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Ruled Surfaces of Adjoint Curve with the Modified Orthogonal Frame

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Abstract — This paper analyzes several specific ruled surfaces generated by the base curve α and its director curve or the α's adjoint curve β and its director curve, where the director curves are frame vectors of the modified orthogonal frame in E^3. Furthermore, this paper studies the flat or minimal properties of the surfaces, as well as their asymptotic and geodesic curves. Afterward, it exemplifies the theoretical results herein. Finally, this paper discusses the need for further research.

Keywords *Modified orthogonal frame, special ruled surface, adjoint curve*

Mathematics Subject Classification (2020) 53A04, 53A05

1. Introduction

The study of curve theory has long been a central topic in differential geometry research [1–4]. One intriguing aspect of this field is exploring specific curve types, such as adjoint curves, defined as the integral of the binormal vector of a curve $\alpha(s)$, parameterized by s, as mentioned in [5]. Adjoint curves have found applications in various fields, including number theory, coding theory, and algebraic geometry [6–9].

A regular curve is characterized by its curvature κ and torsion τ, which uniquely determine the curve at every point, as stated by the fundamental theorem of regular curves. However, the curvature function may vanish at certain points for analytical curves, introducing discontinuities in the principal normal and binormal vectors. This discontinuity makes the curvature function non-differentiable at those points, leading to ambiguities in the Frenet frame due to the vanishing curvature.

To address these challenges, Hord [10] and Sasai [11] introduced an alternative orthogonal frame to handle such points effectively. Sasai [12] further developed a modified orthogonal frame (MOF) for unit-speed analytic curves, offering a simple and practical solution. In this approach, the Frenet vectors are scaled by the curvature function κ, resulting in a new formulation that extends the Frenet derivative equations. This MOF has facilitated research on various frames and ruled surfaces in different spaces [13–19].

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In this paper, we focus on specific ruled surfaces whose base curves are adjoint curves of a considered curve α. The director curves of these surfaces are defined by the tangent, normal, and binormal vectors associated with the MOF in E^3, following the approach in [7]. We also provide several theorems and proofs related to these surfaces. Finally, we exemplify some of these special ruled surfaces to visualize.

2. Preliminaries

This section presents some basic notions to be needed in the following section. Throughout this paper, let $\psi(s,v)$ be a surface in Euclidean 3-space. The unit normal vector field $U(s,v)$ of the surface $\psi(s,v)$ is obtained by

$$U = \frac{\psi_s \times \psi_v}{\|\psi_s \times \psi_v\|}$$

where $\psi_s = \frac{\partial \psi}{\partial s}$ and $\psi_v = \frac{\partial \psi}{\partial v}$ are the partial derivatives of the surface $\psi(s,v)$ with respect to the parameter s and v, respectively. The first fundamental form I of the surface $\psi(s,v)$ is as follows:

$$I = g_{11}ds^2 + 2g_{12}dsdv + g_{22}dv^2$$

where $g_{11} = \langle \psi_s, \psi_s \rangle$, $g_{12} = \langle \psi_s, \psi_v \rangle$, and $g_{22} = \langle \psi_v, \psi_v \rangle$. Moreover, the second fundamental form of the surface $\psi(s,v)$ is defined as follows:

$$II = h_{11}ds^2 + 2h_{12}dsdv + h_{22}dv^2$$

where $h_{11} = \langle \psi_{ss}, U \rangle$, $h_{12} = \langle \psi_{sv}, U \rangle$, and $h_{22} = \langle \psi_{vv}, U \rangle$. The Gaussian curvature K and the mean curvature H of the surface $\psi(s,v)$ are as follows:

$$K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2} \quad \text{and} \quad H = \frac{h_{11}g_{22} - 2g_{12}h_{12} + g_{11}h_{22}}{2\left(g_{11}g_{22} - g_{12}^2\right)} \tag{2.1}$$

Theorem 2.1. [20] On a surface, asymptotic curves are defined as curves along which the normal curvature is zero. This is equivalent to the condition that the second fundamental form vanishes:

$$II = h_{11}ds^2 + 2h_{12}dsdv + h_{22}dv^2 = 0$$

where h_{11}, h_{12}, and h_{22} are the coefficients of the second fundamental form.

Theorem 2.2. [20] For a curve to be geodesic, its geodesic curvature

$$k_g = \nabla_{\dot{\gamma}}\dot{\gamma} = 0$$

where $\nabla_{\dot{\gamma}}\dot{\gamma}$ is the covariant derivative of the tangent vector $\dot{\gamma}$ along itself.

Definition 2.3. [5] Let α be a unit speed curve in E^3 with $\tau \neq 0$. Then, the adjoint curve of α is defined by

$$\beta(s) = \int_{s_0}^{s} B_\alpha(s)ds$$

where B_α is the binormal vector of the curve α.

We express the relations between the MOF $\{T, N, B\}$ and the classical Frenet frame $\{t, n, b\}$ by

$$T = t,\ N = \kappa n,\ \text{and}\ B = \kappa b \tag{2.2}$$

where $\kappa \neq 0$ is the curvature of the curve. The MOF $\{T, N, B\}$ satisfies the following equalities:

$$< T, T >= 1, \quad < N, N >=< B, B >= \kappa^2, \quad \text{and} \quad < T, N >=< T, B >=< N, B >= 0$$

Here, the notation $<,>$ represents the inner product. Using the definitions of T, N, and B and (2.2), the following equalities hold:

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & \dfrac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \dfrac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} \tag{2.3}$$

and

$$\tau = \tau(s) = \frac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2}$$

is the torsion of α. Moreover, κ^2 and τ are analytic. In [1], the differentiation formula for the MOF is denoted by (2.3).

Theorem 2.4. [7] If α is a unit speed curve and β is the adjoint curve of α such that $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\{T_\beta, N_\beta, B_\beta\}$ are the Frenet vectors and $\{\kappa_\alpha, \tau_\alpha\}$ and $\{\kappa_\beta, \tau_\beta\}$ are curvature and torsion of α and β, respectively, then

$$T_\beta = B_\alpha, \quad N_\beta = -N_\alpha, \quad B_\beta = T_\alpha, \quad \kappa_\beta = \tau_\alpha, \quad \text{and} \quad \tau_\beta = \kappa_\alpha$$

Theorem 2.5. [7] If α is a unit speed regular in E^3 and β is the adjoint curve of α according to the MOF with curvature such that $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\{T_\beta, N_\beta, B_\beta\}$ are the MOF and $\{\kappa_\alpha, \tau_\alpha\}$ and $\{\kappa_\beta, \tau_\beta\}$ are the curvature and torsion of α and β, respectively, then

$$T_\beta = \left(\frac{1}{\kappa_\alpha}\right) B_\alpha, \quad N_\beta = -\left(\frac{\tau_\alpha}{\kappa_\alpha^2}\right) N_\alpha, \quad B_\beta = \left(\frac{\tau_\alpha}{\kappa_\alpha}\right) T_\alpha, \quad \kappa_\beta = \frac{\tau_\alpha}{\kappa_\alpha}, \quad \text{and} \quad \tau_\beta = 1$$

3. Some Special Ruled Surfaces According to the MOF

This section investigates ruled surfaces according to the MOF in E^3. It generates new special ruled surfaces to change the base curve with α and its adjoint curve β. Additionally, this section changes the director vectors of these surfaces concerning the MOF vectors T, N, and B.

