Proceedings of International Mathematical Sciences
ISSN: 2717-6355, URL: https://dergipark.org.tr/tr/pub/pims
Volume 6 Issue 2 (2024), Pages 68-76.
Doi: https://doi.org/10.47086/pims.1577951

SOME BEST PROXIMITY POINT RESULTS ON b-METRIC SPACES WITH AN APPLICATION

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Abstract. In this paper, we introduce the concept of ϑ-p-proximal contraction mapping on b-metric spaces. Then, we obtain some best proximity results for these mappings. Also, an example to support the validity and superiority of our result has been given. Lastly, for the existence of solutions of nonlinear fractional differential equations of Caputo type we provide an application.

1. Introduction

In nonlinear analysis, game theory, approximation theory, differential equations, and control systems, fixed point theory is a crucial tool for resolving a variety of issues. As a result, numerous authors have enhanced fixed point theory. The Banach contraction principle [4], which is considered as the foundation of fixed point theory on metric spaces, was presented in this context. Let (Π, η) be a complete metric space and $\kappa : \Pi \to \Pi$ be a contraction mapping, then κ has a unique fixed point. The existence and uniqueness of fixed points in this field have been showed by numerous results [7, 11, 12]. Lately, Popescu [17] extended Banach contraction by introducing a new type of contractive condition called p-contraction. Let (Π, η) be a metric space and $\kappa : \Pi \to \Pi$ be a mapping. If there exists a ϱ in $[0, 1)$ such that

$$\eta(\kappa\check{r}, \kappa\hat{s}) \leq \varrho[\eta(\check{r}, \hat{s}) + |\eta(\check{r}, \kappa\check{r}) - \eta(\hat{s}, \kappa\hat{s})|]$$

for all $\check{r}, \hat{s} \in \Pi$, then κ is said to be a p-contraction mapping. Then, Popescu [17] proved that every p-contraction on a complete metric space has a unique fixed point.

Taking into account nonself mappings $\kappa : A \to B$ where A, B are nonempty subsets of a metric space (Π, η), the fixed point theory has recently been improved. A solution to the equation $\kappa\check{r} = \check{r}$ cannot exist if the intersection of A and B is empty. Then, it is natural to search if there is a point $\check{r}$ in A such that $\eta(\check{r}, \kappa\check{r}) = \eta(A, B)$ which is called a best proximity point of κ [6]. Numerous authors have written about this subject due to the fact that each best proximity point turns into a fixed point in the case of $A = B = \Pi$ [2, 5, 13, 14, 15].

The relevant basic definitions and symbols of best proximity point theory are now restated.

2020 *Mathematics Subject Classification.* Primary: 54H25; 47H10.
Key words and phrases. b-metric spaces; best proximity point; nonlinear fractional differential equations.

Submitted on 01.11.2024, Accepted on 09.12.2024.
Communicated by Mehmet Dik.

Let (Π, η) be a metric space and $\emptyset \neq A, B \subseteq \Pi$. We will use the subsets of A and B, respectively:

$$A_0 = \{\hat{s} \in A : \eta(\hat{s}, \check{r}) = \eta(A, B) \text{ for some } \check{r} \in B\}$$

and

$$B_0 = \{\check{r} \in B : \eta(\hat{s}, \check{r}) = \eta(A, B) \text{ for some } \hat{s} \in A\}$$

where $\eta(A, B) = \inf\{\eta(\hat{s}, \check{r}) : \hat{s} \in A \text{ and } \check{r} \in B\}$.

Definition 1.1. [5] *Let (Π, η) be a metric space and A, B be nonempty subsets of Π. A mapping $\kappa : A \to B$ is said to be proximal contraction, if there exists a real number $\varrho \in [0, 1)$ such that*

$$\left.\begin{array}{r} \eta(\hat{w}_1, \kappa\check{r}_1) = \eta(A, B) \\ \eta(\hat{w}_2, \kappa\check{r}_2) = \eta(A, B) \end{array}\right\} \Rightarrow \eta(\hat{w}_1, \hat{w}_2) \leq \varrho\eta(\check{r}_1, \check{r}_2)$$

for all $\hat{w}_1, \hat{w}_2, \check{r}_1, \check{r}_2 \in A$.

Definition 1.2. [9] *Let (Π, η) be a metric space and $\emptyset \neq A, B \subseteq \Pi$. Assume that $\vartheta : A \times A \to [0, \infty)$ is a function and $\kappa : A \to B$ is a mapping. If the following condition holds, we say that κ is ϑ-proximal admissible*

$$\left.\begin{array}{r} \vartheta(\check{r}_0, \check{r}_1) \geq 1 \\ \eta(\check{r}_1, \kappa\check{r}_0) = \eta(A, B) \\ \eta(\check{r}_2, \kappa\check{r}_1) = \eta(A, B) \end{array}\right\} \Longrightarrow \vartheta(\check{r}_1, \check{r}_2) \geq 1$$

for all $\check{r}_0, \check{r}_1, \check{r}_2 \in A$.

