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A Generalization of Source of Semiprimeness

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Abstract — This paper characterizes the semigroup ideal $\mathcal{L}_R^n(I)$ of a ring R, where I is an ideal of R, defined by $\mathcal{L}_R^0(I) = I$ and $\mathcal{L}_R^n(I) = \{a \in R \mid aRa \subseteq \mathcal{L}_R^{n-1}(I)\}$, for all $n \in \mathbb{Z}^+$, the set of all the positive integers. Moreover, it studies the basic properties of the set $\mathcal{L}_R^n(I)$ and defines n-prime ideals, n-semiprime ideals, n-prime rings, and n-semiprime rings. This study also investigates relationships between the sets $\mathcal{L}_R(I)$ and $\mathcal{L}_R^n(I)$ and exemplifies some of the related properties. It obtains the main results concerning prime rings and prime ideals by the properties of the set $\mathcal{L}_R^n(I)$.

Keywords *Source of semiprimeness, semiprime rings, semiprime ideals, prime rings, prime ideals*

Mathematics Subject Classification (2020) 16N60, 16U80

1. Introduction

Prime and semiprime ideals are essential classes of rings, especially in noncommutative rings. Therefore, many studies have been conducted on rings' prime ideals and semiprime ideals [1–4]. Additionally, numerous generalizations of these structures have been proposed by the concepts of prime and semiprime ideals [4–10]. Moreover, many studies have been undertaken on prime ideals in Noetherian rings [11–15]. Besides, prime ideals play a significant role in the theory of associative algebras [16,17]. In [18], the concept of the source of the semiprimeness of a ring R expressed by S_R has been explored through semiprime ideals, leading to the definition of new structures: The $|S_R|$-reduced rings, the $|S_R|$-domains, and the $|S_R|$-division rings. Further, several properties of these structures have been investigated. Furthermore, Karalarlıoğlu Camcı [19] has introduced the structures of $|S_R|$-semiprime and $|S_R|$-prime rings using the set S_R and analyzed the relationships between these two types of rings. The author has also researched the necessary and sufficient conditions for a ring R to be isomorphic to the subdirect sum of some of the $|S_R|$-prime rings of R and obtained a generalization related to the relationship between the prime radical $\beta(R)$ of R and S_R. In addition, Karalarlıoğlu Camcı [19] has suggested the set $\mathcal{L}_R(A) = \{a \in R : aRa \subseteq A\}$, where A is a non-empty subset of a ring R, and considered some of its basic properties, presented examples to enhance understanding of the set $\mathcal{L}_R(A)$, and investigated the relations between the sets $\mathcal{L}_R(A)$ and S_R.

This study defines the set $\mathcal{L}_R^n(I)$, a generalization of the set $\mathcal{L}_R(I)$ such that I is an ideal of a ring R, analyzes its properties, and exemplifies some of them. Moreover, this generalization proposes the definitions of n-prime ideals, n-semiprime ideals, n-prime rings, and n-semiprime rings, along with theorems and results derived from these novel notions.

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2. Preliminaries

The current section provides the following basic definitions and some properties in [18–22].

Definition 2.1. Let R be a multiplicative semigroup, $I \neq \emptyset$, and $I \subseteq R$. If $ar, ra \in I$, for all $a \in I$ and for all $r \in R$, then I is called a semigroup ideal of R.

Across this study, if R is a ring, then its multiplicative semigroup concerning the second operation of the ring R is considered for the concepts related to semigroup ideals.

Definition 2.2. Let R be a ring and I be a semigroup ideal of R. If $aRb \subseteq I$ implies $a \in I$ or $b \in I$, then I is called a semigroup prime ideal of R.

Definition 2.3. Let R be a ring and I be an ideal of R. If $aRb \subseteq I$ implies $a \in I$ or $b \in I$, then I is called a prime ideal of R.

Definition 2.4. Let R be a ring and I be a semigroup ideal of R. If $aRa \subseteq I$ implies $a \in I$, then I is called a semigroup semiprime ideal of R.

Definition 2.5. Let R be a ring and I be an ideal of R. If $aRa \subseteq I$ implies $a \in I$, then I is called a semiprime ideal of R.