3.1. Tangent Ruled Surface with the Base Curve α

Concerning the MOF, the parameterization of the tangent ruled surface is as follows:

$$\psi_1(s, v) = \alpha(s) + vT_\alpha(s) \tag{3.1}$$

where α is the base curve. If we take the derivatives with respect to the parameter s and v of the tangent ruled surface $\psi_1(s, v)$, respectively, then

$$\begin{cases} \psi_{1_s} = T_\alpha + vN_\alpha, \quad \psi_{1_v} = T_\alpha, \\ \psi_{1_{ss}} = \kappa_\alpha^2 T_\alpha + \left(1 + v\dfrac{\kappa_\alpha'}{\kappa_\alpha}\right) N_\alpha + v\tau_\alpha B_\alpha, \quad \psi_{1_{sv}} = N_\alpha, \quad \text{and} \quad \psi_{1_{vv}} = 0 \end{cases} \tag{3.2}$$

Therefore, $g_{11} = 1 + v^2\kappa_\alpha^2$, $g_{12} = 1$, and $g_{22} = 1$ and thus $g_{11}g_{22} - g_{12}^2 = v^2\kappa_\alpha^2 \neq 0$ where g_{11}, g_{12}, and g_{22} are the coefficients of the first fundamental form of the tangent ruled surface $\psi_1(s, v)$. The unit normal vector field U_1 of the tangent ruled surface $\psi_1(s, v)$ is provided by

$$U_1 = -\frac{1}{\kappa_\alpha} B_\alpha \tag{3.3}$$

Moreover, the coefficients of the second fundamental form as follows: $h_{11} = -v\tau_\alpha\kappa_\alpha$, $h_{12} = 0$, and $h_{22} = 0$. Using (2.1), the Gaussian and mean curvatures of the tangent ruled surface $\psi_1(s, v)$ are obtained as

$$K = 0 \quad \text{and} \quad H = \frac{-\tau_\alpha}{2v\kappa_\alpha} \tag{3.4}$$

Theorem 3.1. Let $\psi_1(s, v)$ be a tangent ruled surface with the MOF in E^3. Then, the tangent ruled surface $\psi_1(s, v)$ is developable.

The proof can be readily observed from (3.4).

Theorem 3.2. Let $\psi_1(s, v)$ be a tangent ruled surface with the MOF in E^3. Then, $\psi_1(s, v)$ cannot be minimal.

The proof can be readily observed from (3.4) by $\tau_\alpha \neq 0$.

Theorem 3.3. Let $\psi_1(s, v)$ be a tangent ruled surface according to the MOF in E^3. Then, the following hold:

i. s-parameter curves of the tangent ruled surface $\psi_1(s, v)$ cannot be asymptotic.

ii. v-parameter curves of the tangent ruled surface $\psi_1(s, v)$ are asymptotic.

PROOF. By the definition of asymptotic curves, $\langle \psi_{1_{ss}}, U \rangle = 0$ and $\langle \psi_{1_{vv}}, U \rangle = 0$.

i. The proof is obvious since $h_{11} \neq 0$.

ii. Since $h_{22} = 0$, v-parameter curves of the $\psi_1(s, v)$ are asymptotic.

□

Theorem 3.4. Let $\psi_1(s, v)$ be a tangent ruled surface with the MOF in E^3. Then,

i. s-parameter curves of $\psi_1(s, v)$ cannot be geodesic.

ii. v-parameter curves of $\psi_1(s, v)$ are geodesic.

PROOF. From the definition of geodesic curves, it must be $\psi_{1_{ss}} \times U = 0$ and $\psi_{1_{vv}} \times U_1 = 0$ for the s and v parameter curves.

i. According to (3.2) and (3.3),

$$\psi_{1_{ss}} \times U = -\kappa_\alpha \left(1 + v\frac{\kappa'_\alpha}{\kappa_\prime}\right) T_\alpha - v\kappa_\alpha N_\alpha$$

Since T_α and N_α are linearly independent, $\psi_{1_{ss}} \times U = 0$ if and only if $\kappa_\alpha = 0$. However, as $\kappa_\alpha \neq 0$, $\psi_1(s, v)$ cannot be a geodesic curve.

ii. From (3.2) and (3.3), $\psi_{1_{vv}} \times U_1 = 0$. Thus, v-parameter curves are geodesic curves.

□

3.2. Tangent Ruled Surface with the Base Curve β

Concerning the MOF, the parameterization of the tangent ruled surface with the adjoint curve β is as follows:

$$\psi_2(s, v) = \beta(s) + vT_\beta(s) \tag{3.5}$$

If we take the derivatives with respect to parameter s and v of the tangent ruled surface $\psi_2(s, v)$, respectively, then

$$\begin{cases} \psi_{2_s} = -v\dfrac{\tau_\alpha}{\kappa_\alpha}N_\alpha + B_\alpha, \quad \psi_{2_v} = \dfrac{1}{\kappa_\alpha}B_\alpha, \quad \psi_{2_{ss}} = v\tau_\alpha\kappa_\alpha T_\alpha + \left(-\tau_\alpha - v\dfrac{\tau'_\alpha}{\kappa_\alpha}\right)N_\alpha + \left(\dfrac{\kappa'_\alpha}{\kappa_\alpha} - v\dfrac{\tau^2_\alpha}{\kappa^2_\alpha}\right)B_\alpha, \\ \psi_{2_{ss}} = v\tau_\alpha\kappa_\alpha T_\alpha + \left(-\tau_\alpha - v\dfrac{\tau'_\alpha}{\kappa_\alpha}\right)N_\alpha + \left(\dfrac{\kappa'_\alpha}{\kappa_\alpha} - v\dfrac{\tau^2_\alpha}{\kappa^2_\alpha}\right)B_\alpha, \quad \psi_{2_{sv}} = -\dfrac{\tau_\alpha}{\kappa_\alpha}N_\alpha, \quad \text{and} \quad \psi_{2_{vv}} = 0 \end{cases} \tag{3.6}$$

Hence,

$$g_{11} = \kappa_\alpha^2 + v^2\tau_\alpha^2, \quad g_{12} = \kappa_\alpha, \quad \text{and} \quad g_{22} = 1, \quad \text{and thus} \quad g_{11}g_{22} - g_{12}^2 = v^2\kappa_\alpha^2 \neq 0 \tag{3.7}$$

The unit normal vector field U_2 of the tangent ruled surface $\psi_2(s,v)$ is provided by $U_2 = -T_\alpha$. The coefficients of the second fundamental form are as follows: $h_{11} = -v\tau_\alpha\kappa_\alpha$, $h_{12} = 0$, and $h_{22} = 0$. Using (2.1), the Gaussian and mean curvatures of the tangent ruled surface $\psi_2(s,v)$ are obtained as follows: $K = 0$ and $H = \dfrac{-\kappa_\alpha}{2v\tau_\alpha}$.

