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On σ-primeness and σ-semiprimeness in rings with involution

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Abstract — This study introduces the structure of $|\mathcal{S}_R^\sigma|$-σ-semiprime ring, $|\mathcal{S}_R^\sigma|$-σ-prime ring, and source of σ-primeness, which have not been previously explored, and presents new results. We define a subset

$$P_{R_\sigma} = \bigcap_{a\in R} S^a_{R_\sigma}$$

of ring R as $S^a_{R_\sigma} = \{b \in R \mid aR\sigma(b) = aRb = (0)\}$, where σ is an involution is referred to as the source of σ-primeness of R. Additionally, we have established some relationships between the prime radical $\beta(R)$ and $\mathcal{S}_R^\sigma$.

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1. Introduction

In [1], McCoy extensively studied the condition under which a ring R is isomorphic to a subdirect sum of prime rings, specifically when the prime radical $\beta(R)$ of R is trivial, i.e., $\beta(R) = (0)$. Recently, there have been numerous studies focused on prime rings and prime radicals. Since every prime ring is a semiprime, semiprime rings play a crucial role in more general results. Building on research in this field over the past few decades, several authors have explored commutativity theorems for prime and semiprime rings [2–9]. Posner [10] introduced a result stating that if a prime ring has a nontrivial derivation that centralizes the entire ring, then the ring must be commutative. In [11], the same conclusion was established for a prime ring with a nontrivial centralizing automorphism. Other researchers have generalized these results by considering mappings that centralize on specific ideals of the ring.

Inspired by these developments, [12] introduced the concept of the source of semiprimeness for a nonempty subset A of ring R, denoted $S_R(A) = \{a \in R | aAa = (0)\}$. S_R refers to $S_R(R)$. This study builds upon these findings by generalizing some of the results in the literature, utilizing the involution σ and the concept of the source of semiprimeness and primeness in rings. In the sequel, defined and some results given in [12] and [13] are generalized for S_R^σ-σ-prime and S_R^σ-σ-semiprime ring. Lastly, the source of σ-primeness is defined, and some of their results are studied.

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2. Preliminaries

An additive mapping $x \to \sigma(x)$ on a ring R is said to be an involution if $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(\sigma(x)) = x$ hold for all $x, y \in R$ [2]. The term ring with involution or σ-ring refers to a ring equipped with an involution. A ring R with an involution is called σ-prime if $xRy = xR\sigma(y) = 0$ implies that $x = 0$ or $y = 0$ for $x, y \in R$. The ring R is said to be σ-semiprime if $xRx = xR\sigma(x) = 0$ implies that $x = 0$ for $x \in R$. Generally, we know every prime ring with an involution is σ-prime, but the converse need not hold. In [14], Oukhtite and Salhi demonstrate that $\sigma(x, y) = (y, x)$ is involution on ring $R \times R$ and $R \times R$ is σ-prime, but not prime. Their foundational work has become a key to studying σ-prime rings which form an overarching class of prime rings. Let I be an ideal of ring R. If $\sigma(I) \subseteq I$, then I is said to be a σ-ideal of R [15]. In [16], the further example analyzes that an ideal I of R may not be a σ-ideal. Let $R = \mathbb{Z} \times \mathbb{Z}$ and $\sigma : R \to R$ defined by $\sigma(a, b) = (b, a)$ for all $a, b \in R$. For an ideal $I = \mathbb{Z} \times \{0\}$ of R, I is not a σ-ideal of R since $\sigma(I) = \{0\} \times \mathbb{Z}$. Let R be a ring and I be an ideal of R. The prime radical of the ideal I is

$$\sqrt{I} = \{r \in R : \text{ For every } m - system\ M \text{ containing } r,\ M \cap I \neq \emptyset\}$$

The prime radical of the ring R is also defined as $\sqrt{(0)} = \beta(R)$ [1]. A ring R is called a $|S_R|$-prime ring if $aRb \subseteq S_R$ implies that $a \in S_R$ or $b \in S_R$, and R is called a $|S_R|$-semiprime ring if $aRa \subseteq S_R$ implies that $a \in S_R$ [12]. Additionally in [17], the authors defined the set $\mathcal{L}_R(A) = \{a \in R \mid\ aRa \subseteq A\}$ where A is a nonempty subset of ring R and it is observed that $\mathcal{L}_R(0) = S_R$. In [13], the source of primeness of the nonempty subset A in ring R is defined as follows:

$$P_R(A) = \bigcap_{a \in R} S_R^a(A)$$

where

$$S_R^a(A) = \{b \in R \ \mid\ aAb = (0)\}.$$

Theorem 2.1. [1] If S is an ideal in the ring R, then the prime radical of the ring S is $\beta(R) \cap S$.