Definition 2.6. Let R be a ring, $A \neq \emptyset$, and $A \subseteq R$. Then, the set $S_R(A) = \{a \in R : aAa = (0)\}$ is called the source of semiprimeness of A in R. If $A = R$, then S_R will be used instead of $S_R(R)$.

Definition 2.7. Let R be a ring. If, for all $a \in R$, $aRa \subseteq S_R$ implies $a \in S_R$, then R is called an $|S_R|$-semiprime ring, and if, for all $a, b \in R$, $aRb \subseteq S_R$ implies $a \in S_R$ or $b \in S_R$, then R is called an $|S_R|$-prime ring.

Proposition 2.8. Let R be a ring. Then, the following properties hold:

i. If I is a semigroup right (left) ideal of R, then $I \subseteq \mathcal{L}_R(I)$.

ii. If I is a semigroup right (left) ideal of R, then $\mathcal{L}_R(I)$ is a semigroup right (left) ideal of R.

iii. If I is a semigroup right (left) ideal of R, then $S_R \subseteq \mathcal{L}_R(I)$.

iv. If I is an ideal of R and $\pi : R \to R/I$ is a natural epimorphism defined by $\pi(r) = r + I$, then $\pi(\mathcal{L}_R(I)) = S_R/I$ and $\pi^{-1}(S_R/I) = \mathcal{L}_R(I)$.

v. For an ideal I of R, I is a semiprime ideal if and only if $I = \mathcal{L}_R(I)$.

3. Main Results

Let R be a ring and I be an ideal of R. In [19], the set $\mathcal{L}_R(I)$ is defined as follows:

$$\mathcal{L}_R(I) = \{a \in R : aRa \subseteq I\}$$

Motivated by this set, the following is introduced:

$$\mathcal{L}_R^0(I) = I \quad \text{and} \quad \mathcal{L}_R^n(I) = \left\{a \in R : aRa \subseteq \mathcal{L}_R^{n-1}(I)\right\}, \text{ for all } n \in \mathbb{Z}^+$$

where $\mathbb{Z}^+$ is the set of all the positive integers. Moreover, $\mathcal{L}_R^1(I)$ is denoted by $\mathcal{L}_R(I)$. Then,

$$\mathcal{L}_R(0) = \{a \in R : aRa \subseteq (0)\} \quad \text{and} \quad \mathcal{L}_R^n(0) = \left\{a \in R : aRa \subseteq \mathcal{L}_R^{n-1}(0)\right\}$$

Consider the set

$$SG_R = \{I \subseteq R : I \text{ is a semigroup ideal of } R\}$$

From Proposition 2.8, the set $\mathcal{L}_R(I) = \{a \in R : aRa \subseteq I\}$ is a semigroup ideal of R. Therefore, $\mathcal{L}_R(I) \in SG_R$. As a consequence,

$$\mathcal{L}_R : SG_R \to SG_R, \ \mathcal{L}_R(I) = \{a \in R : aRa \subseteq I\}$$

can be constructed. Finally, it is operationalizing as

$$\mathcal{L}_R^n(I) = \mathcal{L}_R\left(\mathcal{L}_R^{n-1}(I)\right) = \left\{a \in R : aRa \subseteq \mathcal{L}_R^{n-1}(I)\right\}$$

for all $n \in \mathbb{Z}^+$. Thus, it is noticeable from the induction that

$$\mathcal{L}_R^m\left(\mathcal{L}_R^n(I)\right) = \mathcal{L}_R^{m+n}(I)$$

for all $n, m \in \mathbb{N}$, the set of all the nonnegative integers.

Definition 3.1. Let I be an ideal of a ring R. Then, I is called an n-prime ideal if $\mathcal{L}_R^n(I)$ is a semigroup prime ideal of R.

Definition 3.2. Let I be an ideal of a ring R. Then, I is called an n-semiprime ideal if $\mathcal{L}_R^n(I)$ is a semigroup semiprime ideal of R.

Definition 3.3. Let R be a ring. Then, R is called an n-prime ring if $\mathcal{L}_R^n(0)$ is a semigroup prime ideal of R.