Theorem 3.5. Let $\psi_2(s,v)$ be a tangent ruled surface with the MOF in Euclidean 3-space. Then, $\psi_2(s,v)$ is a flat surface.

Theorem 3.6. Let $\psi_2(s,v)$ be a tangent ruled surface with the MOF in Euclidean 3-space. Then, $\psi_2(s,v)$ cannot be minimal.

PROOF. Since $\kappa_\alpha \neq 0$, the tangent ruled surface $\psi_2(s,v)$ cannot be minimal. □

Theorem 3.7. Let $\psi_2(s,v)$ be a tangent ruled surface with the MOF in E^3. Then,

i. s-parameter curves of the tangent ruled surface $\psi_2(s,v)$ cannot be asymtotic.

ii. v-parameter curves of the tangent ruled surface $\psi_2(s,v)$ are asymptotic curves.

PROOF. From the definition of asymptotic curves, $\langle \psi_{2_{ss}}, U\rangle = 0$ and $\langle \psi_{2_{vv}}, U\rangle = 0$.

i. The proof is obvious since $h_{11} \neq 0$

ii. Since $h_{22} = 0$, v-parameter curves of the $\psi_2(s,v)$ are asymptotic.

□

Theorem 3.8. Let $\psi_2(s,v)$ be a tangent ruled surface with the MOF in E^3. Then,

i. s-parameter curves of $\psi_2(s,v)$ cannot be geodesic.

ii. v-parameter curves of $\psi_2(s,v)$ are geodesic.

PROOF. From the definition of geodesic curves $\psi_{2_{ss}} \times U_2 = 0$ and $\psi_{2_{vv}} \times U_2 = 0$ must be provided for the s and v parameter curves.

i. According to (3.6) and (3.7),

$$\psi_{2_{ss}} \times U_2 = -\left(\frac{\kappa_\alpha'}{\kappa_\alpha} - v\frac{\tau_\alpha^2}{\kappa_\alpha^2}\right)N_\alpha + \left(-v\frac{\tau_\alpha'}{\kappa_\alpha} - \tau_\alpha\right)B_\alpha$$

Since N_α and B_α are linearly independent, $\psi_{2_{ss}} \times U_2 = 0$ if and only if κ_α is a constant and $\tau_\alpha = 0$. However, because $\tau_\alpha \neq 0$, $\psi_2(s,v)$ cannot be a geodesic curve.

ii. From (3.6) and (3.7), $\psi_{2_{vv}} \times U_2 = 0$. Hence, v-parameter curves are geodesic curves.

□

3.3. Normal Ruled Surface with the Base Curve α

Concerning the MOF, the parameterization of the normal ruled surface is as follows:

$$\psi_3(s,v) = \alpha(s) + vN_\alpha(s) \tag{3.8}$$

where α is the base curve. If we take the derivatives with respect to parameter s and v of the normal ruled surface $\psi_3(s,v)$, respectively, then

$$\begin{cases} \psi_{3s} = \left(1 - v\kappa_\alpha^2\right) T_\alpha + v\dfrac{\kappa_\alpha'}{\kappa_\alpha} N_\alpha + v\tau_\alpha B_\alpha, \quad \psi_{3v} = N_\alpha, \\ \psi_{3ss} = \left(-3v\kappa_\alpha'\kappa_\alpha\right) T_\alpha + \left(1 - v\left(\kappa_\alpha^2 + \tau_\alpha^2\right) + v\dfrac{\kappa_\alpha''}{\kappa_\alpha}\right) N_\alpha + v\left(\tau_\alpha' + 2\tau_\alpha\dfrac{\kappa_\alpha'}{\kappa_\alpha}\right) B_\alpha, \\ \psi_{3sv} = -\kappa_\alpha^2 T_\alpha + \dfrac{\kappa_\alpha'}{\kappa_\alpha} N_\alpha + \tau_\alpha B_\alpha, \quad \text{and} \quad \psi_{3vv} = 0 \end{cases} \tag{3.9}$$

Thereby, $g_{11} = 1 - 2v\kappa_\alpha^2 + v^2\kappa_\alpha^2\left(\kappa_\alpha^2 + \tau_\alpha^2\right) + v^2{\kappa_\alpha'}^2$, $g_{12} = v\kappa_\alpha'\kappa_\alpha$, and $g_{22} = \kappa_\alpha^2$, and thus $g_{11}g_{22} - g_{12}^2 = \kappa_\alpha^2\left(\left(1 - v\kappa_\alpha^2\right)^2 + \left(v\kappa_\alpha\tau_\alpha\right)^2\right) \neq 0$. The unit normal vector field U_3 of the normal ruled surface $\psi_3(s,v)$ is provided by

$$U_3 = \frac{1}{\sqrt{\left(1 - v\kappa_\alpha^2\right)^2 + \left(v\kappa_\alpha\tau_\alpha\right)^2}}\left(-v\kappa_\alpha\tau_\alpha T_\alpha + \frac{\left(1 - v\kappa_\alpha^2\right)}{\kappa_\alpha} B_\alpha\right) \tag{3.10}$$

and the coefficients of the second fundamental form are obtained as:

$$h_{11} = \frac{v}{\sqrt{\left(1 - v\kappa_\alpha^2\right)^2 + \left(v\kappa_\alpha\tau_\alpha\right)^2}}\left(v\kappa_\alpha^2\left(\tau_\alpha\kappa_\alpha' - \kappa_\alpha\tau_\alpha'\right) + \left(2\kappa_\alpha'\tau_\alpha + \tau_\alpha'\kappa_\alpha\right)\right)$$

$$h_{12} = \frac{\tau_\alpha\kappa_\alpha}{\sqrt{\left(1 - v\kappa_\alpha^2\right)^2 + \left(v\kappa_\alpha\tau_\alpha\right)^2}}$$

and $h_{22} = 0$. From (2.1), the Gaussian and mean curvatures are as follows, respectively:

$$K = -\frac{\tau_\alpha^2}{\left(\left(1 - v\kappa_\alpha^2\right)^2 + \left(v\kappa_\alpha\tau_\alpha\right)^2\right)^2} \quad \text{and} \quad H = \frac{v\kappa_\alpha\left(\kappa_\alpha v\tau_\alpha\kappa_\alpha' - v\kappa_\alpha^2\tau_\alpha' + \tau_\alpha'\right)}{2\left(\left(1 - v\kappa_\alpha^2\right)^2 + \left(v\kappa_\alpha\tau_\alpha\right)^2\right)^{\frac{3}{2}}} \tag{3.11}$$

Theorem 3.9. Let $\psi_3(s,v)$ be a ruled surface in Euclidean 3-space. Then, $\psi_3(s,v)$ is not a flat surface.

PROOF. Since $\tau_\alpha \neq 0$, $\psi_3(s,v)$ cannot be flat. □

Theorem 3.10. Let $\psi_3(s,v)$ be a normal ruled surface with the MOF in Euclidean 3-space. If the curve α is a cylindirical helix, then $\psi_3(s,v)$ is minimal.

