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Coefficient Inequalities for Two New Subclasses of Bi-univalent Functions Involving Lucas-Balancing Polynomials

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Abstract

In this article, by making use of Lucas-Balancing polynomials two new subclasses of bi-univalent functions are introduced. Then we establish the bounds for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for two new families of analytic and bi-univalent functions in the open unit disk which involve Lucas-Balancing polynomials. Furthermore, we investigate the special cases and consequences for the new family functions. In addition, the Fekete-Szegö problem is handled for the functions belonging to these new subclasses.

Keywords: Analytic and bi-univalent functions, subordination, coefficient inequality, Lucas-Balancing polynomials.

1. *Introduction*

Let A denote the class of all analytic functions of the form

$$f(z) = z + a_2 z^2 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1)$$

in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. It is clear that the functions in A satisfy the conditions and $f(0) = 0$ and $f'(0) = 1$, known as normalization conditions. We show by $\mathcal{S}$ the subclass of A consisting of functions univalent in A.

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The Koebe one quarter theorem (see (Duren 1983)) guarantees that if $f \in \mathcal{S}$, then there exists the inverse function f^{-1} satisfying

$$f^{-1}(f(z)) = z, \ (z \in \mathbb{E}) \quad \text{and} \quad f(f^{-1}(\omega)) = \omega, \ (|\omega| < r_0(f), \ r_0(f) \geq \tfrac{1}{4},$$

where

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + +(2{a_2}^2 - a_3)\omega^3 + \cdots. \qquad (2)$$

One of the most important subclass of analytic and univalent function class on the unit disk $\mathbb{E}$ is the bi-univalent function class and is denoted by Σ. In fact, a function $f \in A$ is called bi-univalent function in $\mathbb{E}$ if both f and f^{-1} are univalent in $\mathbb{E}$. Here, we would like to remind that the problem finding an upper bound for the coefficient $|a_n|$ of the functions belonging to class Σ is still an open problem. A wide range of coefficient estimates for the functions in the class Σ can be found in the literature. For instance, Brannan and Clunie (Brannan and Clunie 1980), and Lewin (Lewin 1967), gave very important bounds on $|a_2|$, respectively. Also, Brannan and Taha (Brannan and Taha 1988), focused on some subclasses of bi-univalent functions and proved certain coefficient estimates. As mentioned above, one of the most attractive open problems in univalent function theory is to find a coefficient estimate on $|a_n|$ $(n \in \mathbb{N}, \ n \geq 3,)$ for the functions in the class Σ. Since this attraction, motivated by the works **(**Brannan and Clunie 1980), (Brannan and Taha 1988), (Lewin 1967), (Srivastava et al. 2010), (Buyankara et al. 2022), (Çağlar et al. 2022), (Çağlar 2019), (Çağlar et al. 2013), (Frasin et al. 2021), (Güney at al. 2018), (Güney at al. 2019), (Orhan et al 2018), (Srivastava et al. 2013), (Toklu 2019), (Toklu et al. 2019), (Zaprawa 2014), (Aktaş and Karaman 2023), (Öztürk and Aktaş 2023), (Öztürk and Aktaş 2024), (Korkmaz and Aktaş 2024), (Aktaş and Hamarat 2023), (Orhan et al. 2023), (Aktaş and Yılmaz 2022), (Yılmaz and Aktaş 2022) and references therein, the authors introduced numerous subclasses of bi-univalent functions and obtained non- sharp estimates on the initial coefficients of functions in these subclasses.

In the univalent function theory, one of the most important notions is subordination principle. Let the function $f \in \mathrm{A}$ and $F \in A$. Then, f is called to be subordinate to F if there exists a Schwarz function ω such that
$\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = F(\omega(z))$ $(z \in \mathbb{E})$. This subordination is shown by

$$f \prec F \ \ or \ \ f(z) \prec F(z) \quad (z \in \mathbb{E}).$$

Especially, if the function F is univalent in $\mathbb{E}$, then subordination is equivalent to

$$f(0) = F(0), \qquad f(\mathbb{E}) \subset F(\mathbb{E}).$$

A comprehensive information about the subordination concept can be found in Monographs written by Miller and Mocanu (see (Miller et al. 2000)).