Definition 3.4. Let R be a ring. Then, R is called an n-semiprime ring if $\mathcal{L}_R^n(0)$ is a semigroup semiprime ideal of R.

Lemma 3.5. Let R be a ring. If P is a prime ideal of R, then P is an n-prime ideal of R.

PROOF. Let R be a ring and P be a prime ideal of R. Since $\mathcal{L}_R(P) = P$, $\mathcal{L}_R^n(P) = P$. Therefore, $\mathcal{L}_R^n(P)$ is a prime ideal of R. Thus, P is an n-prime ideal of R. □

Lemma 3.6. Let R be a ring. If P is a semiprime ideal of R, then P is an n-semiprime ideal of R.

The proof is carried out similarly to the proof of Lemma 3.5.

Example 3.7. Consider the ring $\mathbb{Z}_8 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\}$. Then, $I = \{\overline{0}, \overline{4}\}$ is an ideal of $\mathbb{Z}_8$. Thus, the set

$$\mathcal{L}_{\mathbb{Z}_8}(I) = \{a \in \mathbb{Z}_8 : a\mathbb{Z}_8 a \subseteq I\} = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$$

is a semiprime ideal of $\mathbb{Z}_8$. Thus, I is a 1-semiprime ideal of $\mathbb{Z}_8$ but not a semiprime ideal of $\mathbb{Z}_8$.

Theorem 3.8. Let R be a ring, P be an ideal of R, and A be a semigroup ideal of R such that $P \subseteq A$. Then, A/P is a semigroup ideal of the ring R/P.

PROOF. Since it follows the fact that $A/P \neq \{0 + P\}$, then

$$A/P = \{a + P : a \in A\} \subseteq R/P$$

Therefore,

$$(a + P)(r + P) = ar + P \in A/P$$

and

$$(r + P)(a + P) = ra + P \in A/P$$

for all $a \in A$ and for all $r \in R$. □

Theorem 3.9. Let R be a ring, I be an ideal of R, and $\pi : R \to R/I$ be a natural epimorphism defined by $\pi(r) = r + I$. Then, for all $n \in \mathbb{N}$,

$$\pi^{-1}\left(\mathcal{L}_{R/I}^n(0)\right) = \mathcal{L}_R^n(I)$$

Proof. For $n = 0$,

$$\pi^{-1}\left(\mathcal{L}^0_{R/I}(0)\right) = \pi^{-1}(0) = \operatorname{Ker}\pi = I = \mathcal{L}^0_R(I)$$

Let $x \in \pi^{-1}(\mathcal{L}_{R/I}(0))$, for $n = 1$. Then, $\pi(x) = x + I \in (\mathcal{L}_{R/I}(0))$. It follows that $(x+I)(r+I)(x+I) = (0+I)$, for all $r \in R$. Therefore, $xRx \subseteq I$, for all $r \in R$, because $xrx \in I$, for all $r \in R$. Hence, $x \in \mathcal{L}_R(I)$. Furthermore, if $x \in \mathcal{L}_R(I)$, then $xRx \subseteq I$. Since $xrx \in I$, for all $r \in R$, the equality $(x+I)(r+I)(x+I) = (0+I)$ holds. This requires $\pi(x) = x + I \in (\mathcal{L}_{R/I}(0))$. Thus, $x \in \pi^{-1}(\mathcal{L}_{R/I}(0))$. Hence, $\pi^{-1}(\mathcal{L}_{R/I}(0)) = \mathcal{L}_R(I)$.