Theorem 2.2. [1] If $\beta(R)$ is a prime radical of the ring R, then $\beta(R_n) = (\beta(R))_n$.

Theorem 2.3. [1] If $\beta(R)$ is a prime radical of the ring R, then $\beta(R)$ is a semiprime ideal which is contained in every semiprime ideal in R.

Proposition 2.4. [18] Let R be a ring, $\sigma : R \to R$ be an involution, and $T_n(R)$ be a ring of all $n \times n$ diagonal matrices over R. Then,

$$\gamma : T_n(R) \to T_n(R), [\gamma(A)]_{ij} = \begin{cases} \sigma(a_{ij}), & i = j \\ 0, & i \neq j \end{cases}$$

is an involution on $T_n(R)$. Thus,

i. $\mathcal{S}^{\gamma}_{T_n(R)} \subset T_n(\mathcal{S}^{\sigma}_R)$

ii. If $\mathcal{S}^{\sigma}_R$ is a principal ideal of R, then $\mathcal{S}^{\gamma}_{T_n(R)} = T_n(\mathcal{S}^{\sigma}_R)$.

3. Results

The next part of the work will present the results that generalize the consequences about $|\mathcal{S}_R|$-prime ring, $|\mathcal{S}_R|$-semiprime ring and, source of σ-semiprimeness of R. Unless otherwise stated, σ will represent an involution on the ring R. Begin with the following definition:

Definition 3.1. Let σ be an involution on a ring R.

i. R is said to be a $|\mathcal{S}_R^\sigma|$-σ-prime ring provided for $a, b \in R$, $aRb \subseteq \mathcal{S}_R^\sigma$ and $aR\sigma(b) \subseteq \mathcal{S}_R^\sigma$ implies $a \in \mathcal{S}_R^\sigma$ or $b \in \mathcal{S}_R^\sigma$.

ii. R is called a $|\mathcal{S}_R^\sigma|$-σ-semiprime ring if for $a \in R$, $aRa \subseteq \mathcal{S}_R^\sigma$ and $aR\sigma(a) \subseteq \mathcal{S}_R^\sigma$, then $a \in \mathcal{S}_R^\sigma$.

Following is an immediate result of the above definition:

Remark 3.2. If $\mathcal{S}_R^\sigma$ is a σ-prime ideal of ring R (resp. σ-semiprime ideal), then R is a $|\mathcal{S}_R^\sigma|$-σ-prime ring (resp. $|\mathcal{S}_R^\sigma|$-σ-semiprime ring).

Note that using the definition of σ-prime ring and σ-semiprime ring yields the following basic result:

Remark 3.3. Let R be a ring. Then,

i. If R is a σ-prime ring, then $\mathcal{S}_R^\sigma = \{0\}$.

ii. R is a σ-semiprime ring iff $\mathcal{S}_R^\sigma = \{0\}$.

Besides every σ-prime (σ-semiprime) ring is a prime (semiprime) ring. But the opposite is not always true, as seen in the following examples:

Example 3.4. Let $R = M_2(\mathbb{Z}_4)$ and take the involution $\sigma : M_2(\mathbb{Z}_4) \to M_2(\mathbb{Z}_4)$, $\sigma(A) = A^t$. Since $\mathcal{S}_{\mathbb{Z}_4} = \{\overline{0}, \overline{2}\} = (\overline{2})$, $\mathcal{S}_{\mathbb{Z}_4}^2 = \{0\}$. Moreover, $\mathcal{S}_R = \mathcal{S}_{M_2(\mathbb{Z}_4)} = M_2(\mathcal{S}_{\mathbb{Z}_4})$ and $\mathcal{S}_R^\sigma = \mathcal{S}_R$. However, R is not σ-prime or σ-semiprime because of

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus, $R = M_2(\mathbb{Z}_4)$ is a $\left|\mathcal{S}_{M_2(\mathbb{Z}_4)}\right|$-semiprime ring. Furthermore, if $ARA \subseteq \mathcal{S}_R^\sigma$ and $AR\sigma(A) \subseteq \mathcal{S}_R^\sigma$, for $A \in R$, then $A \in \mathcal{S}_R = \mathcal{S}_R^\sigma$. Therefore, R is a $|\mathcal{S}_R^\sigma|$-σ-semiprime ring.

