})\|+\|\mathfrak{f}''(\mathfrak{B})\|\right)\left(3\alpha^2+8\alpha+7\right)}{(\alpha+2)(24\alpha+24)}$$

for $\alpha > 0$.

The following inequality has been recently obtained by the same author [32].

Theorem 1.5. *Assume that* $\mathfrak{f}$ *is continuously differentiable on* I *with* $\|\mathfrak{f}'\|_{I,+\infty} := \sup_{t\in I}|\mathfrak{f}'(t)| < +\infty$ *and* $\mathfrak{A},\mathfrak{B}$ *are selfadjoint operators with* $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I$. *Then the following inequality holds:*

$$\left\|\int_0^1 \mathfrak{f}((1-\lambda)\mathfrak{A}\otimes 1+\lambda 1\otimes\mathfrak{B})d\lambda-\mathfrak{f}\left(\frac{\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}}{2}\right)\right\|$$

$$\leqslant \|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|^2\frac{\|\mathfrak{f}''\|_{I,+\infty}}{24}.$$

Recently, the following inequality of Ostrowski type was obtained by Stojiljković et al. [33] which generalized the recently obtained results by Dragomir [14].

Theorem 1.6. *The formulation is the same as the one given by Dragomir in his Ostrowski type Theorem given above* (1.1) *with an exception that* $\alpha > 0$, *then*

$$\Big\|\Big(\lambda^\alpha+(1-\lambda)^\alpha\Big)\mathfrak{f}((1-\lambda)\mathfrak{A}\otimes 1+\lambda 1\otimes\mathfrak{B})$$

$$-\alpha\Big((1-\lambda)^\alpha\int_0^1 \mathfrak{f}((1-\lambda)(1-u)\mathfrak{A}\otimes 1+(u+(1-u)\lambda)1\otimes\mathfrak{B})(1-u)^{\alpha-1}du$$

$$+\lambda^\alpha\int_0^1 u^{\alpha-1}\mathfrak{f}(((1-u)+u(1-\lambda))\mathfrak{A}\otimes 1+u\lambda 1\otimes\mathfrak{B})du\Big)\Big\|$$

$$\leqslant \|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|\left(\frac{\lambda^{\alpha+1}}{\alpha+1}+\frac{(1-\lambda)^{\alpha+1}}{\alpha+1}\right)\|\mathfrak{f}'\|_{I,+\infty}.$$

Stojiljković et al., [34] recently obtained a Trapezoid type tensorial inequality which is given by the following.

Theorem 1.7. *Assume that* $\mathfrak{f}$ *is continuously differentiable on* I *with* $\|\mathfrak{f}'\|_{I,+\infty} := \sup_{t\in I}|\mathfrak{f}'(t)| < +\infty$ *and* $\mathfrak{A},\mathfrak{B}$ *are selfadjoint operators with* $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I$. *Then the following inequality holds for* $\alpha > 0$:

$$\Big\|(\mathfrak{f}(\mathfrak{A})\otimes 1+1\otimes\mathfrak{f}(\mathfrak{B})) \tag{1.2}$$

$$-\alpha\left[\int_0^1(1-\lambda)^{\alpha-1}\mathfrak{f}(\lambda 1\otimes\mathfrak{B}+(1-\lambda)\mathfrak{A}\otimes 1)d\lambda+\int_0^1\lambda^{\alpha-1}\mathfrak{f}(\lambda 1\otimes\mathfrak{B}+(1-\lambda)\mathfrak{A}\otimes 1)d\lambda\right]\Big\|$$

$$\leq \|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|\frac{1}{1+\alpha}\left(2-2^{1-\alpha}\right)\|\mathfrak{f}'\|_{I,+\infty}.$$

In order to derive similar inequalities of the tensorial type, we need the following introduction and preliminaries.
Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $\mathfrak{f} : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $\mathfrak{A} = (\mathfrak{A}_1, ..., \mathfrak{A}_k)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of $\mathfrak{A}_i$ is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of $\mathfrak{f}$. If

$$\mathfrak{A}_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of $\mathfrak{A}_i$ for $i = 1, ..., k$ by following , we define

$$\mathfrak{f}(\mathfrak{A}_1, ..., \mathfrak{A}_k) := \int_{I_1} ... \int_{I_k} \mathfrak{f}(\lambda_1, ..., \lambda_k) dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k)$$

as bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.
If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [6] extends the definition of Koranyi [20] for functions of two variables and have the property that