2. Lucas-Balancing Polynomials and Its Generating Function

The notion of Balancing number was defined by Behera and Panda in (Behera et al 1999). Actually, balancing number n and its balancer r are solutions of Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r).$$

It is known that if n is a balancing number, then $8n^2 + 1$ is a perfect square and its positive square root is called a Lucas-Balancing number (Ray 2014). Recently, some properties of these new number sequences have been intensively studied and its some generalizations were defined. Interested readers can find comprehensive information regarding Lucas-Balancing numbers in (Davala and Panda 2015), (Frontczak and Baden-Württemberg 2018), (Frontczak and Baden-Württemberg 2008), (Komatsu and Panda 2016), (Keskin and Karaatlı 2012), (Ray 2014), (Ray 2015), (Ray 2018), (Patel et al. 2018) and references therein. Natural extensions of the Lucas-Balancing numbers is Lucas-Balancing polynomial and it is defined by:

Definition 1.(Frontczak 2019) Let $x \in \mathbb{C}$ and $n \geq 2$. Then, Lucas-Balancing polynomials are defined the following recurrence relation

$$C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x), \qquad (3)$$

where $C_0(x) = 1$ and

$$C_1(x) = 3x. \qquad (4)$$

Using recurrence relation given by (3) we easily obtain that

$$C_2(x) = 18x^2 - 1, \qquad (5)$$

$$C_3(x) = 108x^3 - 9x. \qquad (6)$$

Lemma 1. (Frontczak 2019) The ordinary generating function of the Lucas-Balancing polynomials is given by

$$R(x,z) = \sum_{n=0}^{\infty} C_n(x) z^n = \frac{1-3xz}{1-6xz+z^2}. \qquad (7)$$

3. New Subclasses of Bi-univalent Functions

In this subsection, we introduce some new function subclasses of analytic and bi-univalent function class Σ which is subordinate to Lucas-Balancing polynomials.

Definition 2. A function $f(z) \in \Sigma$ of the form (1) is said to be in the class $B^{C_\Sigma}(R(x,z))$ if the following conditions hold true:

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{1-3xz}{1-6xz+z^2} = R(x,z) \qquad (8)$$

and

$$\frac{2\omega f'(\omega)}{f(\omega)-f(-\omega)} \prec \frac{1-3x\omega}{1-6x\omega+\omega^2} = R(x,\omega), \qquad (9)$$

where $z, \omega \in \mathbb{E}$, g is inverse of f and it is of the form (2).

Our second function class is bi-starlike function class $M^{C_\Sigma}(R(x,z))$ and it is defined as follows:

Definition 3. A function $f(z) \in \Sigma$ of the form (1) is said to be in the class $M^{C_\Sigma}(R(x,z))$ if the following conditions hold true:

$$\frac{2[zf'(z)]'}{[f(z)-f(-z)]'} \prec \frac{1-3xz}{1-6xz+z^2} = R(x,z) \qquad (10)$$

and

$$\frac{2[\omega f'(\omega)]'}{[f(\omega)-f(-\omega)]'} \prec \frac{1-3xz}{1-6x\omega+\omega^2} = R(x,\omega), \qquad (11)$$

where $z, \omega \in \mathbb{E}$, g is inverse of f and it is of the form (2).

In the present paper our main aim is to find upper bounds for the Taylor-Maclaurin coefficients of function subclasses defined by $B^{C_\Sigma}(R(x,z))$ and $M^{C_\Sigma}(R(x,z))$. A rich history for the class Σ can be found in the pioneering work (Srivastava et al. 2010), published by Srivastava et al.

4. Coefficient Estimates for the Classes $B^{C\Sigma}(R(x,z))$ and $M^{C\Sigma}(R(x,z))$

In this section, we present initial coefficients estimates for the function belonging to the subclasses $B^{C\Sigma}(R(x,z))$ and $M^{C\Sigma}(R(x,z))$, respectively.

Theorem 1. Suppose that the function $f(z) \in B^{C\Sigma}(R(x,z))$ and $x \in \mathbb{C}\setminus\left\{\mp\frac{\sqrt{6}}{9}\right\}$. Then,

$$|a_2| \le \frac{3\sqrt{3}|x|\sqrt{|x|}}{\sqrt{2|2-27x^2|}} \tag{12}$$

and

$$|a_3| \le \frac{3|x|}{2}\left(\frac{3|x|}{2}+1\right). \tag{13}$$

Proof. Let the function $f(z) \in B^{C\Sigma}(R(x,z))$ and $g = f^{-1}$ given by (2). In view of Definition 2, from the relations (8) and (9) we can write that

$$\frac{2zf'(z)}{f(z)-f(-z)} = R(x,\tau(z)) \tag{14}$$

and

$$\frac{2\omega f'(\omega)}{f(\omega)-f(-\omega)} = R(x,\varphi(\omega)). \tag{15}$$