Assume that

$$\pi^{-1}\left(\mathcal{L}^n_{R/I}(0)\right) = \mathcal{L}^n_R(I)$$

for an arbitrary $n \in \mathbb{N}$. Let $x \in \pi^{-1}(\mathcal{L}^{n+1}_{R/I}(0))$. Then, $\pi(x) \in (\mathcal{L}^{n+1}_{R/I}(0))$. Namely, $\pi(x)\pi(r)\pi(x) \in (\mathcal{L}^n_{R/I}(0))$. Since π is an epimorphism, $\pi(xrx) \in (\mathcal{L}^n_{R/I}(0))$, for all $r \in R$. Consequently, $xrx \in \pi^{-1}(\mathcal{L}^n_{R/I}(0)) = \mathcal{L}^n_R(I)$, for all $r \in R$. Thus, $xRx \subseteq \mathcal{L}^n_R(I)$ and hence $x \in \mathcal{L}^{n+1}_R(I)$. The converse is similar. Consequently, $\mathcal{L}^{n+1}_R(I) = \pi^{-1}(\mathcal{L}^{n+1}_{R/I}(0))$. □

Lemma 3.10. Let R be a ring, I and P be two ideals of R, and $P \subseteq I$. Then, $\mathcal{L}^n_R(I)/P = \mathcal{L}^n_{R/P}(I/P)$, for all $n \in \mathbb{N}$.

Proof. The proof is straightforward for $n = 0$.

Let $n = 1$. Since I is an ideal of R, $xr, rx \in I$, for all $x \in I$ and for all $r \in R$. Thus, $xrx \in I$ and $x \in \mathcal{L}_R(I)$. Hence, $I \subseteq \mathcal{L}_R(I)$. Moreover, let $x + P \in \mathcal{L}_R(I)/P$. Therefore, $x \in \mathcal{L}_R(I)$. Thereby, $xRx \subseteq I$. In this way, $xRx + P \subseteq I/P$. Herewith, $(x+P)(r+P)(x+P) \in I/P$. Thus, $x + P \in \mathcal{L}_{R/P}(I/P)$. As a result, $\mathcal{L}_R(I)/P \subseteq \mathcal{L}_{R/P}(I/P)$. The converse is similar. Consequently, $\mathcal{L}_R(I)/P = \mathcal{L}_{R/P}(I/P)$.

Suppose that for an arbitrary $n \in \mathbb{N}$,

$$\mathcal{L}^n_R(I)/P = \mathcal{L}^n_{R/P}(I/P)$$

Further, let $y + P \in \mathcal{L}^{n+1}_R(I)/P$. Thus, $y \in \mathcal{L}^{n+1}_R(I)$. Hence, $yRy \subseteq \mathcal{L}^n_R(I)$. Thereby, $yRy + P \subseteq \mathcal{L}^n_R(I)/P = \mathcal{L}^n_{R/P}(I/P)$. Therefore, $(y+P)(r+P)(y+P) \in \mathcal{L}^n_{R/P}(I/P)$, for all $r \in R$. In this way, $y + P \in \mathcal{L}^{n+1}_{R/P}(I/P)$. The converse is similar. In conclusion, $\mathcal{L}^{n+1}_R(I)/P = \mathcal{L}^{n+1}_{R/P}(I/P)$. □

From the aforesaid definitions and theorems, the following significant Theorem is provided.

Theorem 3.11. Let R be a ring and P be an ideal of R. Then, P is an n-prime ideal of R if and only if R/P is an n-prime ring.

Proof. Let R be a ring and P be an ideal of R.

$\Rightarrow$: Assume that P is an n-prime ideal of R. Then, $\mathcal{L}^n_R(P)$ is a semigroup prime ideal of R from Definition 3.1. Thus, R/P is an n-prime ring from Definition 3.3.

$\Leftarrow$: Let R/P is an n-prime ring. Then, $\mathcal{L}^n_{R/P}(\overline{0})$ is a semigroup prime ideal of R/P. From Lemma 3.10, $\mathcal{L}^n_{R/P}(P/P) = \mathcal{L}^n_R(P)/P$. Let $xRy \subseteq \mathcal{L}^n_R(P)$, for all $x, y \in R$. Then, $xry \in \mathcal{L}^n_R(P)$, for all $r \in R$. Hence, since $(xry) + P \in \mathcal{L}^n_R(P)/P$, $(x+P)(R/P)(y+P) \subseteq \mathcal{L}^n_R(P)/P = \mathcal{L}^n_{R/P}(P/P)$. Since $\mathcal{L}^n_{R/P}(P/P)$ is a semigroup prime ideal of R/P, $x + P \in \mathcal{L}^n_{R/P}(P/P) = \mathcal{L}^n_R(P)/P$ or $y + P \in \mathcal{L}^n_{R/P}(P/P) = \mathcal{L}^n_R(P)/P$. Namely, $x \in \mathcal{L}^n_R(P)$ or $y \in \mathcal{L}^n_R(P)$. Thence, $\mathcal{L}^n_R(P)$ is a semigroup prime ideal of R. Consequently, P is an n-prime ideal of R. □