Example 3.5. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \, a, b, c \in \mathbb{Z} \right\}$ and the involution σ is defined as

$$\sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$$

Thus,

$$\mathcal{S}_R^\sigma = \mathcal{S}_R = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \middle| \, b \in \mathbb{Z} \right\}$$

In this case, R is not a σ-prime or a σ-semiprime but R is a $|\mathcal{S}_R^\sigma|$-σ-semiprime ring.

The following present the results that generalize the results about the adapted of the set $L_R(A)$. Let R be a ring, A be a nonempty subset of R, and σ be an involution of R. Let the set $\{a \in R \mid aRa \subseteq A$ and $aR\sigma(a) \subseteq A\ \}$ be denoted as $L_R^\sigma(A)$.

Theorem 3.6. Let A and B be nonempty subsets of R. Then,

i. If $A \subseteq B$, then $L_R^\sigma(A) \subseteq L_R^\sigma(B)$.

ii. $L_R^\sigma(\{0\}) = \mathcal{S}_R^\sigma$.

PROOF. *i.* Suppose that $a \in L_R^\sigma(A)$. Hence, $aRa \subseteq A \subseteq B$ and $aR\sigma(a) \subseteq A \subseteq B$. Thus, $a \in L_R^\sigma(B)$. Consequently, $L_R^\sigma(A) \subseteq L_R^\sigma(B)$.

ii. The definition of $\mathcal{S}_R^\sigma$ yields that $L_R^\sigma(\{0\}) = \{a \in R \mid aRa = aR\sigma(a) = \{0\}\} = \mathcal{S}_R^\sigma$.

□

Remark 3.7. Let A be a nonempty subset of R. Then, $L_R^\sigma(A) \subseteq L_R(A)$.

Note that the unity map is an involution of the commutative ring R. Thence, $L_R^\sigma(A) = L_R(A)$. Consequently, $L_R^\sigma(A)$ is may not be additive [12].

Proposition 3.8. Let I be a semigroup right ideal (semigroup ideal) of multiplicative semigroup R. Then, the following conditions hold:

i. $I \subseteq L_R^\sigma(I)$.

ii. $L_R^\sigma(I)$ is a semigroup right ideal (semigroup ideal) of R.

iii. If I is a left or right ideal, then $\mathcal{S}_R^\sigma \subseteq L_R^\sigma(I)$.

Proof. *i.* Assume that $a \in I$. Since I is a semigroup right ideal, $aRa \subseteq Ia \subseteq I$ and $aR\sigma(a) \subseteq I\sigma(a) \subseteq I$. Therefore, $a \in L_R^\sigma(I)$. As a result, $I \subseteq L_R^\sigma(I)$.

ii. Let $a \in L_R^\sigma(I)$. Hence, $arRar \subseteq (aRa)r \subseteq Ir \subseteq I$ and $arR\sigma(ar) \subseteq aR\sigma(r)\sigma(a) \subseteq aR\sigma(a) \subseteq I$ yields that $ar \in L_R^\sigma(I)$, for $r \in R$. Thus, $L_R^\sigma(I)$ is a semigroup right ideal.

iii. Suppose $a \in \mathcal{S}_R^\sigma$. Then, $aRa = (0) \subseteq I$ and $aR\sigma(a) = (0) \subseteq I$ yields that $a \in L_R^\sigma(I)$. Hence, $\mathcal{S}_R^\sigma \subseteq L_R^\sigma(I)$.

□

Remark 3.9. Let I be a σ-ideal of R. Then, $\gamma : R/I \to R/I$ defined by $\gamma(a+I) = \sigma(a) + I$ is an involution on residue class ring R/I.