Here $\tau(z) = k_1z + k_2z^2 + \cdots$ and $\varphi(\omega) = \varphi_1\omega + \varphi_2\omega^2 + \cdots$ are Schwarz functions such that $\tau(0) = \varphi(\omega) = 0$, $|\tau(z)| < 1$ and $|\varphi(\omega)| < 1$ for all $z, \omega \in \mathbb{E}$. On the other hand, these conditions imply

$$|\tau_j| < 1, \tag{16}$$

$$|\varphi_j| < 1 \tag{17}$$

for all $j \in \mathbb{N}$. Basic computations yield that

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1 + 2a_2z + 2a_3z^2 + \cdots \tag{18}$$

$$\frac{2\omega f'(\omega)}{f(\omega)-f(-\omega)} = 1 - 2a_2\omega + +(4a_2^2 - 2a_3)\omega^2 + \cdots \tag{19}$$

$$R(x,\tau(z)) = C_0(x) + [C_1(x)k_1]z + [C_1(x)k_2 + C_2(x)k_1^2]z^2 + [C_1(x)k_3 + 2C_2(x)k_1k_2 + C_3(x)k_1^3]z^3 + \cdots \tag{20}$$

and

$$R(x,\varphi(\omega)) = C_0(x) + [C_1(x)\varphi_1]\omega + [C_1(x)\varphi_2 + C_2(x)\varphi_1^2]\omega^2 + [C_1(x)\varphi_3 + 2C_2(x)\varphi_1\varphi_2 + C_3(x)\varphi_1^3]\omega^3 + \cdots \tag{21}$$

Now, using equation (14) and comparing the coefficients of (18) and (20), we get

$$2a_2 = C_1(x)k_1, \tag{22}$$

$$2a_3 = C_1(x)k_2 + C_2(x)k_1^2. \tag{23}$$

Similarly, using equation (15) and comparing the coefficients of (19) and (21), we have

$$-2a_2 = C_1(x)\varphi_1, \tag{24}$$

$$4a_2^2 - 2a_3 = C_1(x)\varphi_2 + C_2(x)\varphi_1^2. \tag{25}$$

Now, from equations (22) and (24) we get

$$k_1 = -\varphi_1, \tag{26}$$

and

$$\frac{8a_2^2}{[C_1(x)]^2} = k_1^2 + \varphi_1^2 \tag{27}$$

Also, from the summation of the equations (23) and (25), we easily obtain that

$$4a_2^2 = C_1(x)(k_2+\varphi_2) + C_2(x)(k_1^2+\varphi_1^2), \tag{28}$$

By substituting equation (27) in equation (28) we get

$$a_2^2 = \frac{[C_1(x)]^3(k_2+\varphi_2)}{4(C_1(x))^2 - 8C_2(x)}. \tag{29}$$

Taking into account (4) and (5) in (29) we get

$$a_2^2 = \frac{27x^3(k_2+\varphi_2)}{8-108x^2}. \tag{30}$$

Now, using triangle inequality with the inequalities (16) and (17), we have

$$|a_2|^2 \le \frac{27|x|^3}{|4-54x^2|}. \tag{31}$$

Taking square root both sides of the last inequality, we have (12).

In addition, if we subtract the equation (25) from the equation (23) and consider equation (26), then we obtain

$$a_3 = \frac{C_1(x)(k_2-\varphi_2)}{4} + a_2^2. \tag{32}$$

Considering the equation (27) in (32) and a straightforward calculation yield that

$$a_3 = \frac{C_1(x)(k_2-\varphi_2)}{4} + \frac{[C_1(x)]^2(k_1^2+\varphi_1^2)}{8}. \tag{33}$$

By making use of the equation (4), and triangle inequality with the inequalities (16) and (17) in (33) we deduce the inequality (13). So, the proof is completed.