Lemma 3.12. Let R and S be two rings, $\varphi : R \to S$ be an epimorphism, and I be an ideal of R. Then, $\mathcal{L}^n_S(\varphi(I)) = \varphi(\mathcal{L}^n_R(I))$, for all $n \in \mathbb{N}$.

Proof. Let R and S be two rings, $\varphi : R \to S$ be an epimorphism, and I be an ideal of R. Since $\varphi(I) = \varphi(I)$, then $\mathcal{L}_S^0(\varphi(I)) = \varphi(\mathcal{L}_R^0(I))$, for $n = 0$. Moreover, let $y \in \varphi(\mathcal{L}_R(I))$. Then, $y = \varphi(x)$ and $x \in \mathcal{L}_R(I)$. Thus, $ysy = \varphi(x)\varphi(r)\varphi(x) = \varphi(xrx)$, for all $r \in R$ and for all $s \in S$. Hence, $ySy \subseteq \varphi(I)$. Thereby, $y \in \mathcal{L}_S(\varphi(I))$. The other inclusion is similarly proved. Consequently, $\mathcal{L}_S(\varphi(I)) = \varphi(\mathcal{L}_R(I))$.

Assume that $\mathcal{L}_S^n(\varphi(I)) = \varphi(\mathcal{L}_R^n(I))$, for an arbitrary $n \in \mathbb{N}$. If $y \in \varphi(\mathcal{L}_R^{n+1}(I))$, then $y = \varphi(x)$ and $x \in \mathcal{L}_R^{n+1}(I)$. Thus, $ysy = \varphi(x)\varphi(r)\varphi(x) = \varphi(xrx)$, for all $r \in R$ and for all $s \in S$. Hence, $ySy \subseteq \mathcal{L}_S^n(\varphi(I))$. Thereby, $y \in \mathcal{L}_S^{n+1}(\varphi(I))$. Similarly, $\mathcal{L}_S^{n+1}(\varphi(I)) \subseteq \varphi(\mathcal{L}_R^{n+1}(I))$. Consequently, $\mathcal{L}_S^{n+1}(\varphi(I)) = \varphi(\mathcal{L}_R^{n+1}(I))$. □

Theorem 3.13. Let R and S be two rings and $\varphi : R \to S$ be an epimorphism. If $\operatorname{Ker}\varphi \subseteq P$ is an n-prime ideal of R, then $\varphi(P)$ is an n-prime ideal of S.

Proof. Let $\operatorname{Ker}\varphi \subseteq P$ be an n-prime ideal of R. Then, $\varphi(P)$ is an ideal of S. Since P is an n-prime ideal of R, $\mathcal{L}_R^n(P)$ is a semigroup prime ideal of R. Therefore, $\varphi(\mathcal{L}_R^n(P))$ is also a semigroup ideal of S. Let $a, b \in S$. Then, there exist $x, y \in R$ such that $aSb = \varphi(x)\varphi(R)\varphi(y)$. Thus, $\varphi(xRy) \subseteq \varphi(\mathcal{L}_R^n(P))$. Hence, $\varphi(xry) = \varphi(p)$ such that $p \in \mathcal{L}_R^n(P)$. Thereby, $xry - p \in \operatorname{Ker}\varphi \subseteq P$. Herewith, $xry = p + k$ such that $k \in P$ and $p \in \mathcal{L}_R^n(P)$. In this way, $xry \in \mathcal{L}_R^n(P)$. Since $\mathcal{L}_R^n(P)$ is a semigroup prime ideal of R, $x \in \mathcal{L}_R^n(P)$ or $y \in \mathcal{L}_R^n(P)$. Therefore, $a = \varphi(x) \in \varphi(\mathcal{L}_R^n(P))$ or $b = \varphi(y) \in \varphi(\mathcal{L}_R^n(P))$. Consequently, $\varphi(\mathcal{L}_R^n(P))$ is a prime ideal of S. From Lemma 3.12, since $\varphi(\mathcal{L}_R^n(P)) = \mathcal{L}_R^n(\varphi(P))$, $\mathcal{L}_R^n(\varphi(P))$ is a semigroup prime ideal of S. Thus, $\varphi(P)$ is an n-prime ideal of S. □