On the other hand, Czerwik [8, 10] established an extansion of the famous principle in a different approach than the results found in the literature by introducing a pleasant concept of a b-metric.

Definition 1.3. [10] *Let Π be a non-empty set and $\eta : \Pi \times \Pi \to [0, \infty)$ be a function satisfying for all $\hat{s}, \check{r}, z \in \Pi$,*

b1) $\hat{s} = \check{r}$ *if and only if* $\eta(\hat{s}, \check{r}) = 0$,
b2) $\eta(\hat{s}, \check{r}) = \eta(\check{r}, \hat{s})$,
b3) $\eta(\hat{s}, z) \leq s[\eta(\hat{s}, \check{r}) + \eta(\check{r}, z)]$ *where* $s \geq 1$.

Then, η is called a b-metric on Π with coefficient s. Also, (Π, η) is said to be a b-metric space.

Each metric space is obviously a b-metric space. The opposite might not be right, though. In fact, let $\Pi = \mathbb{R}$ and $\eta : \Pi \times \Pi \to [0, \infty)$ be a function defined by $\eta(\hat{s}, \check{r}) = (\hat{s} - \check{r})^2$ for all $\hat{s}, \check{r} \in \Pi$. Then (Π, η) is a b-metric space with the coefficient $s = 2$. Choose $\hat{s} = 7$, $\check{r} = 4$ and $z = 5$, then

$$\eta(7, 4) = 9 > 5 = \eta(7, 5) + \eta(5, 4).$$

Hence, it is not a metric space.

Let (Π, η) be a b-metric space with the coefficient $s \geq 1$. Let $\{\check{r}_n\}$ be sequence in Π and $\hat{s} \in \Pi$. Then, the sequence $\{\check{r}_n\}$ converges to $\hat{s}$ with respect to τ_η if and only if

$$\lim_{n\to\infty} \eta(\check{r}_n, \check{r}) = 0.$$

The sequence $\{\check{r}_n\}$ is called a Cauchy sequence if for all $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ satisfying $\eta(\check{r}_n, \check{r}_m) < \varepsilon$ for all $m, n \geq n_0$. (Π, η) is called a complete b-metric space if each Cauchy sequence converges to $\check{r} \in \Pi$ with respect to τ_η.

Any b-metric might not be continuous, in contrast to the regular metric. The following definition, which is crucial to our primary findings, helps us get beyond this drawback.

Definition 1.4. [3] *Let (Π,η) be a b-metric space with the coefficient $s \geq 1$ and $\emptyset \neq A, B \subseteq \Pi$ with $A_0 \neq \emptyset$. The pair (A,B) holds the property (M_C) if for every sequences $\{\check{r}_n\}$ in A_0, $\{\hat{s}_n\}$ in B_0 and $\check{r} \in A$, $\hat{s} \in B$, we have*

$$\lim_{n\to\infty} \eta(\check{r}_n, \check{r}) = \lim_{n\to\infty} \eta(\hat{s}_n, \hat{s}) = 0 \implies \lim_{n\to\infty} \eta(\check{r}_n, \hat{s}_n) = \eta(\check{r}, \hat{s}).$$

Now, we recall the following definition.

Definition 1.5. [3]*Let (Π,η) be a b-metric space with the coefficient $s \geq 1$. If each sequence $\{\hat{s}_n\}$ in B such that $\eta(\check{r}, B) \leq \lim_{n\to\infty} \eta(\check{r}, \hat{s}_n) \leq s\eta(\check{r}, B)$ for some $\check{r} \in A$ has a convergent subsequence in B, then B is called an s-approximately compact with respect to A.*

In this paper, we obtain some best proximity results on b-metric spaces by introducing the concept of ϑ-p-proximal contraction mapping. Also, we give an example to support the validity and superiority of our results. Finally, an application to an existence of the solution of nonlinear fractional differential equations for Caputo type is given.

2. Main Results

We begin this section with the following definition.

Definition 2.1. *Let (Π,η) be a b-metric space with $s \geq 1$, $A, B \subseteq \Pi$ with $A_0 \neq \emptyset$. Assume that $\vartheta : A\times A \to [0,\infty)$ is a function and $\kappa : A \to B$ is a mapping. If there exist $\varrho \in \left[0, \frac{1}{2s-1}\right)$ such that*

$$\left.\begin{array}{r} \eta(\hat{w}, \kappa\check{r}) = \eta(A,B) \\ \\ \eta(v, \kappa\hat{s}) = \eta(A,B) \end{array}\right\} \vartheta(\check{r},\hat{s})\eta(\hat{w},v) \leq \varrho\left\{\eta(\check{r},\hat{s}) + |\eta(\check{r},\hat{w}) - \eta(\hat{s},v)|\right\}$$

for all $\check{r}, \hat{s}, \hat{w}, v \in A$, then we say κ is an ϑ-p-contraction mapping.

Now, we give an important condition for our main result.

