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Approximation theorems using the method of $\mathcal{I}_2$-statistical convergence

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Abstract — In this study, we utilize the concept of $\mathcal{I}$-statistical convergence for double sequences to establish a general approximation theorem of Korovkin-type for double sequences of positive linear operators $(PLOs)$ mapping from $H_\omega(X)$ to $C_B(X)$ where $X = [0,\infty) \times [0,\infty)$. We then present an example that demonstrates the applicability of our new main result in cases where classical and statistical approaches are not sufficient. Furthermore, we compute the convergence rate of these double sequences of positive linear operators by employing the modulus of smoothness.

Keywords: *Double sequence, statistical convergence, $\mathcal{I}_2$-statistical convergence, Korovkin theorem, Bleimann-Butzer-Hahn operators*

Subject Classification (2020): 41A25, 41A36

1. Introduction

The investigation of the approximation properties exhibited by positive linear operators has emerged as a critical area of research that continues to capture the interest of scholars working in functional analysis and approximation theory. These operators play a pivotal role in extending classical results related to approximation theory and have proven to be precious tools across a wide range of mathematical disciplines. This includes computational mathematics, stochastic analysis, and studying abstract function spaces. One of the most notable contributions in this area is the development of Korovkin-type approximation theorems, which provide a robust and elegant framework for establishing criteria for convergence [1–13]. These theorems have become essential instruments in studying the behavior of sequences and families of operators, offering deep insights into the structural features of different function spaces and their topological properties.

The field of approximation theory has experienced considerable growth, particularly with the introduction of more sophisticated convergence concepts. Among these advancements is the rise of statistical convergence and, more recently, the generalization of this notion known as $\mathcal{I}$-statistical convergence. These newer frameworks provide a more comprehensive and flexible foundation for analyzing convergence phenomena, particularly in scenarios where traditional convergence methods

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may not be applicable. The strength of $\mathcal{I}$-statistical convergence lies in its ability to handle situations where classical forms of convergence fail to offer useful results. Given this enhanced flexibility in analysis, we explore the approximation properties of double sequences generated by positive linear operators through the lens of $\mathcal{I}$-statistical convergence. Specifically, we establish a novel Korovkin-type approximation theorem tailored to operators that map functions from the weighted space $H_\omega(X)$ to the space $C_B(X)$ of bounded continuous functions. Here, X denotes the unbounded domain $[0,\infty)\times[0,\infty)$, which provides the context for our study.

This study extends Korovkin-type approximation theorems to the domain of double sequences via $\mathcal{I}$-statistical convergence, establishing convergence properties beyond the scope of classical and statistical approaches. We present an analysis of convergence rates for these operators through the modulus of smoothness approximation technique. To demonstrate the practical significance of our theoretical results, we construct an illustrative example that highlights the effectiveness of the $\mathcal{I}$-statistical framework, particularly in cases where traditional methods are insufficient.

2. Preliminaries

This section provides some basic notions to be needed in the following sections. Revisit the concepts of convergence in the sense of Pringsheim, statistical convergence, and $\mathcal{I}$-statistical convergence for double sequences.

Throughout this paper, let $u=\{u_{mn}\}$ be a double sequence with real terms.

Definition 2.1. [14] A sequence $u=\{u_{mn}\}$ is called to be convergent in Pringsheim's sense if, for every $\epsilon>0$, there exists $M=M(\epsilon)\in\mathbb{N}$ such that for all $m,n>M$, the inequality $|u_{mn}-\ell|<\epsilon$ holds, where ℓ is referred as the Pringsheim limit of the sequence, denoted by $P-\lim u_{mn}=\ell$.

We shall refer to such a u as P-convergent for brevity. A double sequence is termed bounded if there is a positive constant N such that $|u_{mn}|\leq N$ for every $(m,n)\in\mathbb{N}^2=\mathbb{N}\times\mathbb{N}$. It is important to highlight that, on the contrary, single sequences, a convergent double sequence is not necessarily bounded.

Definition 2.2. [15] If $G\subseteq\mathbb{N}^2$, then G_{jk} denotes the set $\{(m,n)\in G: m\leq j, n\leq k\}$. The double natural density of G is defined as

$$\delta_2(G):=P-\lim_{j,k}\frac{1}{jk}|G_{jk}|$$

if it exists and assume that the symbol $|.|$ indicates the cardinality of the set. The sequence $u=\{u_{mn}\}$ is statistically convergent to ℓ on the condition that for all $\epsilon>0$, the set $G:=G_\epsilon:=\{m\leq j, n\leq k: \ |u_{mn}-\ell|\geq\epsilon\}$ has natural density zero; i.e.,

$$P-\lim_{j,k}\frac{1}{jk}|\{m\leq j, n\leq k: \ |u_{mn}-\ell|\geq\epsilon\}|=0$$

in this case we indicate with $st_2-\lim u_{mn}=\ell$.