Theorem 3.14. Let R and S be two rings and $\varphi : R \to S$ be an epimorphism. Then, for an ideal I of S,

$$\varphi^{-1}(\mathcal{L}_S(I)) = \mathcal{L}_R\left(\varphi^{-1}(I)\right)$$

Proof. Let R and S be two rings, $\varphi : R \to S$ be an epimorphism, and I be an ideal of S. For all $x \in \varphi^{-1}(\mathcal{L}_S(I))$,

$$\varphi(x) S \varphi(x) = \varphi(x)\varphi(R)\varphi(x) = \varphi(xRx) \subseteq I$$

and

$$xRx \subseteq \varphi^{-1}(\varphi(xRx)) \subseteq \varphi^{-1}(I)$$

Therefore, $x \in \mathcal{L}_R(\varphi^{-1}(I))$ and $\varphi^{-1}(\mathcal{L}_S(I)) \subseteq \mathcal{L}_R(\varphi^{-1}(I))$. Moreover, $xRx \subseteq \varphi^{-1}(I)$, for all $x \in \mathcal{L}_R(\varphi^{-1}(I))$. Thus,

$$\varphi(xRx) = \varphi(x)\varphi(R)\varphi(x) \subseteq \varphi\left(\varphi^{-1}(I)\right) \subseteq I$$

As a result, $\varphi(x) \in \mathcal{L}_S(I)$ and $x \in \varphi^{-1}(\mathcal{L}_S(I))$. Namely, $\mathcal{L}_R(\varphi^{-1}(I)) \subseteq \varphi^{-1}(\mathcal{L}_S(I))$. □

Theorem 3.15. Let R and S be two rings and $\varphi : R \to S$ be an epimorphism. Then, for an ideal I of S,

$$\varphi^{-1}(\mathcal{L}_S^n(I)) = \mathcal{L}_R^n\left(\varphi^{-1}(I)\right), \quad \text{for all } n \in \mathbb{N}$$

Proof. Using Theorem 3.14 and the induction method, the following result is obtained:

$$\varphi^{-1}(\mathcal{L}_S^n(I)) = \mathcal{L}_R^n\left(\varphi^{-1}(I)\right), \quad \text{for all } n \in \mathbb{N}$$

□

Theorem 3.16. Let R be a ring and I be a semigroup ideal of R. Thus, for all $n \in \mathbb{N}$,

$$\mathcal{L}_R^n(I) \subseteq \mathcal{L}_R^{n+1}(I)$$

Proof. Since $\mathcal{L}_R^n(I)$ is a semigroup ideal of R, $aRa \subseteq \mathcal{L}_R^n(I)$, for all $a \in \mathcal{L}_R^n(I)$. Hence, $a \in \mathcal{L}_R^{n+1}(I)$. □

Corollary 3.17. Let R be a ring and I be a semigroup ideal of R. Then,

$$I \subseteq \mathcal{L}_R(I) \subseteq \mathcal{L}_R^2(I) \subseteq \cdots \subseteq \mathcal{L}_R^n(I) \subseteq \mathcal{L}_R^{n+1}(I) \subseteq \cdots$$

4. Conclusion

This study attempts to generalize the set $\mathcal{L}_R(I)$, expressed by $\mathcal{L}_R^n(I)$ such that I is an ideal of a ring R. In this paper, the basic properties of this set are also provided. Furthermore, adopting this generalization, it explores the definitions of n-prime ideals, n-semiprime ideals, n-prime rings, and n-semiprime rings and their properties. Moreover, the relations of this set under epimorphism are mentioned. Future studies could extend these results to different rings, utilizing the generalization of the set $\mathcal{L}_R(I)$, thereby contributing significantly to ring theory. Furthermore, this generalization paves the way for additional extensions, leading to the introduction of new definitions and the development of novel results. In addition, by utilizing the set $\mathcal{L}_R^n(I)$, researchers can define the n-prime radicals, serving as a generalization of the prime radicals of a ring R.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

References

[1] G. Calugareanu, *A new class of semiprime rings*, Houston Journal of Mathematics 44 (1) (2018) 21–30.