Proof. $\gamma((a+I)+(b+I)) = (\sigma(a)+I)+(\sigma(b)+I) = \gamma(a+I)+\gamma(b+I)$, for $a+I,\ b+I \in R/I$. Therefore, γ is an additive mapping. Moreover, from the equations

$$\gamma((a+I)(b+I)) = (\sigma(b)+I)(\sigma(a)+I) = \gamma(b+I)+\gamma(a+I)$$

and

$$\gamma^2(a+I) = \sigma^2(a)+I = a+I$$

γ is an involution on residue class ring R/I. □

Under these circumstances, it is seen that

$$\mathcal{S}_{R/I}^\gamma = \{\,(a+I) \in R/I \mid a \in L_R^\sigma(I)\} = L_R^\sigma(I) + I \tag{3.1}$$

This relation yields the following result:

Proposition 3.10. Let I be a σ-ideal of R and $\pi : R \to R/I,\ \pi(r) = r + I$ be a natural epimorphism. Then,

$$\pi(L_R^\sigma(I)) = \mathcal{S}_{R/I}^\gamma = L_{R/I}^\gamma(0)$$

and

$$\pi^{-1}(\mathcal{S}_{R/I}^\gamma) = \pi^{-1}(L_{R/I}^\gamma(0)) = L_R^\sigma(I)$$

Proof. From (3.1) and Theorem 3.6. *ii*,

$$\pi(L_R^\sigma(I)) = \{\,(a+I) \in R/I \mid aRa \subseteq I \text{ and } aR\sigma(a) \subseteq I\} = \mathcal{S}_{R/I}^\gamma = L_{R/I}^\gamma(0)$$

and

$$\pi^{-1}(\mathcal{S}_{R/I}^\gamma) = \pi^{-1}(L_{R/I}^\gamma(0)) = \{a \in R \mid aRa \subseteq I \text{ and } aR\sigma(a) \subseteq I\} = L_R^\sigma(I)$$

□

Proposition 3.11. If I is a σ-semiprime ideal of R, then $I = L_R^\sigma(I)$.

PROOF. From the Proposition 3.8. *i*, $I \subseteq L_R^\sigma(I)$. Suppose $a \in L_R^\sigma(I)$. Then, $aRa \subseteq I$ and $aR\sigma(a) \subseteq I$. Since I is a σ-semiprime ideal of R, $a \in I$. This means that $L_R^\sigma(I) \subseteq I$. □

Proposition 3.12. R is a $|\mathcal{S}_R^\sigma|$- σ-semiprime ring if and only if $L_R^\sigma(\mathcal{S}_R^\sigma) = \mathcal{S}_R^\sigma$.

PROOF. Let R be a $|\mathcal{S}_R^\sigma|$- σ-semiprime ring and $a \in L_R^\sigma(\mathcal{S}_R^\sigma)$. Then, $aRa \subseteq \mathcal{S}_R^\sigma$ and $aR\sigma(a) \subseteq \mathcal{S}_R^\sigma$. Hence, $a \in \mathcal{S}_R^\sigma$. Consequently, $L_R^\sigma(\mathcal{S}_R^\sigma) \subseteq \mathcal{S}_R^\sigma$. Besides, since $L_R^\sigma(\{0\}) = \mathcal{S}_R^\sigma$ and (0) is an ideal of R, $L_R^\sigma(\{0\})$ is a semigroup right ideal of R. Therefore, $\mathcal{S}_R^\sigma \subseteq L_R^\sigma(\mathcal{S}_R^\sigma)$. As a consequence, $L_R^\sigma(\mathcal{S}_R^\sigma) = \mathcal{S}_R^\sigma$. Conversely, let $aRa \subseteq \mathcal{S}_R^\sigma$ and $aR\sigma(a) \subseteq \mathcal{S}_R^\sigma$. Thus, $a \in L_R^\sigma(\mathcal{S}_R^\sigma) = \mathcal{S}_R^\sigma$. From here, $a \in \mathcal{S}_R^\sigma$. Thence, R is a $|\mathcal{S}_R^\sigma|$- σ-semiprime ring. □

Proposition 3.13. Let I be a σ-ideal of R and $\pi : R \to R/I,\ \pi(r) = r + I$ be a natural epimorphism. If $\mathcal{S}_{R/I}^\gamma$ is a γ-prime ideal of ring R/I, then $\pi^{-1}(\mathcal{S}_{R/I}^\gamma) = L_R^\sigma(I)$ is a σ-prime ideal of R.

Proposition 3.14. The following conditions hold:

i. $\mathcal{S}_R^\sigma$ is a σ-semiprime ideal of R if and only if $\beta(R) = \mathcal{S}_R^\sigma$.

ii. If $\beta(R) = \mathcal{S}_R^\sigma$, then R is a $|\mathcal{S}_R^\sigma|$-σ-semiprime ring.