$$\mathfrak{f}(\mathfrak{A}_1, ...\mathfrak{A}_k) = \mathfrak{f}_1(\mathfrak{A}_1) \otimes ... \otimes \mathfrak{f}_k(\mathfrak{A}_k),$$

whenever $\mathfrak{f}$ can be separated as a product $\mathfrak{f}(t_1, ..., t_k) = \mathfrak{f}_1(t_1)...\mathfrak{f}_k(t_k)$ of k functions each depending on only one variable.
Recall the following property of the tensorial product

$$(\mathfrak{A}\mathfrak{C}) \otimes (\mathfrak{B} \otimes \mathfrak{D}) = (\mathfrak{A} \otimes \mathfrak{B})(\mathfrak{C} \otimes \mathfrak{D})$$

that holds for any $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \in B(\mathscr{H})$.
From the property we can deduce easily the following consequences

$$\mathfrak{A}^n \otimes \mathfrak{B}^n = (\mathfrak{A} \otimes \mathfrak{B})^n, n \geqslant 0,$$

$$(\mathfrak{A} \otimes 1)(1 \otimes \mathfrak{B}) = (1 \otimes \mathfrak{B})(\mathfrak{A} \otimes 1) = A \otimes B,$$

which can be extended, for two natural numbers m, n we have

$$(\mathfrak{A} \otimes 1)^n (1 \otimes \mathfrak{B})^m = (1 \otimes \mathfrak{B})^m (\mathfrak{A} \otimes 1)^n = \mathfrak{A}^n \otimes \mathfrak{B}^m.$$

For more information, consult the following book related to tensors [18]. The following Lemma which we require can be found in a paper of Dragomir [13].

Lemma 1.8. *Assume $\mathfrak{A}$ and $\mathfrak{B}$ are selfadjoint operators with $Sp(\mathfrak{A}) \subset I, Sp(\mathfrak{B}) \subset J$ and having the spectral resolutions . Let $\mathfrak{f}; \mathfrak{h}$ be continuous on $I, \mathfrak{g}, \mathfrak{k}$ continuous on J and ϕ and ψ continuous on an interval K that contains the sum of the intervals $\mathfrak{f}(I) + \mathfrak{g}(J); \mathfrak{h}(I) + \mathfrak{k}(J)$,then*

$$\begin{aligned} &\phi(\mathfrak{f}(\mathfrak{A}) \otimes 1 + 1 \otimes \mathfrak{g}(\mathfrak{B}))\psi(\mathfrak{h}(\mathfrak{A}) \otimes 1 + 1 \otimes \mathfrak{k}(\mathfrak{B})) \\ &= \int_I \int_J \phi(\mathfrak{f}(t) + \mathfrak{g}(s))\psi(\mathfrak{h}(t) + \mathfrak{k}(s)) dE_t \otimes dF_s. \end{aligned} \tag{1.3}$$

In the paper written by Sarikaya and Bardak [27], the following Lemma is given, which is used to obtain inequalities generalized Simpson type inequalities.

Lemma 1.9. *Let $\mathfrak{f} : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a < b$. If $\mathfrak{f}' \in L^1[a,b]$, then the following equality holds:*

$$\begin{aligned}&\frac{(\omega-a)^2}{2(b-a)}\int_0^1\left(\frac{k}{2}-\frac{1}{3}\right)\mathfrak{f}'\left(\frac{1+k}{2}\omega+\frac{1-k}{2}a\right)dk \qquad (1.4)\\ &+\frac{(b-\omega)^2}{2(b-a)}\int_0^1\left(\frac{1}{3}-\frac{k}{2}\right)\mathfrak{f}'\left(\frac{1+k}{2}\omega+\frac{1-k}{2}b\right)dk\\ =&\frac{1}{6}\mathfrak{f}(\omega)+\frac{1}{3(b-a)}\left[(\omega-a)\mathfrak{f}\left(\frac{a+\omega}{2}\right)+(b-\omega)\mathfrak{f}\left(\frac{\omega+b}{2}\right)\right]-\frac{1}{b-a}\int_{\frac{a+\omega}{2}}^{\frac{\omega+b}{2}}\mathfrak{f}(x)dx,\end{aligned}$$

where $\omega = \mu a + (1-\mu)b, \forall \mu \in [0,1]$.