The proof is directly obtained from (3.11).

Theorem 3.11. Let $\psi_3(s,v)$ be a normal ruled surface with the MOF in E^3. Then,

i. s-parameter curves of $\psi_3(s,v)$ are asymptotic curves if and only if the curvatures κ_α and τ_α of the curve α are constant.

ii. v-parameter curves of the $\psi_3(s,v)$ are asymptotic curves.

PROOF. From the definition of asymptotic curves, $\langle \psi_{3ss}, U_3 \rangle = 0$ and $\langle \psi_{3vv}, U_3 \rangle = 0$.

i. From (3.9) and (3.10),

$$h_{11} = \frac{v}{\sqrt{\left(1 - v\kappa_\alpha^2\right)^2 + \left(v\kappa_\alpha\tau_\alpha\right)^2}}\left(v\kappa_\alpha^2\left(\tau_\alpha\kappa_\alpha' - \kappa_\alpha\tau_\alpha'\right) + \left(2\kappa_\alpha'\tau_\alpha + \tau_\alpha'\kappa_\alpha\right)\right) = 0$$

Thus,

$$v\kappa_\alpha^2\left(\tau_\alpha\kappa_\alpha' - \kappa_\alpha\tau_\alpha'\right) + \left(2\kappa_\alpha'\tau_\alpha + \tau_\alpha'\kappa_\alpha\right) = 0$$

since the curvatures κ_α and τ_α of the curve α are constant.

ii. Since $h_{22} = 0$, v-parameter curves of the $\psi_3(s,v)$ are asymptotic.

□

Theorem 3.12. Let $\psi_3(s,v)$ be a ruled surface with the MOF in E^3. Then, *i.* s-parameter curves of $\psi_3(s,v)$ cannot be geodesic.

ii. v-parameter curves of $\psi_3(s,v)$ are geodesic.

PROOF. From the definition of geodesic curves, $\psi_{3_{ss}} \times U_3 = 0$ and $\psi_{3_{vv}} \times U_3 = 0$ for the s and v parameter.

i. According to (3.9) and (3.10),

$$\begin{aligned}\psi_{3_{ss}} \times U_3 = & \left(\kappa_\alpha - v\kappa_\alpha^3 + \left(1 - v\kappa_\alpha^2\right)\left(v\kappa_\alpha'' - v\kappa_\alpha\left(\kappa_\alpha^2 + \tau_\alpha^2\right)\right)\right)T_\alpha \\ & + \left(3\kappa_\alpha' v - 3\kappa_\alpha^2\kappa_\alpha' v^2 - 2v^2\kappa_\alpha'\tau_\alpha^2 - v^2\tau_\alpha\tau_\alpha'\kappa_\alpha\right)N_\alpha \\ & + \left(v\tau_\alpha\kappa_\alpha - v^2\tau_\alpha\kappa_\alpha\left(\kappa_\alpha^2 + \tau_\alpha^2\right) + v^2\kappa_\alpha''\tau_\alpha\right)B_\alpha\end{aligned}$$

Since T_α, N_α, and B_α are linearly independent, $\psi_{3_{ss}} \times U = 0$ if and only if κ_α is a constant and $\tau_\alpha = 0$. However, as $\tau_\alpha \neq 0$, $\psi_3(s,v)$ cannot be a geodesic curve.

ii. From (3.9) and (3.10), $\psi_{3_{vv}} \times U = 0$. Therefore, v-parameter curves are geodesic.

□

3.4. Normal Surfaces with the Adjoint Curve β

Concerning the MOF, the parameterization of the normal ruled surface is as follows:

$$\psi_4(s,v) = \beta(s) + vN_\beta(s) \tag{3.12}$$

where β is the base curve. If we take the derivatives with respect to the parameter s and v of the normal ruled surface $\psi_4(s,v)$, then

$$\begin{cases} \psi_{4_s} = \tau_\alpha T_\alpha + v\left(\dfrac{\kappa_\alpha'\tau_\alpha}{\kappa_\alpha^3} - \dfrac{\tau_\alpha'}{\kappa_\alpha^2}\right)N_\alpha + \left(1 - v\dfrac{\tau_\alpha^2}{\kappa_\alpha^2}\right)B_\alpha \\ \psi_{4_v} = -\dfrac{\tau_\alpha}{\kappa_\alpha^2}N_\alpha \\ \psi_{4_{ss}} = v\left(2\tau_\alpha' - v\tau_\alpha\dfrac{\kappa_\alpha'}{\kappa_\alpha}\right)T_\alpha + \left(\dfrac{\kappa_\alpha'}{\kappa_\alpha} + v\left(4\dfrac{\kappa_\alpha'\tau_\alpha^2}{\kappa_\alpha^3} - 5\dfrac{\tau_\alpha\tau_\alpha'}{\kappa_\alpha^2}\right)\right)B_\alpha \\ \quad + \left(v\left(\tau_\alpha + \dfrac{\kappa_\alpha''\tau_\alpha}{\kappa_\alpha^3} - \dfrac{\tau_\alpha''}{\kappa_\alpha^2} + \dfrac{\kappa_\alpha'\tau_\alpha}{\kappa_\alpha^3} - 2\dfrac{{\kappa_\alpha'}^2\tau_\alpha}{\kappa_\alpha^4} + \dfrac{\kappa_\alpha'\tau_\alpha'}{\kappa_\alpha^3} + \dfrac{\tau_\alpha^3}{\kappa_\alpha^2}\right) - \tau_\alpha\right)N_\alpha \\ \psi_{4_{sv}} = \tau_\alpha T_\alpha + \left(\dfrac{\kappa_\alpha'\tau_\alpha}{\kappa_\alpha^3} - \dfrac{\tau_\alpha'}{\kappa_\alpha^2}\right)N_\alpha - \dfrac{\tau_\alpha^2}{\kappa_\alpha^2}B_\alpha \\ \psi_{4_{vv}} = 0 \end{cases} \tag{3.13}$$

Hence, $g_{11} = v^2\left(\tau_\alpha^2 + \dfrac{{\kappa_\alpha'}^2\tau_\alpha^2}{\kappa_\alpha^4} - 2\dfrac{\kappa_\alpha'\tau_\alpha\tau_\alpha'}{\kappa_\alpha^3} + \dfrac{{\tau_\alpha'}^2}{\kappa_\alpha^2} + \dfrac{\tau_\alpha^4}{\kappa_\alpha^2}\right) + \kappa_\alpha^2 - 2v\tau_\alpha^2$, $g_{12} = \dfrac{v\tau_\alpha}{\kappa_\alpha^2}\left(\tau_\alpha' - \dfrac{\kappa_\alpha'\tau_\alpha}{\kappa_\alpha}\right)$, and $g_{22} = \dfrac{\tau_\alpha^2}{\kappa_\alpha^2}$ and thus $g_{11}g_{22} - g_{12}^2 = \dfrac{\tau_\alpha^2}{\kappa_\alpha^4}\left(\left(\kappa_\alpha^2 - v\tau_\alpha^2\right)^2 + \left(v\tau_\alpha\kappa_\alpha\right)^2\right) \neq 0$. Moreover, the unit normal vector field U_4 is provided by