Kostyrko et al. have defined $\mathcal{I}$-convergence using the ideal $\mathcal{I}$ [16]. This type of convergence can be seen as a general form of statistical convergence.

Definition 2.3. [16] Let a class $\mathcal{I}$ of subsets of U, a non-empty set, is called an *ideal* in U iff (i) $\emptyset\in\mathcal{I}, (ii)$ $E,F\in\mathcal{I}$ implies $E\cup F\in\mathcal{I}$ (additive) and (iii) for each $E\in\mathcal{I}$ and $F\subseteq E$ we have $F\in\mathcal{I}$ (hereditary).

If $\{u\}\in\mathcal{I}$ for each $u\in U$ then an ideal called *admissible.* If $\mathcal{I}$ is a *non-trivial ideal* in U (i.e. $U\notin\mathcal{I}$, $\mathcal{I}\neq\{\emptyset\}$) then the family of sets $\mathcal{F}=\{X\subseteq U:(\exists E\in\mathcal{I})(X=U\setminus E)\}$ is a *filter* in U and we call such a filter, the filter linking with the ideal $\mathcal{I}$. A non-trivial ideal $\mathcal{I}_2$ of $\mathbb{N}^2$ is called s*trongly admissible* if $\{j\}\times\mathbb{N}$ and $\mathbb{N}\times\{j\}$ belong to $\mathcal{I}_2$ for every $j\in\mathbb{N}$. It seems obvious that a strongly admissible ideal is also admissible. Let

$$\mathcal{I}_2^0=\left\{F\subseteq\mathbb{N}^2:(\exists m(F)\in\mathbb{N})(j,k\geq m(F)\Rightarrow(j,k)\notin F)\right\}$$

then $\mathcal{I}_2^0$ is a non-trivial strongly admissible ideal [17] and clearly $\mathcal{I}_2$ is strongly admissible iff $\mathcal{I}_2^0\subseteq\mathcal{I}_2$.

Remark 2.4. Note that if $\mathcal{I}_2$ is the ideal $\mathcal{I}_2^0$ then $\mathcal{I}_2$-convergence coincides with Pringsheim convergence and if we take $\mathcal{I}_2^\delta:=\left\{G\subseteq\mathbb{N}^2:\delta_2(G)=0\right\}$ then $\mathcal{I}_2^\delta$-convergence becomes statistical convergence.

Definition 2.5. [18] A sequence $u=\{u_m\}$ is called $\mathcal{I}-$statistically convergent to $L\in X$, if for every $\epsilon>0$, and every $\eta>0$,

$$\left\{j\in\mathbb{N}:\frac{1}{j}\left|\{m\leq j:|u_m-L|\geq\epsilon\}\right|\geq\eta\right\}\in\mathcal{I}$$

Definition 2.6. [2] A sequence u is called $\mathcal{I}_2$-statistically convergent to θ if for all $\epsilon>0$ and $\eta>0$,

$$\left\{(j,k)\in\mathbb{N}^2:\frac{1}{jk}\ \left|\{m\leq j,n\leq k:\ \ |u_{mn}-\theta|\geq\epsilon\}\right|\geq\eta\right\}\in\mathcal{I}_2$$

symbolically, $\mathcal{I}_{stat}^2-\lim u=\theta$.

For the remainder of the paper, we will denote $\mathcal{I}_2$ as a non-trivial strongly admissible ideal on $\mathbb{N}^2$.

3. New Approximation Theorem

In this part, we provide an approximation theorem of Korovkin-type for double sequences of PLOs acting on two variables, mapping from $H_\omega(X)$ to $C_B(X)$ over the domain $X=[0,\infty)\times[0,\infty)$ via $\mathcal{I}_2$-statistical convergence. Furthermore, we provide an illustrative example demonstrating that our new main result holds in cases where its classical and statistical counterparts are inapplicable.