PROOF. *i.* Since $\mathcal{S}_R^\sigma$ is a semiprime ideal of R, R is a $|\mathcal{S}_R^\sigma|$-σ-semiprime ring. Here, $\mathcal{S}_R^\sigma \subseteq \beta(R)$. Besides, since $\mathcal{S}_R^\sigma$ is a semiprime ideal of R, $\beta(R) \subseteq \mathcal{S}_R^\sigma$. The converse is explicit.

ii. It is seen that from Remark 3.2.

□

In view of Theorem 2.1 and Proposition 3.14. *i*, the following result is obtained:

Proposition 3.15. Let $\mathcal{S}_R^\sigma$ be a semiprime ideal of R. If I is an ideal of R and $\beta(I)$ is a prime radical of I, then $\beta(I) = I \cap \mathcal{S}_R^\sigma$.

Theorem 3.16. Let $\gamma : T_n(R) \to T_n(R)$ be an involution. If $\mathcal{S}_R^\sigma$ is a principal and σ-semiprime ideal of R, then $T_n(R)$ is a $|\mathcal{S}_{T_n(R)}^\gamma|$-$\gamma$-semiprime ring and $|\mathcal{S}_{T_n(R)}^\gamma| = |\mathcal{S}_R^\sigma|^{n^2}$.

PROOF. Since $\mathcal{S}_R^\sigma$ be a σ-semiprime ideal of R, $\beta(R) = \mathcal{S}_R^\sigma$ utilizing Proposition 3.14 .*i*. Moreover, in view of the Theorem 2.2,

$$\beta(T_n(R)) = T_n(\beta(R)) = T_n(\mathcal{S}_R^\sigma) = \mathcal{S}_{T_n(R)}^\gamma$$

From the Theorem 2.3, $\mathcal{S}_{T_n(R)}^\gamma$ is a semiprime ideal of $T_n(R)$ since $\beta(T_n(R))$ is a semiprime ideal. Hence, $T_n(R)$ is a $|\mathcal{S}_{T_n(R)}^\gamma|$-$\gamma$-semiprime ring. Applying Proposition 2.4, $\mathcal{S}_{T_n(R)}^\gamma = T_n(\mathcal{S}_R^\sigma)$. Hence, $|\mathcal{S}_{T_n(R)}^\gamma| = |\mathcal{S}_R^\sigma|^{n^2}$. □

Lemma 3.17. Let $\{I_i\}_{i\in\Lambda}$ be a family of ideals of R. Then, $L_R^\sigma\left(\bigcap\limits_{i\in\Lambda} I_i\right) = \bigcap\limits_{i\in\Lambda} L_R^\sigma(I_i)$.

PROOF. Let $a \in L_R^\sigma\left(\bigcap I_i\right)$. Then, $aRa \subseteq \bigcap I_i$ and $aR\sigma(a) \subseteq \bigcap I_i$. Therefore, $aRa \subseteq I_i$ and $aR\sigma(a) \subseteq I_i$, for all $i \in \Lambda$. Thus, $a \in L_R^\sigma(I_i)$, for all $i \in \Lambda$. Hence, $a \in \bigcap\limits_{i\in\Lambda} L_R^\sigma(I_i)$. Similarly, $\bigcap\limits_{i\in\Lambda} L_R^\sigma(I_i) \subseteq L_R^\sigma\left(\bigcap\limits_{i\in\Lambda} I_i\right)$. □

Lemma 3.18. Let I be a σ-ideal of R. If I is a σ-semiprime ideal of R, then $L_R^{\sigma}(I^2) = I$.

Proof. Let I be a σ-semiprime ideal of R. Since $I^2 \subseteq I$, adopting the Theorem 3.6. *i.* and Proposition 3.11, $L_R^{\sigma}(I^2) \subseteq L_R^{\sigma}(I) = I$. Conversely, assume that $a \in I$. Since I is a σ-ideal of R, $aRa \subseteq I^2$ and $aR\sigma(a) \subseteq I^2$. Thus, $a \in L_R^{\sigma}(I^2)$. □

Definition 3.19. Let R be a ring, $\emptyset \neq A \subseteq R$, and $a \in R$. We define $S_{R_\sigma}^a(A)$ as follows:

$$S_{R_\sigma}^a(A) = \{b \in R \mid aA\sigma(b) = aAb = (0)\}$$

$P_{R_\sigma}(A) = \bigcap_{a \in R} S_{R_\sigma}^a(A)$ is called the source of σ-primeness of the subset A in R. We write $S_{R_\sigma}^a$ instead of $S_{R_\sigma}^a(R)$. In particular, we can similarly define the source of σ-primeness of the ring R as follows:

$$P_{R_\sigma} = \bigcap_{a \in R} S_{R_\sigma}^a$$

Evidently, $S_{R_\sigma}^a \subseteq S_R^a$ for all $a \in R$. Hence, $P_{R_\sigma} \subseteq P_R$. First, let mention some inferences which are easy to see, but these will help understand the set.