$$U_4 = \frac{1}{\sqrt{\left(\kappa_\alpha^2 - v\tau_\alpha^2\right)^2 + \left(v\tau_\alpha\kappa_\alpha\right)^2}}\left(\left(\kappa_\alpha^2 - v\tau_\alpha^2\right)T_\alpha - v\tau_\alpha B_\alpha\right) \tag{3.14}$$

and the coefficients of the second fundamental form are obtained as:

$$h_{11} = \frac{1}{\sqrt{\left(\kappa_\alpha^2 - v\tau_\alpha^2\right)^2 + \left(v\tau_\alpha\kappa_\alpha\right)^2}}\left(5\frac{v^2\kappa_\alpha'\tau_\alpha^3}{\kappa_\alpha} - 7v^2\tau^2\tau_\alpha' + \kappa_\alpha\kappa_\alpha'\tau_\alpha v\left(1 - v\right) + 2v\kappa_\alpha^2\tau_\alpha'\right)$$

$$h_{12} = \frac{\tau_\alpha \kappa_\alpha^2}{\sqrt{(\kappa_\alpha^2 - v\tau_\alpha^2)^2 + (v\tau_\alpha\kappa_\alpha)^2}}$$

and $h_{22} = 0$. By (2.1), the Gaussian and mean curvatures are provided as follows:

$$K = -\frac{\kappa_\alpha^8}{\left((\kappa_\alpha^2 - v\tau_\alpha^2)^2 + (v\tau_\alpha\kappa_\alpha)^2\right)^2}$$

and

$$H = \frac{\kappa_\alpha}{2\tau_\alpha^2} \frac{\left(v^2\tau_\alpha^2\left(5\kappa_\alpha'\tau_\alpha^3 - 7\kappa_\alpha\tau_\alpha'\right) + \kappa_\alpha'\tau_\alpha v\kappa_\alpha^2\left(1 - v^2 + 2\tau_\alpha^2\right) + 2v\tau_\alpha'\kappa_\alpha^3\left(1 - \tau_\alpha^2\right)\right)}{\left((\kappa_\alpha^2 - v\tau_\alpha^2)^2 + (v\tau_\alpha\kappa_\alpha)^2\right)^{\frac{3}{2}}} \tag{3.15}$$

Theorem 3.13. Let $\psi_4(s,v)$ be a normal ruled surface with the MOF in Euclidean 3-space. Then, $\psi_4(s,v)$ is not a flat surface.

Proof. Since $\kappa_\alpha \neq 0$, $\psi_4(s,v)$ cannot be flat. □

Theorem 3.14. Let $\psi_4(s,v)$ be a normal ruled surface in Euclidean 3-space. If the curve α is a cylindrical helix, then $\psi_4(s,v)$ is minimal.

Proof. Let the curve α be a cylindrical helix. Then, κ_α and τ_α are constant. By (3.15),

$$v^2\tau_\alpha^2\left(5\kappa_\alpha'\tau_\alpha^3 - 7\kappa_\alpha\tau_\alpha'\right) + \kappa_\alpha'\tau_\alpha v\kappa_\alpha^2\left(1 - v^2 + 2\tau_\alpha^2\right) + 2v\tau_\alpha'\kappa_\alpha^3\left(1 - \tau_\alpha^2\right) = 0$$

Since $\kappa_\alpha' = 0$ and $\tau_\alpha' = 0$, $H = 0$. Therefore, $\psi_4(s,v)$ is minimal. □

Theorem 3.15. Let $\psi_4(s,v)$ be a normal ruled surface with the MOF in E^3. Then,

i. s-parameter curves of $\psi_4(s,v)$ are asymptotic curves if and only if the curvatures $\kappa_\alpha, \tau_\alpha$ of the curve α are constant, $\dfrac{\kappa_\alpha}{\tau_\alpha} = \dfrac{5v}{1-v}$, or $\dfrac{\kappa_\alpha}{\tau_\alpha} = \dfrac{7v}{2}$.

ii. v-parameter curves of the $\psi_4(s,v)$ are asymptotic curves.

Proof. From the definition of asymptotic curves, $\langle \psi_{4_{ss}}, U\rangle = 0$ and $\langle \psi_{4_{vv}}, U\rangle = 0$.

i. From (3.13) and (3.14), $h_{11} = 0$. Thus,

$$\left(5\frac{v^2\kappa_\alpha'\tau_\alpha^3}{\kappa_\alpha} - 7v^2\tau^2\tau_\alpha' + \kappa_\alpha\kappa_\alpha'\tau_\alpha v(1-v) + 2v\kappa_\alpha^2\tau_\alpha'\right) = 0$$

since the curvatures κ_α and τ_α of the curve α are constant.

ii. Since $h_{22} = 0$, v-parameter curves of the $\psi_4(s,v)$ are asymptotic.

□

Theorem 3.16. Let $\psi_4(s,v)$ be a normal ruled surface with the MOF in E^3. Then,

i. s-parameter curves of $\psi_4(s,v)$ cannot be geodesic.

ii. v-parameter curves of $\psi_4(s,v)$ are geodesic.

The proof is similar to the previous theorem about the normal ruled surface $\psi_3(s,v)$.

3.5. Binormal Ruled Surface with the Curve α

Concerning the MOF, the parameterization of the binormal ruled surface is as follows:

$$\psi_5(s,v) = \alpha(s) + vB_\alpha(s) \tag{3.16}$$

where the base curve α. If we take the derivatives with respect to parameter s and v of the binormal ruled surface $\psi_5(s,v)$, then $\psi_{5_s} = T_\alpha - v\tau_\alpha N_\alpha + v\frac{\kappa'_\alpha}{\kappa_\alpha}B_\alpha$, $\psi_{5_v} = B_\alpha$, $\psi_{5_{ss}} = \kappa_\alpha^2 vT_\alpha + \left(1 - v\tau'_\alpha - 2v\tau_\alpha\frac{\kappa'_\alpha}{\kappa_\alpha}\right)N_\alpha + v\left(-\tau_\alpha^2 + \frac{\kappa''_\alpha}{\kappa_\alpha}\right)B_\alpha$, $\psi_{5_{sv}} = -\tau_\alpha N_\alpha + \frac{\kappa'_\alpha}{\kappa_\alpha}B_\alpha$, and $\psi_{5_{vv}} = 0$. Hence, $g_{11} = 1 + v^2\left({\kappa'_\alpha}^2 + \tau_\alpha^2\kappa_\alpha^2\right)$, $g_{12} = v\kappa'_\alpha\kappa_\alpha$, and $g_{22} = \kappa_\alpha^2$ and thus $g_{11}g_{22} - g_{12}^2 = \kappa_\alpha^2\left(1 + (v\kappa_\alpha\tau_\alpha)^2\right) \neq 0$. Moreover, the unit normal vector field is provided by