This paper delves into a novel area of mathematics: tensorial inequalities of the Simpson type for differentiable functions within a tensorial Hilbert space. This field is young and ripe for exploration, and obtaining new bounds for various combinations of convex functions is crucial for its advancement. The paper is structured logically. The "Main Results" section unveils the key findings that contribute to the novelty of this work. Subsequently, the "Examples and Consequences" section showcases practical applications of the obtained results. By choosing specific convex functions, we generate numerous tensorial Simpson-type inequalities and bounds. Finally, the "Conclusion" section summarizes the paper's contributions and highlights its significance for the development of tensorial inequalities. In the following theorem, you'll find a fundamental result that serves as the foundation for deriving further inequalities throughout the paper.

2. Main results

The following Lemma will be crucial in obtaining the inequalities which follow.

Lemma 2.1. *Assume that $\mathfrak{f}$ is continuously differentiable on I, A and B are self-adjoint operators with $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I$, then*

$$\begin{aligned}&\frac{1}{6}\mathfrak{f}(\lambda\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B})+\frac{1-\lambda}{3}\mathfrak{f}\left(\frac{(1+\lambda)\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B}}{2}\right) \qquad (2.1)\\ &+\frac{\lambda}{3}\mathfrak{f}\left(\frac{\lambda\mathfrak{A}\otimes 1+(2-\lambda)1\otimes\mathfrak{B}}{2}\right)-\frac{1}{2}\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\\ =&\frac{(1-\lambda)^2}{2}(1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1)\int_0^1\left(\frac{k}{2}-\frac{1}{3}\right)\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)dk\\ &+\frac{\lambda^2(1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1)}{2}\int_0^1\left(\frac{1}{3}-\frac{k}{2}\right)\mathfrak{f}'\left(\left(\frac{(1+k)\lambda}{2}\mathfrak{A}\otimes 1\right)+\left(1-\frac{\lambda(1+k)}{2}\right)1\otimes\mathfrak{B}\right)dk.\end{aligned}$$

Proof. We will start the proof with Lemma 1.9 (eq. (1.4)). Introducing the substitutions on the right hand side and simplifying the integral, then assuming that $\mathfrak{A}$ and $\mathfrak{B}$ have the spectral resolutions

$$\mathfrak{A} = \int tdE(t) \text{ and } \mathfrak{B} = \int sdF(s).$$

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$\int_I \int_I \left(\frac{1}{6}\mathfrak{f}(\lambda t+(1-\lambda)s)+\frac{1-\lambda}{3}\mathfrak{f}\left(\frac{(1+\lambda)t+(1-\lambda)s}{2}\right)\right.$$

$$+\frac{\lambda}{3}\mathfrak{f}\left(\frac{\lambda t+(2-\lambda)s}{2}\right)-\frac{1}{2}\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)t+s\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\Bigg)dE_t\otimes dF_s$$

$$=\int_I\int_I\left(\frac{(1-\lambda)^2}{2}(s-t)\int_0^1\left(\frac{k}{2}-\frac{1}{3}\right)\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)dk\right.$$

$$\left.+\frac{\lambda^2(s-t)}{2}\int_0^1\left(\frac{1}{3}-\frac{k}{2}\right)\mathfrak{f}'\left(\left(\frac{(1+k)\lambda}{2}t\right)+\left(1-\frac{\lambda(1+k)}{2}\right)s\right)dk\right)dE_t\otimes dF_s.$$

By utilizing the Fubini's Theorem and Lemma 1.8 (eq. (1.3)) for appropriate choices of the functions involved, we have successively

$$\int_I\int_I\mathfrak{f}(\lambda t+(1-\lambda)s)dE_t\otimes dF_s=\mathfrak{f}\left(\lambda\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B}\right),$$

$$\int_I\int_I\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)t+s\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\Bigg)dE_t\otimes dF_s$$

$$=\int_0^1\int_I\int_I\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)t+s\left(\frac{\phi+(1-\lambda)}{2}\right)\right)dE_t\otimes dF_s d\phi$$

$$=\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1+\left(\frac{\phi+(1-\lambda)}{2}1\otimes\mathfrak{B}\right)d\phi,\right.$$

$$\int_I\int_I(s-t)\int_0^1\left(\frac{k}{2}-\frac{1}{3}\right)\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)dk dE_t\otimes dF_s$$

$$=\int_0^1\left(\frac{k}{2}-\frac{1}{3}\right)\int_I\int_I(s-t)\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)dE_t\otimes dF_s dk$$

$$=(1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1)\int_0^1\left(\frac{k}{2}-\frac{1}{3}\right)\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)dk.$$

Following the same principle for other terms, the equality follows.