We show by $C_B(X)$ the space of all bounded and continuous real-valued functions on X. This space is equipped with the supremum norm

$$\|f\|_X=\sup_{(x,y)\in X}|f(x,y)|\,,\ \ (f\in C_B(X))$$

Consider the space $H_\omega(X)$ consisting of all real valued functions g on X and providing

$$|g(u,t)-g(x,y)|\leq\omega\left(\left|\frac{u}{1+u}-\frac{x}{1+x}\right|,\left|\frac{t}{1+t}-\frac{y}{1+y}\right|\right)$$

In this context, ω represents a function of the modulus of continuity type, as it fulfil the conditions, for $\delta,\delta_1,\delta_2>0$, as follows:

i. ω is nonnegative, increasing function on X with regard to δ_1,δ_2

ii. $\omega(\delta,\delta_1+\delta_2)\leq\omega(\delta,\delta_1)+\omega(\delta,\delta_2)$

iii. $\omega(\delta_1+\delta_2,\delta)\leq\omega(\delta_1,\delta)+\omega(\delta_2,\delta)$

iv. $\lim\limits_{\delta_1,\delta_2\to0}\omega(\delta_1,\delta_2)=0$

Then, it is clear from (iv) that all functions belonging to $H_\omega(X)$ are continuous on X. Moreover, all functions $g\in H_\omega(X)$ fulfil the inequality

$$|g(u,t)|\leq|g(0,0)|+\omega(1,1)\,,\ \ x,y\geq0$$

and hence the function g is bounded on X. As a result, $H_\omega(X)\subseteq C_B(X)$.

Furthermore, we use the test functions below

$$g_0(u,t)=1,\ g_1(u,t)=\frac{u}{1+u},\ g_2(u,t)=\frac{t}{1+t} \text{ and } g_3(u,t)=\left(\frac{u}{1+u}\right)^2+\left(\frac{t}{1+t}\right)^2$$

The main result of the relevant section is given in the following theorem:

Theorem 3.1. Let $\{S_{mn}\}$ be a double sequence of PLOs moving from $H_\omega(X)$ into $C_B(X)$. Suppose that the following conditions are valid:

$$\mathcal{I}^2_{stat}-\lim \left\|S_{mn}(g_l)-g_l\right\|_X=0, l\in\{0,1,2,3\} \tag{3.1}$$

Then, for any $g\in H_\omega(X)$,

$$\mathcal{I}^2_{stat}-\lim \left\|S_{mn}(g)-g\right\|_X=0 \tag{3.2}$$

Proof. Suppose that (3.1) holds. Let $g\in H_\omega(X)$ and $(x,y)\in X$ be fixed. Since $g\in H_\omega(X)$, for all $(u,t)\in X$, we write

$$|g(u,t)-g(x,y)|\le\epsilon+\frac{2K}{\delta^2}\left\{\left(\frac{u}{1+u}-\frac{x}{1+x}\right)^2+\left(\frac{t}{1+t}-\frac{y}{1+y}\right)^2\right\}$$

where $K:=\|g\|_X$. After some easy calculations, since S_{mn}, is PLOs, we obtain

$$\begin{aligned}|S_{mn}(g;x,y)-g(x,y)| \le\ & \epsilon+\tau\{|S_{mn}(g_0;x,y)-g_0(x,y)| \\ &+|S_{mn}(g_1;x,y)-g_1(x,y)|+|S_{mn}(g_2;x,y)-g_2(x,y)| \\ &+|S_{mn}(g_3;x,y)-g_3(x,y)|\}\end{aligned}$$

where $\tau:=\max\left\{\epsilon+K+\frac{2K}{\delta^2},\frac{4K}{\delta^2},\frac{2K}{\delta^2}\right\}$. Now, taking supremum over $(x,y)\in X$ we have

$$\|S_{mn}(g)-g\|_X\le\epsilon+\tau\sum_{l=0}^{3}\|S_{mn}(g_l)-g_l\|_X \tag{3.3}$$

For a given $\beta>0$, choose $\epsilon>0$ such that $\epsilon<\beta$. Then, setting

$$U:=\{m\le j,n\le k:\|S_{mn}(g)-g\|_X\ge\beta\}$$

$$U_l:=\left\{m\le j,n\le k:\|S_{mn}(g_l)-g_l\|_X\ge\frac{\beta-\epsilon}{4\tau}\right\},l\in\{0,1,2,3\}$$