i. $aR0 = aR\sigma(0) = (0)$ for all $a \in R$. Hence $P_{R_\sigma} = \bigcap_{a \in R} S_{R_\sigma}^a \neq \emptyset$.

ii. $S_{R_\sigma}^0(A) = R$.

iii. $S_{A_\sigma}^a \subseteq S_{R_\sigma}^a(A)$. If $b \in S_{A_\sigma}^a$, then $b \in A$ such that $aAb = aA\sigma(b) = (0)$. Since $A \subseteq R$, we have $b \in R$ and $aAb = aA\sigma(b) = (0)$. This means that $b \in S_{R_\sigma}^a(A)$.

If $x \in P_{R_\sigma}(A)$, then $aAx = aA\sigma(x) = (0)$, for all $a \in R$. Hence, $RAx = RA\sigma(x) = (0)$. Therefore, $P_{R_\sigma}(A) = \{x \in R : RAx = RA\sigma(x) = (0)\}$.

Theorem 3.20. Let $\emptyset \neq A, B \subseteq R$. Then, $P_{(R\times R)_\sigma}(A \times B) = P_{R_\sigma}(A) \times P_{R_\sigma}(B)$.

Proof. $P_{(R\times R)_\sigma}(A \times B) = \{(x, y) \in R \times R \mid (R \times R)(A \times B)(x, y) = (R \times R)(A \times B)\sigma(x, y) = (0, 0)\}$. Assume that $(x, y) \in P_{(R\times R)_\sigma}(A \times B)$. Then, $(R \times R)(A \times B)(x, y) = (R \times R)(A \times B)\sigma(x, y) = (0, 0)$. Namely, $RAx = RA\sigma(x) = (0)$, $RBy = RB\sigma(y) = (0)$. Hence, $x \in P_{R_\sigma}(A)$, $y \in P_{R_\sigma}(B)$. Thus, $(x, y) \in P_{R_\sigma}(A) \times P_{R_\sigma}(B)$. Similarly, the reverse is also seen. □

Lemma 3.21. Let A and B are nonempty subsets of R. Then, the following conditions hold:

i. If $A \subseteq B$, then $P_{R_\sigma}(B) \subseteq P_{R_\sigma}(A)$. In particular, $P_{R_\sigma} \subseteq P_{R_\sigma}(A)$.

ii. If A is a subring of R, then $A \cap P_{R_\sigma}(A) \subseteq P_{A_\sigma}$.

Proof. *i.* Let $x \in P_{R_\sigma}(B)$. We have $x \in \bigcap_{a \in R} S_{R_\sigma}^a(B)$ and $aBx = aB\sigma(x) = (0)$, for all $a \in R$. Since $A \subseteq B$, we get $aAx = aA\sigma(x) = (0)$ for all $a \in R$. This means that $x \in S_{R_\sigma}^a(A)$ for all $a \in R$. Hence, we get $x \in \bigcap_{a \in R} S_{R_\sigma}^a(A)$ and $x \in P_{R_\sigma}(A)$. This gives up $P_{R_\sigma}(B) \subseteq P_{R_\sigma}(A)$. Specially, $P_{R_\sigma} \subseteq P_{R_\sigma}(A)$ is satisfied for $A \subseteq R$.

ii. Let $x \in A \cap P_{R_\sigma}(A)$. Then $x \in A$ and $x \in P_{R_\sigma}(A)$. Hence, we get $x \in A$ and $x \in \bigcap_{a \in R} S_{R_\sigma}^a(A)$. Using $x \in A$, $x \in S_{A_\sigma}^a$ for all $a \in A$. This expression gives us $x \in \bigcap_{a \in R} S_{A_\sigma}^a = P_{A_\sigma}$. Thus, $A \cap P_{R_\sigma}(A) \subseteq P_{A_\sigma}$.

□

It is well known that every prime ring is a semiprime ring. Consider the relationship between the source of σ-primeness and σ-semiprimeness.