□

Theorem 2.2. *Assume that* $\mathfrak{f}$ *is continuously differentiable on* I *with* $\|\mathfrak{f}'\|_{I,+\infty}:=\sup_{t\in I}|\mathfrak{f}'(t)|<+\infty$ *and* $\mathfrak{A},\mathfrak{B}$ *are selfadjoint operators with* $Sp(\mathfrak{A}),Sp(\mathfrak{B})\subset I,\lambda\in[0,1]$ *, then*

$$\left\|\frac{1}{6}\mathfrak{f}(\lambda\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B})+\frac{1-\lambda}{3}\mathfrak{f}\left(\frac{(1+\lambda)\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B}}{2}\right)\right. \tag{2.2}$$

$$\left.+\frac{\lambda}{3}\mathfrak{f}\left(\frac{\lambda\mathfrak{A}\otimes 1+(2-\lambda)1\otimes\mathfrak{B}}{2}\right)-\frac{1}{2}\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\right\|$$

$$\leq\frac{5\left\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\right\|(2\lambda-1)^2+1)}{144}\left\|\mathfrak{f}'\right\|_{I,+\infty}.$$

Proof. If we take the operator norm of the previously obtained Lemma (2.1) and use the triangle inequality, we get

$$\left\|\frac{1}{6}\mathfrak{f}(\lambda\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B})+\frac{1-\lambda}{3}\mathfrak{f}\left(\frac{(1+\lambda)\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B}}{2}\right)\right.$$

$$\left.+\frac{\lambda}{3}\mathfrak{f}\left(\frac{\lambda\mathfrak{A}\otimes 1+(2-\lambda)1\otimes\mathfrak{B}}{2}\right)-\frac{1}{2}\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\right\|$$

$$\leq \frac{(1-\lambda)^2}{2}\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|\int_0^1\left\|\frac{k}{2}-\frac{1}{3}\right\|\left\|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right\|dk$$

$$+\frac{\lambda^2\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|}{2}\int_0^1\left\|\frac{1}{3}-\frac{k}{2}\right\|\left\|\mathfrak{f}'\left(\left(\frac{(1+k)\lambda}{2}\mathfrak{A}\otimes 1\right)+\left(1-\frac{\lambda(1+k)}{2}\right)1\otimes\mathfrak{B}\right)\right\|dk.$$

Realize here that by Lemma 1.8,

$$\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right|$$

$$=\int_I\int_I\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)\right|dE_t\otimes dF_s.$$

Since

$$\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)\right|\leqslant\|\mathfrak{f}'\|_{I,+\infty}.$$

If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right|$$

$$=\int_I\int_I\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)\right|dE_t\otimes dF_s$$

$$\leqslant\|\mathfrak{f}'\|_{I,+\infty}\int_I\int_I dE_t\otimes dF_s=\|\mathfrak{f}'\|_{I,+\infty}.$$

From which we get the following,

$$\int_0^1\left\|\frac{1}{3}-\frac{k}{2}\right\|\left\|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right\|dk$$

$$\leqslant\|\mathfrak{f}'\|_{I,+\infty}\int_0^1\left\|\frac{1}{3}-\frac{k}{2}\right\|dk=\frac{5\|\mathfrak{f}'\|_{I,+\infty}}{36}.$$

Evaluation of the second part is analogous, summing everything up we obtain the desired equality. □

Theorem 2.3. *Assume that* $\mathfrak{f}$ *is continuously differentiable on* I *and* $|\mathfrak{f}'|$ *is convex and* $\mathfrak{A},\mathfrak{B}$ *are selfadjoint operators with* $Sp(\mathfrak{A}),Sp(\mathfrak{B})\subset I,\lambda\in[0,1]$, *then*

$$\left\|\frac{1}{6}\mathfrak{f}(\lambda\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B})+\frac{1-\lambda}{3}\mathfrak{f}\left(\frac{(1+\lambda)\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B}}{2}\right)\right.\tag{2.3}$$

$$\left.+\frac{\lambda}{3}\mathfrak{f}\left(\frac{\lambda\mathfrak{A}\otimes 1+(2-\lambda)1\otimes\mathfrak{B}}{2}\right)-\frac{1}{2}\int_0^1\mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\right\|$$

$$\leq\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\|\,\frac{\|\mathfrak{f}'(\mathfrak{A})\|(\lambda(\lambda(122\lambda-93)+3)+29)+\|\mathfrak{f}'(\mathfrak{B})\|(\lambda((273-122\lambda)\lambda-183)+61)}{1296}.$$