From (3.3), we obtain

$$U\subseteq\bigcup_{l=0}^{3}U_l$$

which gives

$$\frac{|U|}{jk}\le\sum_{l=0}^{3}\frac{|U_l|}{jk}$$

For every $\delta>0$, we have

$$\left\{(m,n):\frac{\left|\left\{m\le j,n\le k:\|S_{mn}(g)-g\|_X\ge\beta\right\}\right|}{jk}\ge\delta\right\}\subseteq\bigcup_{l=0}^{3}\left\{(m,n):\frac{\left|\left\{m\le j,n\le k:\|S_{mn}(g_l)-g_l\|_X\ge\frac{\beta-\epsilon}{4\tau}\right\}\right|}{jk}\ge\frac{\delta}{3}\right\}$$

Since the ideal $\mathcal{I}_2$ possesses the properties of additivity and heredity, the proof of the theorem is thereby completed. □

By making appropriate choices, as indicated in Remark 2.4, we derive the following statistical of Theorem 3.1.

Corollary 3.2. Let $\{S_{mn}\}$ be a double sequence of PLOs moving from $H_\omega(X)$ into $C_B(X)$. Suppose that the following conditions apply:

$$st_2 - \lim \|S_{mn}(g_l) - g_l\|_X = 0, \ l \in \{0,1,2,3\}$$

Then, for any $g \in H_\omega(X)$,

$$st_2 - \lim \|S_{mn}(g) - g\|_X = 0$$

4. Application of Approximation Theorem

In the present section, we construct a sequence of PLOs that demonstrates the power of Theorem 3.1. Specifically, this sequence satisfies the hypotheses of Theorem 3.1 while failing to satisfy the conditions required by Corollary 3.2.

Example 4.1. Let us take the following Bleimann-Butzer-Hahn operators of two variables [10]:

$$L_{mn}(g;x,y) = \frac{1}{(1+x)^m (1+y)^n} \sum_{k=0}^{m} \sum_{s=0}^{n} f\left(\frac{k}{m-k+1}, \frac{s}{n-s+1}\right) \binom{m}{k} \binom{n}{s} x^k y^s \tag{4.1}$$

where $g \in H_\omega(X)$, and $X = [0,\infty) \times [0,\infty)$. It is known that

$$L_{mn}(g_0;x,y) = 1$$

$$L_{mn}(g_1;x,y) = \frac{m}{m+1}\frac{x}{1+x}$$

$$L_{mn}(g_2;x,y) = \frac{n}{n+1}\frac{y}{1+y}$$

$$L_{mn}(g_3;x,y) = \frac{m(m-1)}{(m+1)^2}\left(\frac{x}{1+x}\right)^2 + \frac{m}{(m+1)^2}\frac{x}{1+x} + \frac{n(n-1)}{(n+1)^2}\left(\frac{y}{1+y}\right)^2 + \frac{n}{(n+1)^2}\frac{y}{1+y}$$

Besides, let $E \in \mathcal{I}_2$ be infinite set and define $u = \{u_{mn}\}$ by

$$u_{mn} = \begin{cases} \frac{1}{mn}, & j - \sqrt{j} + 1 \le m \le j, \ k - \sqrt{k} + 1 \le n \le k, \ (j,k) \notin E \\ mn, & 1 \le m \le j, \ 1 \le n \le k, \ (j,k) \in E \\ 0, & \text{otherwise} \end{cases} \tag{4.2}$$

For every $\epsilon > 0$, since $v_{jk} := \frac{1}{jk} |\{(m,n) : |u_{mn} - 0| \ge \epsilon\}| \le \frac{\sqrt{j}\sqrt{k}}{jk}$ tends to zero in Pringsheim's sense and $(j,k) \notin E$, so for every $\delta > 0$, we get

$$\{(i,j) : v_{jk} \ge \delta\} \subseteq E \cup \left\{\left(\mathbb{N}^2 \backslash E\right) \cap ((\{1,2,...,j_1\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,...,j_1\}))\right\} \in \mathcal{I}_2$$

for some $j_1 \in \mathbb{N}$. Hence, we obtain $\mathcal{I}^2_{stat} - \lim u = 0$. Note that, $\{u_{mn}\}$ is neither statistically convergent nor classically convergent to zero. Using (4.1) and (4.2), we now define the following PLOs on $H_\omega(X)$:

$$S_{mn}(g;x,y) = (1 + u_{mn}) L_{mn}(g;x,y) \tag{4.3}$$

Since $\mathcal{I}^2_{stat} - \lim u = 0$, we can show that the sequence $\{S_{mn}\}$, as defined by (4.3) satisfies all conditions of Theorem 3.1. Consequently, for all $g \in H_\omega(X)$, we have

$$\mathcal{I}^2_{stat} - \lim \|S_{mn}(g) - g\|_X = 0$$

Since u is neither classically convergent nor statistically convergent to zero, the sequence $\{S_{mn}(g)\}$ cannot convergence to g on X in the usual or statistical sense.