Theorem 2. Suppose that the function $f(z) \in M^{C_\Sigma}(R(x,z))$ and $x \in \mathbb{C}\backslash\left\{\mp\frac{2\sqrt{2}}{\sqrt{117}}\right\}$. Then,

$$|a_2| \leq \frac{3\sqrt{3}|x|\sqrt{|x|}}{\sqrt{|2(8-117x^2)|}} \quad (34)$$

and

$$|a_3| \leq \frac{|x|}{16}(8+9|x|). \quad (35)$$

Proof. Let the function $f(z) \in M^{C_\Sigma}(R(x,z))$ and $g = f^{-1}$ given by (2). In view of Definition 3, from the relations (10) and (11) we can write that

$$\frac{2[zf'(z)]'}{[f(z)-f(-z)]'} = R(x,p(z)) \quad (36)$$

and

$$\frac{2[\omega f'(\omega)]'}{[f(\omega)-f(-\omega)]'} = R(x,d(\omega)). \quad (37)$$

By virtue of the relations (32) and (33), there are two Schwarz functions $p(z) = p_1 z + p_2 z^2 + \cdots$ and $d(\omega) = d_1\omega + d_2\omega^2 + \cdots$ are Schwarz functions such that $p(0) = d(0) = 0$ and $|p(z)| < 1, |d(\omega)| < 1$ for all $z, \omega \in \mathbb{E}$. On the other hand, these conditions imply that

$$|p_j| < 1, \quad (38)$$

$$|d_j| < 1 \quad (39)$$

for all $j \in \mathbb{N}$. A straightforward calculation yields that

$$\frac{2[zf'(z)]'}{[f(z)-f(-z)]'} = 1 + 4a_2 z + 6a_3 z^2 + \cdots \quad (40)$$

and

$$\frac{2[\omega f'(\omega)]'}{[f(\omega)-f(-\omega)]'} = 1 - 4a_2\omega + (12{a_2}^2 - 6a_3)\omega^2 + \cdots. \quad (41)$$

$$R(x,p(z)) = C_0(x) + [C_1(x)p_1]z + [C_1(x)p_2 + C_2(x){p_1}^2]z^2 + [C_1(x)p_3 + 2C_2(x)p_1p_2 + C_3(x){p_1}^3]z^3 + \cdots \quad (42)$$

and

$$R(x,d(\omega)) = C_0(x) + [C_1(x)d_1]\omega + [C_1(x)d_2 + C_2(x){d_1}^2]\omega^2 + [C_1(x)d_3 + 2C_2(x)d_1d_2 + C_3(x){d_1}^3]\omega^3 + \cdots \quad (43)$$

Now, using equation (36) and comparing the coefficients of (40) and (42), we get

$$4a_2 = C_1(x)p_1, \quad (44)$$

$$6a_3 = C_1(x)p_2 + C_2(x){p_1}^2. \quad (45)$$

Similarly, using equation (37) and comparing the coefficients of (41) and (43), we have

$$-4a_2 = C_1(x)d_1, \quad (46)$$

$$12{a_2}^2 - 6a_3 = C_1(x)d_2 + C_2(x){d_1}^2. \quad (47)$$

Now, from equations (44) and (46) we get

$$p_1 = -d_1, \quad (48)$$

and

$$\frac{32{a_2}^2}{[C_1(x)]^2} = {p_1}^2 + {d_1}^2. \quad (49)$$

Also, from the summation of the equations (45) and (47), we easily obtain that

$$12{a_2}^2 = C_1(x)(p_2 + d_2) + C_2(x)({p_1}^2 + {d_1}^2), \quad (50)$$

By substituting equation (49) in equation (50) we get

$${a_2}^2 = \frac{[C_1(x)]^3(p_2+d_2)}{12(C_1(x))^2 - 32C_2(x)}. \quad (51)$$

Plugging equations (4) and (5) into (51), we get that

$${a_2}^2 = \frac{27x^3(p_2+d_2)}{4(8-117x^2)}. \quad (52)$$

Now, using triangle inequality with the inequalities (38) and (39), we have

$$|a_2|^2 \leq \frac{27|x|^3}{|2(8-117x^2)|}. \quad (53)$$

Taking square root both sides of the last inequality, we have (34).

In addition, if we subtract the equation (47) from the equation (45) and consider equation (48), then we obtain

$$a_3 = \frac{C_1(x)(p_2-d_2)}{12} + {a_2}^2. \quad (54)$$

Considering the equation (49) in (54) and a straightforward calculation yield that

$$a_3 = \frac{C_1(x)(p_2-d_2)}{12} + \frac{[C_1(x)]^2({p_1}^2+{d_1}^2)}{32}. \quad (55)$$

By making use of the equation (4), and triangle inequality with the inequalities (38) and (39) in (55), we deduce the inequality (35). So, the proof is completed.

5. Fekete-Szegö inequalities for the class $B^{C_\Sigma}(R(x,z))$ and $M^{C_\Sigma}(R(x,z))$

Our result regarding Fekete-Szegö inequality for the function class $B^{C_\Sigma}(R(x,z))$ is the following.