(H) If $\{\check{r}_n\} \subseteq A_0$ is a sequence satisfying $\vartheta(\check{r}_n, \check{r}_{n+1}) \geq 1$ and $\check{r}_n \to \check{r} \in A$, then there is a subsequence $\{\check{r}_{n_k}\}$ of $\{\check{r}_n\}$ satisfying $\vartheta(\check{r}_{n_k}, \check{r}) \geq 1$ for all $k \in \mathbb{N}$.

Theorem 2.1. *Let (Π,η) be a complete b-metric space with $s \geq 1$, $A, B \subseteq \Pi$ with $A_0 \neq \emptyset$. Assume that the following conditions hold:*

i) *the condition (H) holds and $\kappa : A \to B$ is ϑ-p-proximal contraction mapping with $\kappa(A_0) \subseteq B_0$,*
ii) *κ is an ϑ-proximal admissible,*
iii) *the pair (A,B) satisfies the property (M_C),*
iv) *there are $\check{r}_0, \check{r}_1 \in A_0$ such that $\eta(\check{r}_1, \kappa\check{r}_0) = \eta(A,B)$ and $\vartheta(\check{r}_0, \check{r}_1) \geq 1$,*
v) *B is an s-approximately compact with respect to A.*

Then, κ has a best proximity point $\check{r}^*$ in A.If for another best proximity point $\hat{s}^* \in A$, $\vartheta(\check{r}^*, \hat{s}^*) \geq 1$, then $\check{r}^* = \hat{s}^*$.

Proof. From the condition (iv), there are $\check{r}_0, \check{r}_1 \in A_0$ such that $\eta(\check{r}_1, \kappa\check{r}_0) = \eta(A,B)$ and $\vartheta(\check{r}_0, \check{r}_1) \geq 1$. Since $\kappa\check{r}_1 \in \kappa(A_0) \subseteq B_0$, there exists $\check{r}_2 \in A_0$ such that

$$\eta(\check{r}_2, \kappa\check{r}_1) = \eta(A,B).$$

Since κ is an ϑ-proximal admissible, we get $\vartheta(\check{r}_1, \check{r}_2) \geq 1$. Similarly, since $\kappa\check{r}_2 \in \kappa(A_0) \subseteq B_0$, there exists $\check{r}_3 \in A_0$ such that

$$\eta(\check{r}_3, \kappa\check{r}_2) = \eta(A,B).$$

Since κ is an ϑ-proximal admissible, we get $\vartheta(\check{r}_2, \check{r}_3) \geq 1$. Repeating this process, we can construct a sequence $\{\check{r}_n\}$ in A such that

$$\eta(\check{r}_{n+1}, \kappa\check{r}_n) = \eta(A, B) \text{ and } \vartheta(\check{r}_n, \check{r}_{n+1}) \geq 1 \tag{2.1}$$

for all $n \geq 1$. Then, we have

$$\begin{aligned}
\eta(\check{r}_n, \check{r}_{n+1}) &\leq \vartheta(\check{r}_n, \check{r}_{n+1})\eta(\check{r}_n, \check{r}_{n+1}) \\
&\leq \varrho\left\{\eta(\check{r}_{n-1}, \check{r}_n) + |\eta(\check{r}_{n-1}, \check{r}_n) - \eta(\check{r}_n, \check{r}_{n+1})|\right\}
\end{aligned}$$

Suppose that there exists $n_0 \in \mathbb{N}$ such that $\eta(\check{r}_{n_0-1}, \check{r}_{n_0}) \leq \eta(\check{r}_{n_0}, \check{r}_{n_0+1})$, then we have

$$\begin{aligned}
\eta(\check{r}_{n_0}, \check{r}_{n_0+1}) &\leq \vartheta(\check{r}_{n_0}, \check{r}_{n_0+1})\eta(\check{r}_{n_0}, \check{r}_{n_0+1}) \\
&\leq \varrho\left\{\eta(\check{r}_{n_0-1}, \check{r}_{n_0}) + \left|\eta(\check{r}_{n_0-1}, \check{r}_{n_0}) - \eta(\check{r}_{n_0}, \check{r}_{n_0+1})\right|\right\} \\
&= \varrho\left\{\eta(\check{r}_{n_0-1}, \check{r}_{n_0}) + \eta(\check{r}_{n_0}, \check{r}_{n_0+1}) - \eta(\check{r}_{n_0-1}, \check{r}_{n_0})\right\} \\
&= \varrho\eta(\check{r}_{n_0}, \check{r}_{n_0+1}) \\
&< \eta(\check{r}_{n_0}, \check{r}_{n_0+1}).
\end{aligned}$$