[2] A. Hamed, A. Malek, *S-prime ideals of a commutative ring*, Contributions to Algebra and Geometry 61 (3) (2020) 533–542.

[3] A. Tarizadeh, M. Aghajani, *On purely-prime ideals with applications*, Communications in Algebra 49 (2) (2021) 824–835.

[4] D. D. Anderson, E. Smith, *Weakly prime ideals*, Houston Journal of Mathematics 29 (4) (2003) 831–840.

[5] K. K. Pathak, J. Goswami, *S-Semiprime ideals and weakly S-semiprime ideals of rings*, Palestine Journal of Mathematics 12 (4) (2023) 115–124.

[6] D. D. Anderson, M. Bataineh, *Generalizations of prime ideals*, Communications in Algebra 36 (2) (2008) 686–696.

[7] A. Abouhalaka, *A note on weakly semiprime ideals and their relationship to prime radical in noncommutative rings*, Journal of Mathematics 2024 (2024) Article ID 9142090 6 pages.

[8] A. Badawi, *On weakly semiprime ideals of commutative rings*, Contributions to Algebra and Geometry 57 (3) (2016) 589–597.

[9] C. Beddani, W. Messirdi, *2-prime ideals and their applications*, Journal of Algebra and Its Applications 15 (03) (2016) 1650051 11 pages.

[10] S. Koc, Ü. Tekir, G. Ulucak, *On strongly quasi primary ideals*, Bulletin of the Korean Mathematical Society 56 (3) (2019) 729–743.

[11] D. D. Anderson, T. Dumitrescu, *S-Noetherian rings*, Communications in Algebra 30 (9) (2002) 4407–4416.

[12] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bulletin of the Australian Mathematical Society 75 (3) (2007) 417–429.

[13] Z. Bilgin, M. L. Reyes, Ü. Tekir, *On right S-Noetherian rings and S-Noetherian modules*, Communications in Algebra 46 (2) (2018) 863–869.

[14] S. M. Bhatwadekar, P. K. Sharma, *Unique factorization and birth of almost primes*, Communications in Algebra 33 (1) (2005) 43–49.

[15] H. Ahmed, H. Sana, *S-Noetherian rings of the forms $A[X]$ and $A[[X]]$*, Communications in Algebra 43 (9) (2015) 3848–3856.

[16] K. Ajaykumar, B. S. Kiranagi, R. Rangarajan, *Pullback of Lie algebra and Lie group bundles and their homotopy invariance*, Journal of Algebra and Related Topics 8 (1) (2020) 15–26.

[17] R. Kumar, *On characteristic ideal bundles of a Lie algebra bundle*, Journal of Algebra and Related Topics 9 (2) (2021) 23–28.

[18] N. Aydın, Ç. Demir, D. Karalarlıoğlu Camcı, *The source of semiprimeness of rings*, Communications of the Korean Mathematical Society 33 (4) (2018) 1083–1096.

[19] D. Karalarlıoğlu Camcı, *Source of semiprimeness and multiplicative (generalized) derivations in rings*, Doctoral Dissertation Çanakkale Onsekiz Mart University (2017) Çanakkale.

[20] N. H. McCoy, The theory of rings, Chelsea Publishing Company, New York, 1973.

[21] Y. S. Park, J. P. Kim, *Prime and semiprime ideals in semigroups*, Kyungpook Mathematical Journal 32 (3) (1992) 629–633.

[22] W. M. Boothby, An introduction to differentiable manifolds and Riemannian geometry, 2nd Edition, Academic Press, San Diego, 2002.