$$U_5 = \frac{1}{\sqrt{1 + (v\kappa_\alpha\tau_\alpha)^2}}\left(-v\kappa_\alpha\tau_\alpha T_\alpha - \frac{1}{\kappa_\alpha}N_\alpha\right)$$

and the coefficients of the second fundamental form are obtained as:

$$h_{11} = \frac{1}{\sqrt{1 + (v\kappa_\alpha\tau_\alpha)^2}}\left(v\left(\kappa_\alpha\tau'_\alpha + 2\tau_\alpha\kappa'_\alpha - \kappa_\alpha^3 v\tau_\alpha\right) - \kappa_\alpha\right)$$

$$h_{12} = \frac{\tau_\alpha\kappa_\alpha}{\sqrt{1 + (v\kappa_\alpha\tau_\alpha)^2}}$$

and $h_{22} = 0$. From (2.1), the Gaussian and mean curvatures are as follows, respectively:

$$K = -\frac{\tau_\alpha^2}{\left(1 + (v\kappa_\alpha\tau_\alpha)\right)^2} \quad \text{and} \quad H = \frac{v\left(\kappa_\alpha\tau'_\alpha - \kappa_\alpha^3 v\tau_\alpha\right) - \kappa_\alpha}{2\left(1 + (v\kappa_\alpha\tau_\alpha)^2\right)^{\frac{3}{2}}} \tag{3.17}$$

Theorem 3.17. Let $\psi_5(s,v)$ be a binormal ruled surface with the MOF in Euclidean 3-space. Then, $\psi_5(s,v)$ is not a flat surface.

PROOF. Since $\tau_\alpha \neq 0$, $\psi_5(s,v)$ cannot be flat. □

Theorem 3.18. Let $\psi_5(s,v)$ be a ruled surface in Euclidean 3-space. If the curve α is a cylindrical helix, then $\psi_5(s,v)$ is minimal

The result is directly obtained from (3.17).

Theorem 3.19. Let $\psi_5(s,v)$ be a ruled surface in E^3 with the MOF. Then,

i. s-parameter curves of $\psi_5(s,v)$ are asymptotic curves if and only if the curvatures κ_α and τ_α of the curve α are constant and $\tau_\alpha = \frac{1}{v^2\kappa_\alpha^2}$.

ii. v-parameter curves of $\psi_5(s,v)$ are asymptotic curves.

The proof is similar to Theorem 3.11.

Theorem 3.20. Let $\psi_5(s,v)$ be a ruled surface with the MOF in E^3. Then,

i. s-parameter curves of $\psi_5(s,v)$ cannot be geodesic.

ii. v-parameter curves of $\psi_5(s,v)$ are geodesic.

The proof is similar to Theorem 3.12.

3.6. Binormal Ruled Surface with the Adjoint Curve β

Concerning the MOF, the parameterization of the binormal ruled surface is as follows:

$$\psi_6(s,v) = \beta(s) + vB_\beta(s) \tag{3.18}$$

where the base curve is the adjoint curve β. If we take the derivatives with respect to the parameter s and v of the binormal ruled surface $\psi_6(s,v)$, then $\psi_{6_s} = v\left(\frac{\tau'_\alpha}{\kappa_\alpha} - \frac{\kappa'_\alpha\tau_\alpha}{\kappa_\alpha^2}\right)T_\alpha + v\frac{\tau_\alpha}{\kappa_\alpha}N_\alpha + B_\alpha$ and

$\psi_{6v} = \dfrac{\tau_\alpha}{\kappa_\alpha} T_\alpha$ and thus

$$\psi_{6ss} = v\left(\frac{\tau''_\alpha}{\kappa_\alpha} - 2\frac{\kappa'_\alpha \tau'_\alpha}{\kappa^2_\alpha} - \frac{\kappa''_\alpha \tau_\alpha}{\kappa^2_\alpha} + 2\frac{{\kappa'_\alpha}^2 \tau_\alpha}{\kappa^3_\alpha} - \kappa_\alpha\tau_\alpha\right) T_\alpha + \left(-\tau_\alpha + v\left(2\frac{\tau'_\alpha}{\kappa_\alpha} - \frac{\kappa'_\alpha\tau_\alpha}{\kappa^2_\alpha}\right)\right) N_\alpha + \left(\frac{\kappa'_\alpha}{\kappa_\alpha} + v\frac{\tau^2_\alpha}{\kappa_\alpha}\right) B_\alpha$$

$$\psi_{6sv} = \tau_\alpha T_\alpha + \left(\frac{\kappa'_\alpha \tau_\alpha}{\kappa^3_\alpha} - \frac{\tau'_\alpha}{\kappa^2_\alpha}\right) N_\alpha - \frac{\tau^2_\alpha}{\kappa^2_\alpha} B_\alpha$$

and $\psi_{6vv} = 0$ Hence, $g_{11} = \kappa^2_\alpha + v^2\left(\tau^2_\alpha + \left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'^2\right)$, $g_{12} = \frac{v\tau_\alpha}{\kappa_\alpha}\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)'$, and $g_{22} = \frac{\tau^2_\alpha}{\kappa^2_\alpha}$ and thus $g_{11}g_{22} - g^2_{12} = \frac{\tau^2_\alpha}{\kappa^2_\alpha}\left(\kappa^2_\alpha + v^2\tau^2_\alpha\right) \neq 0$. Moreover, the unit normal vector field U_6 is provided by

$$U_6 = \frac{1}{\sqrt{\kappa^2_\alpha + v^2\tau^2_\alpha}}\left(N_\alpha - \frac{\tau_\alpha}{\kappa_\alpha} v B_\alpha\right)$$

and the coefficients of the second fundamental form are obtained as:

$$h_{11} = \frac{1}{\sqrt{\kappa^2_\alpha + v^2\tau^2_\alpha}}\left(-\tau_\alpha\left(\kappa^2_\alpha + v^2\tau^2_\alpha\right) + 2v\left(\kappa_\alpha\tau'_\alpha - \kappa'_\alpha\tau_\alpha\right)\right)$$

$$h_{12} = \frac{\tau_\alpha\kappa_\alpha}{\sqrt{\kappa^2_\alpha + v^2\tau^2_\alpha}}$$

and $h_{22} = 0$. By (2.1), the Gaussian and mean curvatures are as follows:

$$K = -\frac{\kappa^4_\alpha}{\left(\kappa^2_\alpha + v^2\tau^2_\alpha\right)^2} \quad \text{and} \quad H = -\frac{\tau_\alpha}{\sqrt{\kappa^2_\alpha + v^2\tau^2_\alpha}} \tag{3.19}$$

Theorem 3.21. Let $\psi_6(s,v)$ be a binormal ruled surface according to the MOF in Euclidean 3-space. Then, $\psi_6(s,v)$ is not a flat surface.

Proof. Since $\kappa_\alpha \neq 0$, $\psi_6(s,v)$ cannot be flat. □

Theorem 3.22. Let $\psi_6(s,v)$ be a ruled surface in Euclidean 3-space. Then, $\psi_6(s,v)$ cannot be minimal.

The result is obtained directly from 3.19.