This is a contradiction. Then, we assume that $\eta(\check{r}_n, \check{r}_{n+1}) < \eta(\check{r}_{n-1}, \check{r}_n)$ for all $n \geq 1$. Therefore, we get

$$\begin{aligned}
\eta(\check{r}_n, \check{r}_{n+1}) &\leq \vartheta(\check{r}_n, \check{r}_{n+1})\eta(\check{r}_n, \check{r}_{n+1}) \\
&\leq \varrho\left\{\eta(\check{r}_{n-1}, \check{r}_n) + \eta(\check{r}_n, \check{r}_{n-1}) - \eta(\check{r}_n, \check{r}_{n+1})\right\} \\
&= 2\varrho\eta(\check{r}_{n-1}, \check{r}_n) - \varrho\eta(\check{r}_n, \check{r}_{n+1})
\end{aligned}$$

and so,

$$\eta(\check{r}_n, \check{r}_{n+1}) \leq \left(\frac{2\varrho}{\varrho+1}\right)\eta(\check{r}_{n-1}, \check{r}_n)$$

for all $n \geq 1$. Using the last inequality, we have

$$\begin{aligned}
\eta(\check{r}_n, \check{r}_{n+1}) &\leq \left(\frac{2\varrho}{\varrho+1}\right)\eta(\check{r}_{n-1}, \check{r}_n) \\
&\leq \left(\frac{2\varrho}{\varrho+1}\right)^2 \eta(\check{r}_{n-2}, \check{r}_{n-1}) \\
&\;\;\vdots \\
&\leq \left(\frac{2\varrho}{\varrho+1}\right)^n \eta(\check{r}_0, \check{r}_1)
\end{aligned}$$

for all $n \in \mathbb{N}$. Now, assume $n \in \mathbb{N}$ and $p \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\eta(\check{r}_n, \check{r}_{n+p}) &\leq s\eta(\check{r}_n, \check{r}_{n+1}) + s^2\eta(\check{r}_{n+1}, \check{r}_{n+2}) + \ldots + s^p\eta(\check{r}_{n+p-1}, \check{r}_{n+p}) \\
&\leq \frac{1}{s^{n-1}}\left\{\begin{array}{c} \left(\frac{2\varrho s}{\varrho+1}\right)^n \eta(\check{r}_0, \check{r}_1) + \left(\frac{2\varrho s}{\varrho+1}\right)^{n+1} \eta(\check{r}_0, \check{r}_1) + \cdots \\ + \left(\frac{2\varrho s}{\varrho+1}\right)^{n+p-1} \eta(\check{r}_0, \check{r}_1) \end{array}\right\} \\
&= \left(\frac{2\varrho s}{\varrho+1}\right)^n \left\{1 + \frac{2\varrho s}{\varrho+1} + \cdots + \left(\frac{2\varrho s}{\varrho+1}\right)^{p-1}\right\}\eta(\check{r}_0, \check{r}_1) \\
&\leq \frac{\left(\frac{2\varrho s}{\varrho+1}\right)^n}{1 - \frac{2\varrho s}{\varrho+1}}\eta(\check{r}_0, \check{r}_1).
\end{aligned}$$

Thus, $\{\check{r}_n\}$ is a Cauchy sequence in A. Since (Π,η) is a complete b-metric space and A is a closed subset of Π, there exists a $\check{r}^*\in A$ such that $\check{r}_n\to\check{r}^*$. Moreover, we have

$$\begin{aligned}\eta(\check{r}^*,B) &\le \eta(\check{r}^*,\kappa\check{r}_n)\\ &\le s\eta(\check{r}^*,\check{r}_{n+1})+s\eta(\check{r}_{n+1},\kappa\check{r}_n)\\ &= s\eta(\check{r}^*,\check{r}_{n+1})+s\eta(A,B)\\ &\le s\eta(\check{r}^*,\check{r}_{n+1})+s\eta(\check{r}^*,B).\end{aligned}$$

Therefore, we get

$$\eta(\check{r}^*,B)\le\lim_{n\to\infty}\eta(\check{r}^*,\kappa\check{r}_n)\le s\eta(\check{r}^*,B).$$

Since B is s-approximately compact with respect to A, there exists a subsequence $\{\kappa\check{r}_{n_k}\}$ of $\{\kappa\check{r}_n\}$ such that $\kappa\check{r}_{n_k}\to\hat{s}^*\in B$ as $k\to\infty$. Therefore by taking $k\to\infty$ in $\eta(\check{r}_{n_k+1},\kappa\check{r}_{n_k})=\eta(A,B)$, since the pair (A,B) satisfies the property (M_C), we have $\eta(\check{r}^*,\hat{s}^*)=\eta(A,B)$, and so $\check{r}^*\in A_0$. Also, since $\kappa\check{r}^*\in\kappa(A_0)\subseteq B_0$, there exists $z\in A_0$ such that

$$\eta(z,\kappa\check{r}^*)=\eta(A,B). \tag{2.2}$$

On the other hand, using the condition (H) we can say that there exists a subsequence $\{\check{r}_{n_r}\}$ of $\{\check{r}_n\}$ such that $\vartheta(\check{r}_{n_r},\check{r}^*)\ge 1$ for all $r\in\mathbb{N}$. Also, since κ is an ϑ-proximal admissible mapping, we have $\vartheta(\check{r}_{n_r+1},z)\ge 1$ for all $r\in\mathbb{N}$. Now, from (2.1), (2.2) and the condition of ϑ-p-proximal contraction, we obtain