5. Rate of $\mathcal{I}_2$-statistical Convergence

The rate of convergence for double sequences of PLOs, in terms of $\mathcal{I}_2$-statistical convergence, is described by the modulus of smoothness. Define the following modulus of smoothness for the bivariate case, similarly to the one in [19], (see [20]):

$$\overline{\omega}_2(g;\delta_1,\delta_2)=\sup\left\{|g(u,t)-g(x,y)|:(u,t),(x,y)\in X \text{ and } \left|\frac{u}{1+u}-\frac{x}{1+x}\right|\leq\delta_1,\left|\frac{t}{1+t}-\frac{y}{1+y}\right|\leq\delta_2\right\} \quad (5.1)$$

where $\delta_1,\delta_2>0$. It is clear that if $g\in H_\omega(K)$ then, we have

i. $\lim\limits_{\delta_1,\delta_2\to 0}\bar{\omega}_2(g;\delta_1,\delta_2)=0$

ii. $|g(u,t)-g(x,y)|\leq\bar{\omega}_2(g;\delta_1,\delta_2)\left(1+\frac{\left|\frac{u}{1+u}-\frac{x}{1+x}\right|}{\delta_1}\right)\left(1+\frac{\left|\frac{t}{1+t}-\frac{y}{1+y}\right|}{\delta_2}\right)$

Theorem 5.1. Let $\{S_{mn}\}$ be a double sequence of PLOs moving from $H_\omega(X)$ into $C_B(X)$. Suppose that the following conditions are valid:

i. $\mathcal{I}^2_{stat}-\lim\|S_{mn}(g_0)-g_0\|_X=0$

ii. $\mathcal{I}^2_{stat}-\lim\bar{\omega}_2(g;\gamma_{mn},\eta_{mn})=0$

where $\gamma_{mn}:=\sqrt{\left\|S_{mn}\left(\left(\frac{u}{1+u}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X}$ and $\eta_{mn}:=\sqrt{\left\|S_{mn}\left(\left(\frac{v}{1+v}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X}$

Then, for any $g\in H_\omega(X)$,

$$\mathcal{I}^2_{stat}-\lim\|S_{mn}(g)-g\|_X=0$$

Proof. Let $g\in H_\omega(X)$ and $(x,y)\in X$ be fixed. Since S_{mn} is linear and positive and also, thanks to (5.1),

$$|S_{mn}(g;x,y)-g(x,y)|\leq\tau|S_{mn}(g_0;x,y)-g_0(x,y)|+S_{mn}(|g(u,t)-g(x,y)|;x)$$

where $\tau:=\|g\|_X$. We get, with the help of the Cauchy-Schwartz inequality,

$$\begin{aligned}
|S_{mn}(g;x,y)-g(x,y)| &\leq \tau|S_{mn}(g_0;x,y)-g_0(x,y)|\\
&\quad+S_{mn}\left(\bar{\omega}_2(g;\delta_1,\delta_2)\left(1+\frac{\left|\frac{u}{1+u}-\frac{x}{1+x}\right|}{\delta_1}\right)\left(1+\frac{\left|\frac{t}{1+t}-\frac{y}{1+y}\right|}{\delta_2}\right);x,y\right)\\
&\leq \tau|S_{mn}(g_0;x,y)-g_0(x,y)|\\
&\quad+\bar{\omega}_2(g;\delta_1,\delta_2)\left\{S_{mn}(g_0;x,y)+\frac{1}{\delta_1}S_{mn}\left(\left|\frac{u}{1+u}-\frac{x}{1+x}\right|;x,y\right)\right.\\
&\quad+\frac{1}{\delta_2}S_{mn}\left(\left|\frac{t}{1+t}-\frac{y}{1+y}\right|;x,y\right)\\
&\quad\left.+\frac{1}{\delta_1\delta_2}S_{mn}\left(\left|\frac{u}{1+u}-\frac{x}{1+x}\right|\left|\frac{t}{1+t}-\frac{y}{1+y}\right|;x,y\right)\right\}\\
&\leq \bar{\omega}_2(g;\delta_1,\delta_2)\left\{S_{mn}(g_0;x,y)+\frac{1}{\delta_1}\sqrt{S_{mn}\left(\left(\frac{u}{1+u}-\frac{x}{1+x}\right)^2;x,y\right)}\sqrt{S_{mn}(g_0;x,y)}\right\}\\
&\quad+\frac{1}{\delta_2}\sqrt{S_{mn}\left(\left(\frac{t}{1+t}-\frac{y}{1+y}\right)^2;x,y\right)}\sqrt{S_{mn}(g_0;x,y)}\\
&\quad+\frac{1}{\delta_1\delta_2}\sqrt{S_{mn}\left(\left(\frac{u}{1+u}-\frac{x}{1+x}\right)^2;x,y\right)}\sqrt{S_{mn}\left(\left(\frac{t}{1+t}-\frac{y}{1+y}\right)^2;x,y\right)}\\
&\quad+\tau|S_{mn}(g_0;x,y)-g_0(x,y)|
\end{aligned}$$