Theorem 3.22. Let $\emptyset \neq A \subseteq R$. Then, $P_{R_\sigma}(A) \subseteq {S_R}^{\sigma}(A)$.

Proof. If $b \in P_{R_\sigma}(A)$, then $b \in \bigcap_{a\in R} S^a_{R_\sigma}(A)$. In particular, $b \in S^b_{R_\sigma}(A)$. Therefore, $bAb = bA\sigma(b) = (0)$. Hence, $b \in {S_R}^\sigma(A)$. □

Proposition 3.23. If I is a right ideal of R, then $S^a_{R_\sigma}(I)$ is a σ-ideal of R for all $a \in R$.

Proof. Let $x, y \in S^a_{R_\sigma}(I)$. Then, $aIx = aIy = aI\sigma(x) = aI\sigma(y) = (0)$ for all $a \in R$. From here $aI(x-y) = aIx-aIy = (0)$ and $aI\sigma(x-y) = aI\sigma(x)-aI\sigma(y) = (0)$. We obtain $x-y \in S^a_{R_\sigma}(I)$. Besides that, we have $aI(xr) = (aIx)r = (0)$, $aI(rx) = a(Ir)x \subseteq aIx = (0)$, $aI\sigma(rx) = aI\sigma(x)\sigma(r) = (0)$, and $aI\sigma(xr) = aI\sigma(r)\sigma(x) = a(I\sigma(r))\sigma(x) \subseteq aI\sigma(x) = (0)$ for any $r \in R$. Thus, we get $xr, rx \in S^a_{R_\sigma}(I)$. Hereby, $S^a_{R_\sigma}(I)$ is an ideal of R. Moreover, if $\sigma(x) \in S^a_{R_\sigma}(I)$, then $aI\sigma(x) = aIx = (0)$. Hence, $x \in S^a_{R_\sigma}(I)$. □

Corollary 3.24. Let I be a right ideal of R. Then, $P_{R_\sigma}(I)$ is a σ-ideal of R.

In the following Lemma, if R is a σ-prime ring, its relation for the set source of σ-primeness is examined.

Lemma 3.25. The following are provided:

i. If R is a σ-prime ring, then $P_{R_\sigma} = \{0\}$.

ii. The source of σ-primeness P_{R_σ} is contained by every σ-prime ideal of the R.

Proof. *i.* Let R be a σ-prime ring and $x \in P_{R_\sigma}$. From definition of the set P_{R_σ}, we have $RRx = RR\sigma(x) = (0)$. Since R is a σ-prime ring, we obtained $x = 0$. Namely, $P_{R_\sigma} = \{0\}$.

ii. Let P be a σ-prime ideal in R. If $x \in P_{R_\sigma}$, then $RRx = RR\sigma(x) = (0) \subseteq P$. Since P is a σ-prime ideal of R, we get $x \in P$. Hence, we get $P_{R_\sigma} \subseteq P$.

□

4. Conclusion

In this paper, we stated the source of $|\mathcal{S}^\sigma_R|$-σ-prime ring, the source of $|\mathcal{S}^\sigma_R|$-σ-semiprime ring, and source of σ-primeness of ring R where σ an involution on R. Below are the conclusions reached:

i. If $\mathcal{S}^\sigma_R$ is a semiprime ideal of ring R (resp. prime ideal), then R is a $|\mathcal{S}^\sigma_R|$-σ-semiprime ring (resp. $|\mathcal{S}^\sigma_R|$-σ-prime ring).

ii. If R is a σ-prime ring, then $\mathcal{S}^\sigma_R = \{0\}$.

iii. R is a σ-semiprime ring iff $\mathcal{S}^\sigma_R = \{0\}$.

iv. Let R be a ring and A and B be nonempty subsets of R. If $A \subseteq B$, then $L^\sigma_R(A) \subseteq L^\sigma_R(B)$ and $L^\sigma_R(\{0\}) = \mathcal{S}^\sigma_R$.

v. Let A be a nonempty subset of R. Then, $L^\sigma_R(A) \subseteq L_R(A)$.

vi. If I is a semiprime ideal of R, then $I = L^\sigma_R(I)$.

vii. Let $\mathcal{S}^\sigma_R$ be a semiprime ideal of R. If I is an ideal of R and $\beta(I)$ is a prime radical of I, then $\beta(I) = I \cap \mathcal{S}^\sigma_R$.