$$\begin{aligned}\eta(\check{r}_{n_r+1},z) &\le \vartheta(\check{r}_{n_r+1},z)\eta(\check{r}_{n_r+1},z)\\ &\le \varrho\left(\eta(\check{r}_{n_r},\check{r}^*)+\left|\eta(\check{r}_{n_r},\check{r}_{n_r+1})-\eta(\check{r}^*,z)\right|\right)\end{aligned}$$

for all $r\in\mathbb{N}$. Thus, taking limit as $r\to\infty$ we have

$$\eta(\check{r}^*,z)\le\varrho\eta(\check{r}^*,z),$$

which gives $\check{r}^*=z$. From (2.2), the point $\check{r}^*$ is best proximity point of the mapping κ. Now, assume that $\check{r}^*$ and $\hat{s}^*$ are different best proximity points of κ in A and $\vartheta(\check{r}^*,\hat{s}^*)\ge 1$. Then, we get

$$\eta(\check{r}^*,\kappa\check{r}^*)=\eta(A,B)$$

and

$$\eta(\hat{s}^*,\kappa\hat{s}^*)=\eta(A,B).$$

Since the mapping κ is ϑ-p-proximal contraction, we have

$$\begin{aligned}\eta(\check{r}^*,\hat{s}^*) &\le \vartheta(\check{r}^*,\hat{s}^*)\,\eta(\check{r}^*,\hat{s}^*)\\ &\le \varrho\left(\eta(\check{r}^*,\hat{s}^*)+|\eta(\check{r}^*,\check{r}^*)-\eta(\hat{s}^*,\hat{s}^*)|\right)\\ &= \varrho\eta(\check{r}^*,\hat{s}^*)\end{aligned}$$

which gives $\check{r}^*=\hat{s}^*$. This is contradiction. Hence, κ has only one best proximity point. □

Example 2.1. *Let $\Pi=\mathbb{R}^2$ and $\eta:\Pi\times\Pi\to\mathbb{R}$ be a function defined by*

$$\eta((\check{r}_1,\check{r}_2),(\hat{s}_1,\hat{s}_2))=\max\{\check{r}_1,\hat{s}_1\}+(\check{r}_2-\hat{s}_2)^2$$

for all $(\check{r}_1,\check{r}_2),(\hat{s}_1,\hat{s}_2)\in\Pi$. Then, (Π,η) is a b-metric space with coefficient $s=2$. Now, consider the sets $A=[0,1]\times\{0\}$ and $B=[0,1]\times\{1\}$. We get $\eta(A,B)=1$, A is closed, $A_0=\{(0,0)\}$ and $B_0=\{(0,1)\}$. Also, it can be seen that B is an s-approximately compact with respect to A and the pair (A,B) satisfies the property (M_C). Define a function ϑ : $A\times A\to[0,\infty)$ and a mapping $\kappa:A\to B$ as

$$\vartheta(\check{r},\hat{s}) = \begin{cases} 1 & , \quad \check{r},\hat{s} \in A_0 \\ \\ 0 & , \quad \textit{otherwise} \end{cases}$$

and

$$\kappa(\check{r}_1, 0) = \left(\frac{\check{r}_1}{2}, 1\right),$$

respectively. Then, we can choose $\check{r}_0 = (0,0) = \check{r}_1 \in A_0$ *such that* $\eta(\check{r}_1, \kappa\check{r}_0) = \eta(A,B)$ *and* $\vartheta(\check{r}_0,\check{r}_1) \geq 1$*, and so the condition (iv) hold. Also, it is clear that* $\kappa : A \to B$ *is an* ϑ*-proximal admissible and* ϑ*-p-proximal contraction mapping with* $\kappa(A_0) \subseteq B_0$*. Hence, all hypotheses of Theorem 2.1 are satisfied, and so there is a unique best proximity point* $\check{r} = (0,0)$ *of* κ*.*

If we take $A = B = \Pi$ in Theorem 2.1, then we obtain the following fixed point result.

Corollary 2.2. *Let* (Π,η) *be a complete b-metric space with the coefficient* $s \geq 1$ *and* $\kappa : \Pi \to \Pi$ *be a continuous mapping. If the following conditions hold,*

i) *for all* $\hat{s},\check{r} \in \Pi$*, it is satisfied*

$$\vartheta(\check{r},\hat{s})\eta(\kappa\check{r},\kappa\hat{s}) \leq \varrho\left\{\eta(\check{r},\hat{s}) + |\eta(\check{r},\kappa\check{r}) - \eta(\hat{s},\kappa\hat{s})|\right\},$$