Taking supremum over $(x,y)\in X$, we obtain

$$\begin{aligned}
\left\| S_{mn}(g)-g\right\|_X \leq\; & \bar{\omega}_2(f;\delta_1,\delta_2)\left\| S_{m,n}(g_0)-g_0\right\|_X \\
& +\bar{\omega}_2(f;\delta_1,\delta_2)\frac{1}{\delta_1}\sqrt{\left\| S_{mn}\left(\left(\frac{u}{1+u}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X}\sqrt{\left\| S_{mn}(g_0)\right\|_X} \\
& +\bar{\omega}_2(f;\delta_1,\delta_2)\frac{1}{\delta_2}\sqrt{\left\| S_{mn}\left(\left(\frac{v}{1+v}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X}\sqrt{\left\| S_{mn}(g_0)\right\|_X} \\
& +\bar{\omega}_2(f;\delta_1,\delta_2)\frac{1}{\delta_1\delta_2}\sqrt{\left\| S_{mn}\left(\left(\frac{u}{1+u}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X}\sqrt{\left\| S_{mn}\left(\left(\frac{v}{1+v}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X} \\
& +\tau\left\| S_{mn}(g_0)-g_0\right\|_X+\bar{\omega}_2(f;\delta_1,\delta_2)
\end{aligned}$$

Let $\delta_1 := \gamma_{mn} := \sqrt{\left\| S_{mn}\left(\left(\frac{u}{1+u}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X}$ and $\delta_2 := \eta_{mn} := \sqrt{\left\| S_{mn}\left(\left(\frac{v}{1+v}-\frac{\cdot}{1+\cdot}\right)^2\right)\right\|_X}$. Thus,

$$\begin{aligned}
\left\| S_{mn}(g)-g\right\|_X \leq\; & \bar{\omega}_2(f;\delta_1,\delta_2)\left\| S_{mn}(g_0)-g_0\right\|_X \\
& +2\bar{\omega}_2(f;\delta_1,\delta_2)\sqrt{\left\| S_{mn}(g_0)\right\|_X} \\
& +\tau\left\| S_{mn}(g_0)-g_0\right\|_X+2\bar{\omega}_2(f;\delta_1,\delta_2)
\end{aligned}$$

As a result, by the conditions of the theorem, for any $g\in H_\omega(X)$,

$$\mathcal{I}^2_{stat}-\lim\left\| S_{mn}(g)-g\right\|_X=0$$

□

6. Conclusion

This paper introduces a novel perspective on Korovkin-type approximation in $H_\omega(X)$. We develop a new approximation theorem by combining $\mathcal{I}$-statistical convergence with test functions such as $g_0(u,v)=1$, $g_1(u,v)=\frac{u}{1+u}$, $g_2(u,v)=\frac{v}{1+v}$ and $g_3(u,v)=\left(\frac{u}{1+u}\right)^2+\left(\frac{v}{1+v}\right)^2$. This approach significantly improves over conventional methods, particularly in its enhanced ability to handle approximations in unbounded domains. The theoretical advancements are further demonstrated through a concrete example, highlighting the practical relevance of the proposed framework. Later, we focus on the speed of convergence, utilizing the modulus of smoothness to achieve this. Our findings suggest that this method opens new avenues for approximation theory, potentially expanding the reach of Korovkin-type theorems and facilitating their application to more complex operators and diverse settings. Future research could build on these results by exploring the method's applicability to a broader range of function spaces and examining its impact in other areas of analysis.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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