Proof. Since $|\mathfrak{f}'|$ is convex on I, then we get

$$\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)\right|\leqslant\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)|\mathfrak{f}'(t)|+\frac{1+k}{2}(1-\lambda)|\mathfrak{f}'(s)|$$

for all $k \in [0,1]$ and $t, s \in I$.
If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$\begin{aligned}
&\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda + \frac{1-k}{2}\right)\mathfrak{A}\otimes 1 + \frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right| \\
&= \int_I\int_I \left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda + \frac{1-k}{2}\right)t + \frac{1+k}{2}(1-\lambda)s\right)\right| dE_t \otimes dF_s \\
&\leqslant \int_I\int_I \left[\left(\frac{1+k}{2}\lambda + \frac{1-k}{2}\right)|\mathfrak{f}'(t)| + \frac{1+k}{2}(1-\lambda)|\mathfrak{f}'(s)|\right] dE_t \otimes dF_s \\
&= \left(\frac{1+k}{2}\lambda + \frac{1-k}{2}\right)|\mathfrak{f}'(\mathfrak{A})|\otimes 1 + \frac{1+k}{2}(1-\lambda)1\otimes|\mathfrak{f}'(\mathfrak{B})|
\end{aligned}$$

for all $k \in [0,1]$.
If we take the norm in the inequality, we get the following

$$\begin{aligned}
&\left\|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda + \frac{1-k}{2}\right)\mathfrak{A}\otimes 1 + \frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right\| \\
&\leqslant \left\|\left(\frac{1+k}{2}\lambda + \frac{1-k}{2}\right)|\mathfrak{f}'(\mathfrak{A})|\otimes 1 + \frac{1+k}{2}(1-\lambda)1\otimes|\mathfrak{f}'(\mathfrak{B})|\right\| \\
&\leqslant \left(\frac{1+k}{2}\lambda + \frac{1-k}{2}\right)\left\||\mathfrak{f}'(\mathfrak{A})|\otimes 1\right\| + \frac{1+k}{2}(1-\lambda)\left\|1\otimes|\mathfrak{f}'(\mathfrak{B})|\right\| \\
&= \left(\frac{1+k}{2}\lambda + \frac{1-k}{2}\right)\left\|\mathfrak{f}'(\mathfrak{A})\right\| + \frac{1+k}{2}(1-\lambda)\left\|\mathfrak{f}'(\mathfrak{B})\right\|.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\int_0^1 \left\|\frac{k}{2} - \frac{1}{3}\right\| \left\|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda + \frac{1-k}{2}\right)\mathfrak{A}\otimes 1 + \frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right\| dk \\
&\leqslant \int_0^1 \left\|\frac{k}{2} - \frac{1}{3}\right\| \left(\left(\frac{1+k}{2}\lambda + \frac{1-k}{2}\right)\left\|\mathfrak{f}'(\mathfrak{A})\right\| + \frac{1+k}{2}(1-\lambda)\left\|\mathfrak{f}'(\mathfrak{B})\right\|\right) dk \\
&= \frac{\left\|\mathfrak{f}'(\mathfrak{A})\right\|(61\lambda + 29) + 61\left\|\mathfrak{f}'(\mathfrak{B})\right\|(1-\lambda))}{648}.
\end{aligned}$$

Simplifying the other term and adding them, we obtain the desired inequality.

□

We recall that the function $\mathfrak{f} : I \to \mathbb{R}$ is quasi-convex, if

$$\mathfrak{f}((1-\lambda)t + \lambda s) \leqslant \max(\mathfrak{f}(t), \mathfrak{f}(s)) = \frac{1}{2}(\mathfrak{f}(t) + f(s) + |\mathfrak{f}(s) - \mathfrak{f}(t)|)$$

holds for all $t, s \in I$ and $\lambda \in [0,1]$.