viii. $\mathcal{S}^\sigma_R$ is a semiprime ideal if and only if $\beta(R) = \mathcal{S}^\sigma_R$. Moreover, if $\beta(R) = \mathcal{S}^\sigma_R$, then R is a $|\mathcal{S}^\sigma_R|$-σ-semiprime ring.

ix. Let A and B be nonempty subsets of R. Then the following conditions hold:

a. If $A \subseteq B$, then $P_{R_\sigma}(B) \subseteq P_{R_\sigma}(A)$. In particular, $P_{R_\sigma} \subseteq P_{R_\sigma}(A)$.

b. If A is a subring of R, then $A \cap P_{R_\sigma}(A) \subseteq P_{A_\sigma}$.

x. Let $\emptyset \neq A \subseteq R$. Then, $P_{R_\sigma}(A) \subseteq S_R{}^{\sigma}(A)$.

xi. Let $a \in R$. If I is a right ideal of R, then $S^a_{R_\sigma}(I)$ is a σ-ideal of R.

The generalizations obtained in this paper on prime and semiprime rings equipped with involution will allow more general results to be acquired in future studies on derivations on prime and semiprime rings with involution. Moreover, analyzing the source of σ-primeness, exploring its properties and theorems while examining its connection to the prime radical is interesting yet practical, making it a potential research subject for those interested.

Author Contributions

All the authors equally contributed to this work.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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References

[1] N. H. McCoy, The theory of rings, The Macmillan & Co LTD, New York, 1964.

[2] I. N. Herstein, Rings with involution, University of Chicago Press, Chicago, 1976.

[3] L. Oukhtite, L. Taoufiq, *Some properties of derivations on rings with involution*, International Journal of Modern Mathematical Sciences 4 (3) (2009) 309–315.

[4] M. Ashraf, S. Ali, *On left multipliers and commutativity of prime rings*, Demonstratio Mathematica 41 (4) (2008) 764–771.

[5] H. E. Bell, W. S. Martindale III, *Centralizing mappings of semiprime rings*, Canadian Mathematical Bulletin 30 (1) (1987) 92–101.

[6] L. Molnár, *On centralizers of an H-algebra*, Publicationes Mathematicae Debrecen 46 (1-2) (1995) 89–95.

[7] L. Oukhtite, *On Jordan ideals and derivations in rings with involution*, Commentationes Mathematicae Universitatis Carolinae 51 (3) (2010) 389–395.

[8] L. Oukhtite, *Left multipliers and Lie ideals in rings with involution*, International Journal of Open Problems in Computer Science and Mathematics 3 (3) (2010) 267–277.

[9] L. Oukhtite, *Posner's second theorem for Jordan ideals in ring with involution*, Expositiones Mathematica 29 (4) (2011) 415–419.

[10] E. C. Posner, *Derivations in prime rings*, Proceedings of the American Mathematical Society 8 (6) (1957) 1093–1100.

[11] J. Mayne, *Centralizing automorphisms of prime rings*, Canadian Mathematical Bulletin 19 (1) (1976) 113–115.

[12] N. Aydın, Ç. Demir, D. Karalarlıoğlu Camcı, *The source of semiprimeness of rings*, Communications of the Korean Mathematical Society 33 (4) (2018) 1083–1096.

[13] D. Yeşil, D. Karalarlıoğlu Camcı, *The source of primeness of rings*, Journal of New Theory (41) (2022) 100–104.

[14] L. Oukhtite, S. Salhi, *Centralizing automorphisms and Jordan left derivations on σ-prime rings*, Advances in Algebra 1 (1) (2008) 19–26.

[15] L. Oukhtite, S. Salhi, *On commutativity of σ-prime rings*, Glasnik Matematicki Series III 41 (61) (2006) 57–64.

[16] N. U. Rehman, R. M. Al-Omary, A. Z. Ansari, *On Lie ideals of $*$-prime rings with generalized derivations*, Boletin de la Sociedad Matematica Mexicana 21 (2015) 19–26.

[17] D. Karalarlıoğlu Camcı, *Source of semiprimeness and multiplicative (generalized) derivations in rings*, Doctoral Dissertation Çanakkale Onsekiz Mart University (2017) Çanakkale.

[18] D. Karalarlıoğlu Camcı, D. Yeşil, B. Albayrak, *Source of semiprimeness of $*$-prime rings*, Journal of Amasya University the Institute of Sciences and Technology 5 (1) (2024) 43–48.