ii) *If* $\{\check{r}_n\} \subseteq \Pi$ *is a sequence such that* $\vartheta(\check{r}_n,\check{r}_{n+1}) \geq 1$ *and* $\check{r}_n \to \check{r} \in \Pi$*, then there exists a subsequence* $\{\check{r}_{n_k}\}$ *of* $\{\check{r}_n\}$ *such that* $\vartheta(\check{r}_{n_k},\check{r}) \geq 1$ *for all* $k \in \mathbb{N}$*.*

iii) κ *is an* ϑ*-admissible,*

iv) *there is* $\check{r}_0 \in \Pi$ *such that* $\vartheta(\check{r}_0,\kappa\check{r}_0) \geq 1$*,*

then, κ *has a fixed point* $\check{r}^*$ *in* Π*.If for another fixed point* $\hat{s}^* \in \Pi$*,* $\vartheta(\check{r}^*,\hat{s}^*) \geq 1$*, then* $\check{r}^* = \hat{s}^*$*.*

If we take $s = 1$ in Corollary 2.2, then we obtain the following fixed point result.

Corollary 2.3. *Let* (Π,η) *be a complete metric space and* $\kappa : \Pi \to \Pi$ *be a mapping. If the following conditions hold,*

i) *for all* $\hat{s},\check{r} \in \Pi$*, it is satisfied*

$$\vartheta(\check{r},\hat{s})\eta(\kappa\check{r},\kappa\hat{s}) \leq \varrho\left\{\eta(\check{r},\hat{s}) + |\eta(\check{r},\kappa\check{r}) - \eta(\hat{s},\kappa\hat{s})|\right\}, \tag{2.3}$$

ii) *If* $\{\check{r}_n\} \subseteq \Pi$ *is a sequence such that* $\vartheta(\check{r}_n,\check{r}_{n+1}) \geq 1$ *and* $\check{r}_n \to \check{r} \in \Pi$*, then there exists a subsequence* $\{\check{r}_{n_k}\}$ *of* $\{\check{r}_n\}$ *such that* $\vartheta(\check{r}_{n_k},\check{r}) \geq 1$ *for all* $k \in \mathbb{N}$*.*

iii) κ *is an* ϑ*-admissible,*

iv) *there is* $\check{r}_0 \in \Pi$ *such that* $\vartheta(\check{r}_0,\kappa\check{r}_0) \geq 1$*,*

then, κ *has a fixed point* $\check{r}^*$ *in* Π*. If for another fixed point* $\hat{s}^* \in \Pi$*,* $\vartheta(\check{r}^*,\hat{s}^*) \geq 1$*, then* $\check{r}^* = \hat{s}^*$*.*

3. Application

For nonlinear fractional differential equations of Caputo type, we provide adequate requirements for their existence and uniqueness in this section. For a continuous function $g : [0,\infty) \to \mathbb{R}$ of order $\alpha > 0$, the Caputo derivative is defined as

$${}^{C}D^{\alpha}(g(\gamma)) = \frac{1}{\Gamma(n-\alpha)}\int_0^{\gamma}(\gamma - s)^{n-\alpha-1}g^{(n)}(s)ds, \alpha > 0, n-1 < \alpha < n$$

where Γ is the gamma function and n is an integer.

The following nonlinear fractional differential equation of Caputo type

$${}^{C}D^{\alpha}(g(\gamma)) = f(\gamma,\check{r}(\gamma)) \tag{3.1}$$

with integral boundary conditions

$$\check{r}(0) = 0 \text{ and } \check{r}(1) = \int_0^\theta \check{r}(\hat{w})d\hat{w}$$

where $1 < \alpha \leq 2$, $0 < \theta < 1$, $\check{r} \in C[0,1]$ which is the space of all continuous real-valued functions defined on $[0,1]$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Since f is a continuous, the equation (3.1) is equivalent to the following integral equation [1, 16]:

$$\begin{aligned} \check{r}(\gamma) &= \frac{1}{\Gamma(\alpha)} \int_0^\gamma (\gamma - \hat{w})^{\alpha-1} f(\hat{w}, \check{r}(\hat{w}))d\hat{w} \\ &\quad - \frac{2\gamma}{(2-\theta^2)\Gamma(\alpha)} \int_0^1 (1-\hat{w})^{\alpha-1} f(\hat{w}, \check{r}(\hat{w}))d\hat{w} \\ &\quad + \frac{2\gamma}{(2-\theta^2)\Gamma(\alpha)} \int_0^\theta \left(\int_0^{\hat{w}} (\hat{w} - r)^{\alpha-1} f(r, \check{r}(r))dr \right) d\hat{w}. \end{aligned} \tag{3.2}$$