Theorem 2.4. *Assume that $\mathfrak{f}$ is continuously differentiable on I with $|\mathfrak{f}'|$ is quasi-convex on I, $\mathfrak{A}$ and $\mathfrak{B}$ are selfadjoint operators with $Sp(\mathfrak{A}), Sp(\mathfrak{B}) \subset I, \alpha$, then*

$$\begin{aligned}
&\left\|\frac{1}{6}\mathfrak{f}(\lambda\mathfrak{A}\otimes 1 + (1-\lambda)1\otimes\mathfrak{B}) + \frac{1-\lambda}{3}\mathfrak{f}\left(\frac{(1+\lambda)\mathfrak{A}\otimes 1 + (1-\lambda)1\otimes\mathfrak{B}}{2}\right)\right. \\
&\left. + \frac{\lambda}{3}\mathfrak{f}\left(\frac{\lambda\mathfrak{A}\otimes 1 + (2-\lambda)1\otimes\mathfrak{B}}{2}\right) - \frac{1}{2}\int_0^1 \mathfrak{f}\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1 + 1\otimes\mathfrak{B}\left(\frac{\phi + (1-\lambda)}{2}\right)\right) d\phi\right\| \\
&\leq \frac{5\left\|1\otimes\mathfrak{B} - \mathfrak{A}\otimes 1\right\|(2\lambda-1)^2 + 1)}{288}\left\|\mathfrak{f}'\right\|_{I,+\infty}. \qquad (2.4)
\end{aligned}$$

Proof. Since $|\mathfrak{f}'|$ is quasi-convex on I, then we get

$$\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)\right| \leqslant \frac{1}{2}(|\mathfrak{f}'(t)|+|\mathfrak{f}'(s)|+||\mathfrak{f}'(t)|-|\mathfrak{f}'(s)||)$$

for all $k\in[0,1]$ and $t,s\in I$. If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\begin{aligned}
&\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right| \\
&=\int_I\int_I\left|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)t+\frac{1+k}{2}(1-\lambda)s\right)\right|dE_t\otimes dF_s \\
&\leqslant \frac{1}{2}\int_I\int_I(|\mathfrak{f}'(t)|+|\mathfrak{f}'(s)|+||\mathfrak{f}'(t)|-|\mathfrak{f}'(s)||)dE_t\otimes dF_s \\
&=\frac{1}{2}(|\mathfrak{f}'(\mathfrak{A})|\otimes 1+1\otimes|\mathfrak{f}'(\mathfrak{B})|+||\mathfrak{f}'(\mathfrak{A})|\otimes 1-1\otimes|\mathfrak{f}'(\mathfrak{B})||)
\end{aligned}$$

for all $k\in[0,1]$.
If we take the norm, then we get

$$\begin{aligned}
&\left\|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right\| \\
&\leqslant\left\|\frac{1}{2}(|\mathfrak{f}'(\mathfrak{A})|\otimes 1+1\otimes|\mathfrak{f}'(\mathfrak{B})|+||\mathfrak{f}'(\mathfrak{A})|\otimes 1-1\otimes|\mathfrak{f}'(\mathfrak{B})||)\right\| \\
&\leqslant\frac{1}{2}\left(\||\mathfrak{f}'(\mathfrak{A})|\otimes 1+1\otimes|\mathfrak{f}'(\mathfrak{B})|\|+\||\mathfrak{f}'(\mathfrak{A})|\otimes 1-1\otimes|\mathfrak{f}'(\mathfrak{B})|\|\right)
\end{aligned}$$

Which when applied in our case, we get

$$\begin{aligned}
&\int_0^1\left\|\frac{k}{2}-\frac{1}{3}\right\|\left\|\mathfrak{f}'\left(\left(\frac{1+k}{2}\lambda+\frac{1-k}{2}\right)\mathfrak{A}\otimes 1+\frac{1+k}{2}(1-\lambda)1\otimes\mathfrak{B}\right)\right\|dk \\
&\leqslant\int_0^1\left\|\frac{k}{2}-\frac{1}{3}\right\|\left(\frac{1}{2}\left(\||\mathfrak{f}'(\mathfrak{A})|\otimes 1+1\otimes|\mathfrak{f}'(\mathfrak{B})|\|+\||\mathfrak{f}'(\mathfrak{A})|\otimes 1-1\otimes|\mathfrak{f}'(\mathfrak{B})|\|\right)\right)dk.
\end{aligned}$$

Which when simplified, we obtain the desired inequality. □

3. Some examples and consequences

In the following sequel we provide examples to the obtained Theorems in Main section. Examples consist of taking f to be an exponential operator and applying various conditions as given by the Theorems.