Theorem 3. Suppose that the function $f(z) \in B^{C_\Sigma}(R(x,z))$, $\mu \in \mathbb{R}$ and $x \in \mathbb{C} \setminus \left\{0, \mp\frac{\sqrt{6}}{9}\right\}$. Then, we have

$$|a_3 - \mu a_2{}^2| \leq \begin{cases} \frac{3}{2}|x|, & if \quad |1-\mu| \leq \frac{|2-27x^2|}{|9x^2|}, \\ \frac{27|x|^3|1-\mu|}{|4-54x^2|}, & if \quad |1-\mu| \geq \frac{|2-27x^2|}{|9x^2|}, \end{cases} \tag{56}$$

Proof. Let the function $f(z) \in B^{C_\Sigma}(R(x,z))$ and $\mu \in \mathbb{R}$. By equations (29) and (32) in Definition 2, we can write that

$$a_3 - \mu a_2{}^2 = \frac{C_1(x)(k_2-\varphi_2)}{4} + a_2{}^2 - \mu a_2{}^2$$

$$=(1-\mu)a_2{}^2 + \frac{C_1(x)(k_2-\varphi_2)}{4}$$

$$=(1-\mu)\frac{[C_1(x)]^3(k_2+\varphi_2)}{4(C_1(x))^2-8C_2(x)} + \frac{C_1(x)(k_2-\varphi_2)}{4}$$

$$= C_1(x)\left\{\left(h_1(\mu)+\frac{1}{4}\right)k_2 + \left(h_1(\mu)-\frac{1}{4}\right)\varphi_2\right\}, \tag{57}$$

where $h_1(\mu) = \frac{(1-\mu)[C_1(x)]^2}{4(C_1(x))^2-8C_2(x)}$. Now, taking modulus and using triangle inequality with (16), (17), (4) and (5) in (57), we complete the proof.

For $\mu = 1$ in Theorem 3, we obtain the following corollary.

Corollary 1. If the function $f(z) \in B^{C_\Sigma}(R(x,z))$. Then,

$$|a_3 - a_2{}^2| \leq \frac{3}{2}|x|. \tag{58}$$

Our next result regarding Fekete-Szegö inequality for the function class $M^{C_\Sigma}(R(x,z))$ is the following.

Theorem 4. Suppose that the function $f(z) \in M^{C_\Sigma}(R(x,z))$, $\mu \in \mathbb{R}$ and $x \in \mathbb{C} \setminus \left\{0, \frac{\sqrt{8}}{\sqrt{117}}\right\}$. Then, we have

$$|a_3 - \mu a_2{}^2| \leq \begin{cases} \frac{|x|}{2}, & if \quad |1-\mu| \leq \frac{|8-117x^2|}{|27x^2|}, \\ \frac{27|x|^3|1-\mu|}{|2(8-117x^2)|}, & if \quad |1-\mu| \geq \frac{|8-117x^2|}{|27x^2|}, \end{cases} \tag{59}$$

Proof. Let the function $f(z) \in M^{C_\Sigma}(R(x,z))$ and $\mu \in \mathbb{R}$. By equations (51) and (54) in definition 3, we can write that

$$a_3 - \mu a_2{}^2 = \frac{C_1(x)(p_2-d_2)}{12} + a_2{}^2 - \mu a_2{}^2$$

$$=(1-\mu)a_2{}^2 + \frac{C_1(x)(p_2-d_2)}{12}$$

$$=(1-\mu)\frac{[C_1(x)]^3(p_2+d_2)}{12(C_1(x))^2-32C_2(x)} + \frac{C_1(x)(p_2-d_2)}{12}$$

$$= C_1(x)\left\{\left(h_2(\mu)+\frac{1}{12}\right)p_2 + \left(h_2(\mu)-\frac{1}{12}\right)d_2\right\}, \tag{60}$$

where $h_2(\mu) = \frac{(1-\mu)[C_1(x)]^2}{12(C_1(x))^2-32C_2(x)}$. Now, taking modulus and using triangle inequality with (38), (39), (4) and (5) in (60), we complete the proof.

İf we take $\mu = 1$ in the Theorem 4, we have the following corollary.

Corollary 2. If the function $f(z) \in M^{C_\Sigma}(R(x,z))$. Then,

$$|a_3 - a_2{}^2| \leq \frac{|x|}{2}. \tag{61}$$

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