Theorem 3.1. *Let $\chi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function. Suppose the following conditions hold:*

i) *the mapping $\kappa : C[0,1] \to C[0,1]$*

$$\begin{aligned} \kappa\check{r}(\gamma) &= \frac{1}{\Gamma(\alpha)} \int_0^\gamma (\gamma - \hat{w})^{\alpha-1} f(\hat{w}, \check{r}(\hat{w}))d\hat{w} \\ &\quad - \frac{2\gamma}{(2-\theta^2)\Gamma(\alpha)} \int_0^1 (1-\hat{w})^{\alpha-1} f(\hat{w}, \check{r}(\hat{w}))d\hat{w} \\ &\quad + \frac{2\gamma}{(2-\theta^2)\Gamma(\alpha)} \int_0^\theta \left(\int_0^{\hat{w}} (\hat{w} - r)^{\alpha-1} f(r, \check{r}(r))dr \right) d\hat{w} \end{aligned}$$

for all $\check{r} \in C[0,1]$ and $\gamma \in [0,1]$, is a continuous mapping where $1 < \alpha \leq 2$, $0 < \theta < 1$.

ii) *there exists $\check{r}_0 \in C[0,1]$ such that $\chi(\check{r}_0(\gamma), \kappa\check{r}_0(\gamma)) \geq 0$ for all $\gamma \in [0,1]$.*

iii) *if for each $\gamma \in [0,1]$ and $\check{r}, \hat{s} \in C[0,1]$, $\chi(\check{r}(\gamma), \hat{s}(\gamma)) \geq 0$, then $\chi(\kappa\check{r}(\gamma), \kappa\hat{s}(\gamma)) \geq 0$.*

iv) *for each sequence $\{\check{r}_n\} \subseteq C[0,1]$ such that for all $\gamma \in [0,1]$, $\{\check{r}_n(\gamma)\}$ converges to $\check{r}(\gamma)$ for some $\check{r} \in C[0,1]$ and $\chi(\check{r}_n(\gamma), \check{r}_{n+1}(\gamma)) \geq 0$ for all $n \geq 1$, then there exists a subsequence $\{\check{r}_{n_k}\}$ of $\{\check{r}_n\}$ such that $\chi(\check{r}_{n_k}(\gamma), \check{r}(\gamma)) \geq 0$ for all $\gamma \in [0,1]$ and $k \geq 1$.*

v) *there exists q in $[0,1)$ such that*

$$|f(\hat{w}, \check{r}(\hat{w})) - f(\hat{w}, \hat{s}(\hat{w}))| \leq \frac{\Gamma(\alpha+1)}{5} \left\{ \varrho\left(|\check{r}(\hat{w}) - \hat{s}(\hat{w})|^2\right) + N(\check{r}, \kappa) \right\}^{\frac{1}{2}}$$

where $\varrho \in [0,1)$ and $N(\check{r}, \kappa) = \left|\sup_{\hat{w}\in[0,1]} |\check{r}(\hat{w}) - \kappa\check{r}(\hat{w})|^2 - \sup_{\hat{w}\in[0,1]} |\hat{s}(\hat{w}) - \kappa\hat{s}(\hat{w})|^2\right|$. Then, the problem (3.1) has a unique solution.

Proof. Let $\Pi = C[0,1]$ and $\eta : \Pi \times \Pi \to [0,\infty)$ a function defined by

$$\eta(\hat{w}, v) = \sup_{\gamma\in[0,1]} |\hat{w}(\gamma) - v(\gamma)|^2$$

for all $\gamma \in [0,1]$ and $\hat{w}, v \in \Pi$. Hence, (Π, η) is a complete b-metric space with $s = 2$. We shall show that κ satisfies the inequality (2.3). Let's take $\check{r}, \hat{s} \in \Pi$ with $\chi(\check{r}(\gamma), \hat{s}(\gamma)) \geq 0$