Corollary 3.1. *If $\mathfrak{A},\mathfrak{B}$ are selfadjoint operators with $Sp(\mathfrak{A}),Sp(\mathfrak{B})\subset[m,M]$ and $1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1$ is invertible, then by (2.2), we get*

$$\begin{aligned}
&\left\|\frac{1}{6}\exp(\lambda\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B})+\frac{1-\lambda}{3}\exp\left(\frac{(1+\lambda)\mathfrak{A}\otimes 1+(1-\lambda)1\otimes\mathfrak{B}}{2}\right)\right. \\
&\left.+\frac{\lambda}{3}\exp\left(\frac{\lambda\mathfrak{A}\otimes 1+(2-\lambda)1\otimes\mathfrak{B}}{2}\right)-\frac{1}{2}\int_0^1\exp\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1+1\otimes\mathfrak{B}\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\right\| \\
&\leq\frac{5\left\|1\otimes\mathfrak{B}-\mathfrak{A}\otimes 1\right\|\left((2\lambda-1)^2+1\right)}{144}\exp(M). \qquad (3.1)
\end{aligned}$$

Corollary 3.2. *Since for* $\mathfrak{f}(t) = \exp(t), t \in \mathbb{R}$, $|\mathfrak{f}'|$ *is convex, then by* (2.3)

$$\left\|\left|\frac{1}{6}\exp(\lambda\mathfrak{A}\otimes 1 + (1-\lambda)1\otimes\mathfrak{B}) + \frac{1-\lambda}{3}\exp\left(\frac{(1+\lambda)\mathfrak{A}\otimes 1 + (1-\lambda)1\otimes\mathfrak{B}}{2}\right)\right.\right.$$
$$\left.\left.+\frac{\lambda}{3}\exp\left(\frac{\lambda\mathfrak{A}\otimes 1 + (2-\lambda)1\otimes\mathfrak{B}}{2}\right) - \frac{1}{2}\int_0^1 \exp\left(\left(\frac{1+\lambda-\phi}{2}\right)\mathfrak{A}\otimes 1 + 1\otimes\mathfrak{B}\left(\frac{\phi+(1-\lambda)}{2}\right)\right)d\phi\right|\right\|$$
$$\leq \|1\otimes\mathfrak{B} - \mathfrak{A}\otimes 1\| \frac{\|\exp(\mathfrak{A})\|\left(\lambda(\lambda(122\lambda-93)+3)+29\right) + \|\exp(\mathfrak{B})\|\left(\lambda((273-122\lambda)\lambda-183)+61\right)}{1296}. \tag{3.2}$$

Setting $\lambda = \frac{1}{2}$, *we obtain*

$$\left\|\left|\frac{1}{6}\exp\left(\frac{\mathfrak{A}\otimes +1\otimes\mathfrak{B}}{2}\right) + \frac{1}{6}\exp\left(\frac{3\mathfrak{A}\otimes 1 + 1\otimes\mathfrak{B}}{4}\right)\right.\right.$$
$$\left.\left.+\frac{1}{6}\exp\left(\frac{\mathfrak{A}\otimes 1 + 3\cdot 1\otimes\mathfrak{B}}{2}\right) - \frac{1}{2}\int_0^1 \exp\left(\left(\frac{\frac{3}{2}-\phi}{2}\right)\mathfrak{A}\otimes 1 + 1\otimes\mathfrak{B}\left(\frac{\phi+\frac{1}{2})}{2}\right)\right)d\phi\right|\right\|$$
$$\leq \|1\otimes\mathfrak{B} - \mathfrak{A}\otimes 1\| \frac{5(\|\exp(\mathfrak{A})\| + \|\exp(\mathfrak{B})\|}{288}. \tag{3.3}$$

4. Conclusion

Tensors have become important in various fields, for example in physics because they provide a concise mathematical framework for formulating and solving physical problems in fields such as mechanics, electromagnetism, quantum mechanics, and many others. As such inequalities are crucial in numerical aspects. Reflected in this work is the tensorial Sarikaya and Bardak's Lemma, which as a consequence enabled us to obtain Simpson type inequalities in Hilbert space. New Simpson type inequalities are given, examples of specific convex functions and their inequalities using our results are given in the section some examples and consequences. Plans for future research can be reflected in the fact that the obtained inequalities in this work can be sharpened or generalized by using other methods. An interesting perspective can be seen in incorporating other techniques for Hilbert space inequalities with the techniques shown in this paper. One direction is the technique of the Mond-Pecaric inequality, on which we will work on.

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