Theorem 3.23. Let $\psi_6(s,v)$ be a ruled surface with the MOF in E^3. Then,

i. s-parameter curves of $\psi_6(s,v)$ cannot be asymptotic curves.

ii. v-parameter curves of $\psi_6(s,v)$ are asymptotic curves.

The proof is similar to Theorem 3.11.

Theorem 3.24. Let $\psi_6(s,v)$ be a ruled surface in E^3 with the MOF. Then,

i. s-parameter curves of $\psi_6(s,v)$ cannot be geodesic.

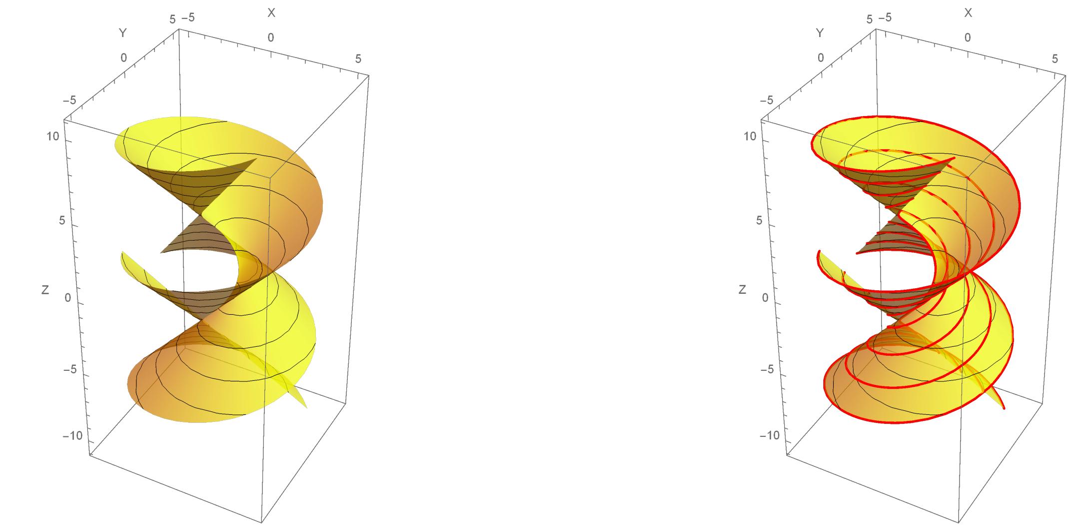
ii. v-parameter curves of $\psi_6(s,v)$ are geodesic.

The proof is similar to Theorem 3.12.

Example 3.25. Consider the curve α and and the adjoint curve β are provided by the following parametric equations, respectively:

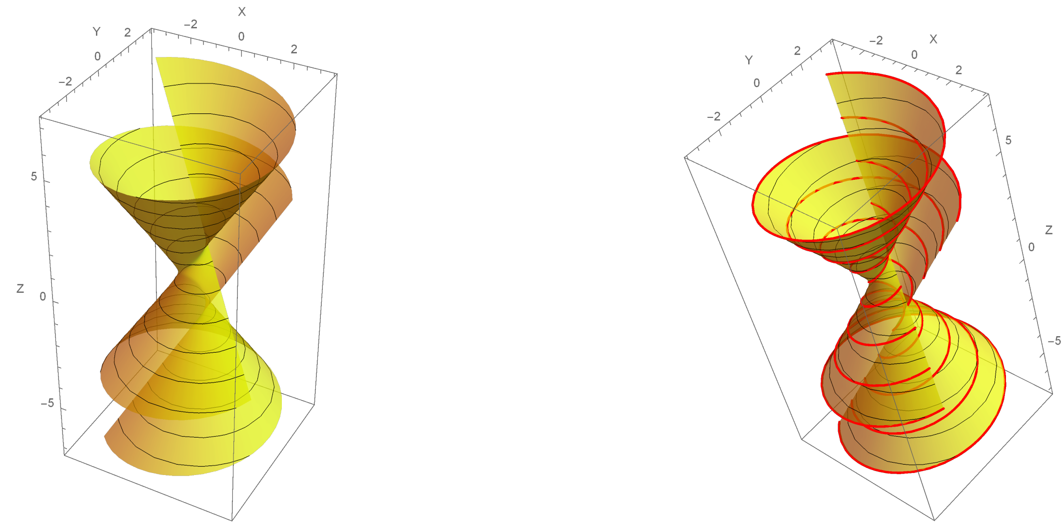
$$\alpha(s) = \left(\cos\left(\frac{\sqrt{7}s}{4}\right), \sin\left(\frac{\sqrt{7}s}{4}\right), \frac{3s}{4}\right) \text{ and } \beta(s) = \left(-\frac{3\sqrt{7}}{16}\cos\left(\frac{\sqrt{7}}{4}s\right), -\frac{3\sqrt{7}}{16}\sin\left(\frac{\sqrt{7}}{4}s\right), -\frac{7\sqrt{7}}{64}s\right)$$

According to the curves α and β, the graphs of some ruled surfaces are as in Figures 1-6.



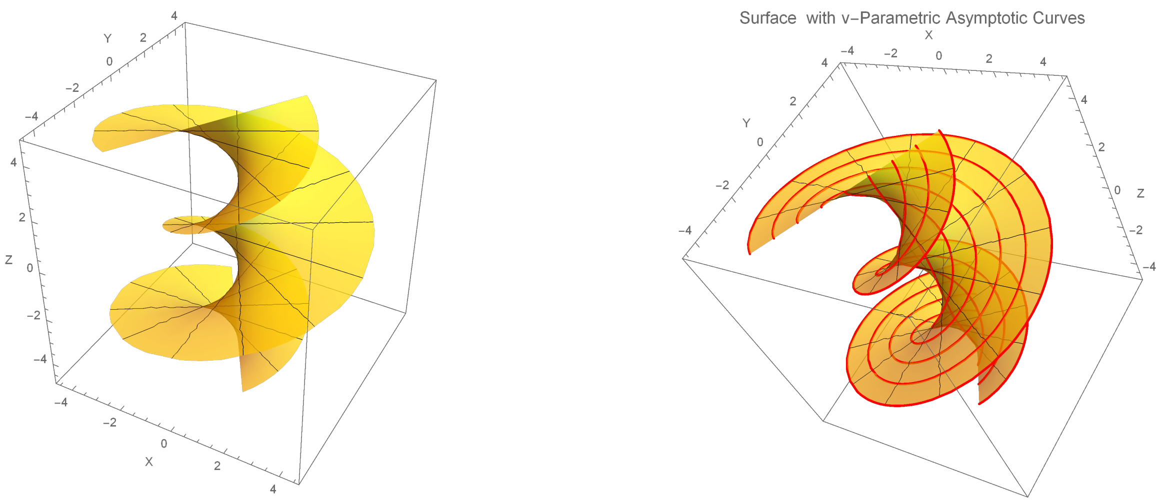
(a) Tangent ruled surface $\psi_1(s,v)$ **(b)** $v-$parameter curves of $\psi_1(s,v)$

Figure 1. Graph of the tangent ruled surface $\psi_1(s,v)$ in (3.1) whose director curve is α with the MOF