for all $\gamma \in [0,1]$.Then, we have

$$
\begin{aligned}
|\kappa\check{r}(\gamma)-\kappa\hat{s}(\gamma)| &= \left|\begin{array}{c}
\frac{1}{\Gamma(\alpha)}\int_0^{\gamma}(\gamma-\hat{w})^{\alpha-1}f(\hat{w},\check{r}(\hat{w}))d\hat{w} \\
-\frac{2\gamma}{(2-\theta^2)\Gamma(\alpha)}\int_0^{1}(1-\hat{w})^{\alpha-1}f(\hat{w},\check{r}(\hat{w}))d\hat{w} \\
+\frac{2\gamma}{(2-\theta^2)\Gamma(\alpha)}\int_0^{\theta}\left(\int_0^{\hat{w}}(\hat{w}-r)^{\alpha-1}f(r,\check{r}(r))dr\right)d\hat{w} \\
-\frac{1}{\Gamma(\alpha)}\int_0^{\gamma}(\gamma-\hat{w})^{\alpha-1}f(\hat{w},\hat{s}(\hat{w}))d\hat{w} \\
+\frac{2\gamma}{(2-\theta^2)\Gamma(\alpha)}\int_0^{1}(1-\hat{w})^{\alpha-1}f(\hat{w},\hat{s}(\hat{w}))d\hat{w} \\
-\frac{2\gamma}{(2-\theta^2)\Gamma(\alpha)}\int_0^{\theta}\left(\int_0^{\hat{w}}(\hat{w}-r)^{\alpha-1}f(r,\hat{s}(r))dr\right)d\hat{w}
\end{array}\right| \\
&\le \frac{1}{\Gamma(\alpha)}\left\{\int_0^{\gamma}|\gamma-\hat{w}|^{\alpha-1}\left(|f(\hat{w},\check{r}(\hat{w}))-f(\hat{w},\hat{s}(\hat{w}))|\right)d\hat{w}\right\} \\
&\quad+\frac{2\gamma}{(2-\theta^2)\Gamma(\alpha)}\left\{\int_0^{1}(1-\hat{w})^{\alpha-1}\left(|f(\hat{w},\check{r}(\hat{w}))-f(\hat{w},\hat{s}(\hat{w}))|\right)d\hat{w}\right\} \\
&\quad+\frac{2\gamma}{(2-\theta^2)\Gamma(\alpha)}\left\{\int_0^{\theta}\left(\int_0^{\hat{w}}|\hat{w}-r|^{\alpha-1}\left(|f(r,\check{r}(r))-f(r,\hat{s}(r))|\right)dr\right)d\hat{w}\right\} \\
&\le \int_0^{\gamma}\left(\begin{array}{c}\frac{|\gamma-\hat{w}|^{\alpha-1}}{\Gamma(\alpha)}\frac{\Gamma(\alpha+1)}{5} \\ \times\left\{\varrho\left(|\check{r}(\hat{w})-\hat{s}(\hat{w})|^2+N(\check{r},\kappa)\right)\right\}^{\frac{1}{2}}\end{array}\right)d\hat{w} \\
&\quad+\frac{2\gamma}{(2-\theta^2)}\int_0^{1}\left(\begin{array}{c}\frac{(1-\hat{w})^{\alpha-1}}{\Gamma(\alpha)}\frac{\Gamma(\alpha+1)}{5} \\ \times\left\{\varrho\left(|\check{r}(\hat{w})-\hat{s}(\hat{w})|^2+N(\check{r},\kappa)\right)\right\}^{\frac{1}{2}}\end{array}\right)d\hat{w} \\
&\quad+\frac{2\gamma}{(2-\theta^2)}\int_0^{\theta}\left(\int_0^{\hat{w}}\left[\begin{array}{c}\frac{|\hat{w}-r|^{\alpha-1}}{\Gamma(\alpha)}\frac{\Gamma(\alpha+1)}{5} \\ \times\left\{\varrho\left(|\check{r}(\hat{w})-\hat{s}(\hat{w})|^2+N(\check{r},\kappa)\right)\right\}^{\frac{1}{2}}\end{array}\right]dr\right)d\hat{w} \\
&\le \frac{\Gamma(\alpha+1)}{5}\left\{\varrho\left(\eta(\check{r},\hat{s})+|\eta(\check{r},\kappa\check{r})-\eta(\hat{s},\kappa\hat{s})|\right)\right\}^{\frac{1}{2}} \\
&\quad\times\sup_{\gamma\in[0,1]}\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{2\gamma}{(2-\theta^2)}\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}\right)\right\} \\
&\le \left\{\varrho\left(\eta(\check{r},\hat{s})+|\eta(\check{r},\kappa\check{r})-\eta(\hat{s},\kappa\hat{s})|\right)\right\}^{\frac{1}{2}}
\end{aligned}
$$

which implies that

$$\eta(\kappa\check{r},\kappa\hat{s}) \le \varrho\left(\eta(\check{r},\hat{s})+|\eta(\check{r},\kappa\check{r})-\eta(\hat{s},\kappa\hat{s})|\right) \tag{3.3}$$

$\check{r},\hat{s}\in\Pi$ with $\chi(\check{r}(\gamma),\hat{s}(\gamma))\ge 0$ for all $\gamma\in[0,1]$. Now, consider the mapping $\alpha:\Pi\times\Pi\to[0,\infty)$ defined by

$$\alpha(\check{r},\hat{s})=\begin{cases} 1 \quad , & \chi(\check{r}(\gamma),\hat{s}(\gamma))\ge 0 \text{ for all } \gamma\in[0,1] \\ 0 \quad , & \text{otherwise}\end{cases}.$$

Then, the inequality (3.3) is satisfied for all $\check{r},\hat{s}\in\Pi$ with $\alpha(\check{r},\hat{s})\ge 1$, that is, for all $\check{r},\hat{s}\in\Pi$,

$$\alpha(\check{r},\hat{s})\eta(\kappa\check{r},\kappa\hat{s}) \le \varrho\left(\eta(\check{r},\hat{s})+|\eta(\check{r},\kappa\check{r})-\eta(\hat{s},\kappa\hat{s})|\right)$$

is satisfied. Also, from (iii), the mapping κ is α-admissible mapping. Using the conditions (ii) and (iv) we say that all conditions of Corollary 2.3 are met, and so κ has a fixed point. Therefore, there is a solution to the nonlinear fractional differential equation of Caputo type (3.1) □

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