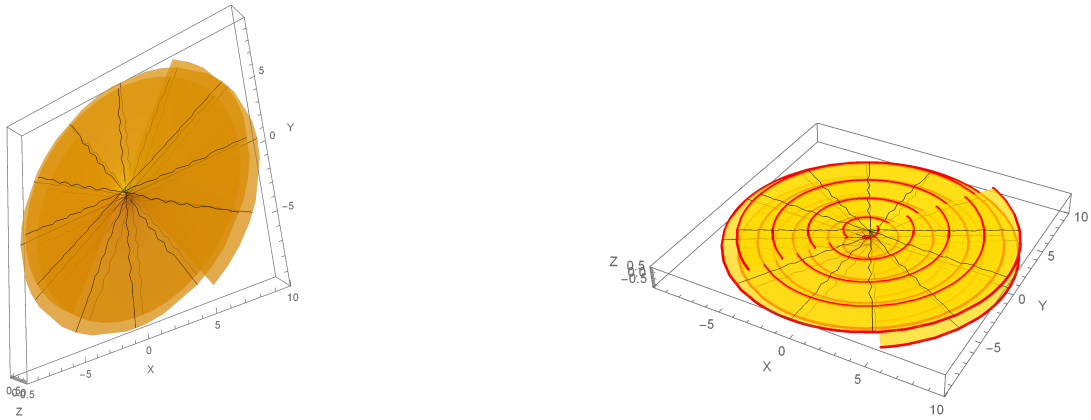
(a) Tangent ruled surface $\psi_2(s,v)$ **(b)** $v-$parameter curves of $\psi_2(s,v)$

Figure 2. Graph of the tangent ruled surface $\psi_2(s,v)$ in (3.5) whose director curve is β with the MOF



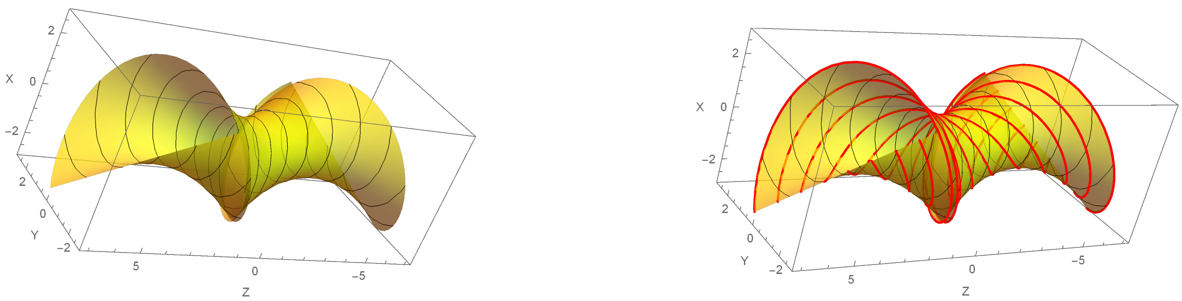
(a) Normal ruled surface $\psi_3(s,v)$ **(b)** $v-$parameter curves of $\psi_3(s,v)$

Figure 3. Graph of the normal ruled surface $\psi_3(s,v)$ in (3.8) whose the director curve is α with the MOF



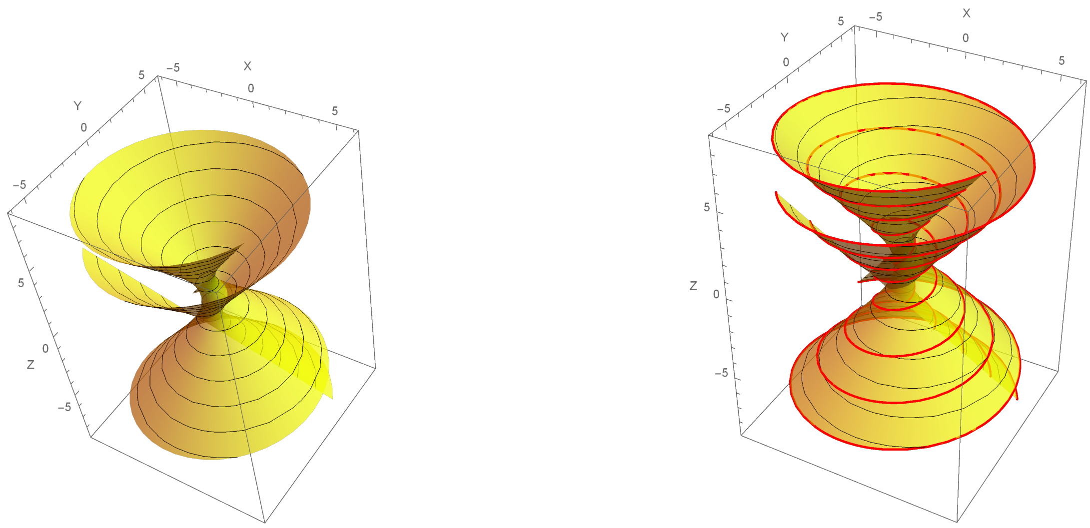
(a) Normal ruled surface $\psi_4(s,v)$ **(b)** $v-$parameter curves of $\psi_4(s,v)$

Figure 4. Graph of the tangent ruled surface $\psi_4(s,v)$ in (3.12) whose director curve is β with the MOF



(a) Binormal ruled surface $\psi_5(s,v)$ **(b)** $v-$parameter curves of $\psi_5(s,v)$

Figure 5. Graph of the binormal ruled surface $\psi_5(s,v)$ in (3.16) whose the director curve is α with the MOF



(a) binormal ruled surface $\psi_6(s,v)$ **(b)** $v-$parameter curves of $\psi_6(s,v)$

Figure 6. Graph of the binormal ruled surface $\psi_6(s,v)$ in (3.18) whose director curve is β with the MOF

4. Results

We calculated the Gaussian curvature K and the mean curvature H of some special ruled surfaces generated by the curve α and its adjoint curve β according to the MOF in E^3. While the tangent ruled surfaces are flat, the normal and binormal ruled surfaces are not flat. Even if the frame of the tangent ruled surface changes, its state of being minimal does not change, so it cannot be minimal.

We found a minimal condition for the normal and binormal ruled surfaces. Additionally, we searched s-parameter and v-parameter curves of some special ruled surfaces. Hence, we got some conditions for the s-parameter curves of some special ruled surfaces to be asymptotic and the v-parameter curves of some special ruled surfaces to be geodesic.

5. Conclusion

This study utilized the MOF to investigate the curvature characteristics and minimality of certain ruled surfaces based on a base curve and its adjoint in Euclidean 3-space. It was determined that tangent-ruled surfaces are flat, while normal and binormal surfaces are not. Additionally, only specific conditions allow for minimality in normal and binormal ruled surfaces. Future studies could explore applying these findings to different classes of ruled surfaces or extending the approach to non-Euclidean spaces. Further research may also analyze the potential applications of these geometric properties in advanced modeling, which could provide insights into mathematical physics and computer-aided geometric design.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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