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# SOME SPACES OF A-IDEAL CONVERGENT SEQUENCES DEFINED BY MUSIELAK-ORLICZ FUNCTION

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Abstract. We introduce basic properties of some sequence spaces using ideal convergent and Musielak Orlicz function $\mathcal{M} = (M_k)$. Including relations related to these spaces are investigated in this paper.

## 1. Introduction, Definitions and Notations

Throughout this article $w$, $c$, $c_0$, $l_\infty$, $l_p$ denote the spaces of all, convergent, null, bounded and $p$ -absolutely summable sequences, where $1 \leq p < \infty$.

Firstly, the notion of $I$ -convergence was introduced by Kostryrko et all [1] and it is the generalization of statistical convergence.
$A = (a_{nk})$ be an infinite matrix of complex entries $a_{nk}$ and $x = (x_k)$ be a sequence in $w$. If $A_n(x) = \sum_{k=1}^{\infty} a_{nk}x_k$ converges for each, then we write $n \in \mathbb{N}$.

**Definition 1.1.** If $X$ is a non-empty set then a family of sets $I \subseteq 2^X$ is ideal if and only if for each $A, B \in I$ we have $A \cup B \in I$ and for each $A \in I$ and each $B \subset A$ we have $B \in I$.[1]

**Definition 1.2.** A non-empty family of sets $F \subset 2^X$ is said to be a filter on $X$ if and only if $\emptyset \notin F$, for each $A, B \in F$ we have $A \cap B \in F$ and for each $A \in F$ and each $B \supset A$ we have $B \in F$.[1]

**Definition 1.3.** An ideal $I \neq \emptyset$ is called non-trivial if $I \neq \emptyset$ and $X \notin I$.[1]

**Definition 1.4.** A non-trivial $I \subseteq 2^X$ is called admissible ideal if and only if $\{\{x\} : x \in X\} \subset I$.[1]

**Definition 1.5.** A sequence $x = (x_n) \in w$ is said to be *I-convergent* to $L$ if there exists $L \in \mathbb{C}$ such that for all $\varepsilon > 0$ , the set $\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in I$. We say $x$, $I-$ *convergent* to $L$ and we write $I - \lim x = L$. The number $L$ is called $I-$ *limit* of $x$.[2]

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**Definition 1.6.** An Orlicz function $M$ is a function which is continuous, nondecreasing, and convex with $M(0) = 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.
Lindenstrauss and Tzafriri [4] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}$$

which is called an Orlicz sequence space. The space $l_M$ becomes a Banach space with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

The space $l_M$ is closely related to the space $l_p$ which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$. Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [5], Bhardwaj and Singh [6] and many others. It is well known that since $M$ is a convex function and $M(0) = 0$ then $M(tx) \leq tM(x)$ for all $t$ with $0 < t < 1$. Dutta and Başar [18] have recently introduced and studied the Orlicz sequence spaces $l'_M(C, \Lambda)$ and $h_M(C, \Lambda)$ generated by Cesàro mean of order one associated with a fixed multiplier sequence of non-zero scalars. The readers may refer to [17] for relevant terminology and details on the algebraic and topological properties on sequence spaces. An Orlicz function $M$ is said to satisfy $\Delta_2 - condition$ for all values of $u$, if there exists constant $K > 0$ such that $M(2u) \leq KM(u) \quad (u \geq 0)$. The $\Delta_2 - condition$ is equivalent to the inequality $M(Lu) \leq KLM(u)$ satisfying for all values of $u$ and for $L > 1$ [7]. A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see [8], [9]. The sequence $N = (N_k)$ defined by

$$N_k(v) = \sup\left\{|v|\, u - (M_k) : u \geq 0\right\}, \quad k = 1, 2, ...$$

is called the complementary function of a Musileak-Orlicz function $\mathcal{M} = (M_k)$. For a given Musileak-Orlicz function $\mathcal{M} = (M_k)$, the Musileak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$t_{\mathcal{M}} = \left\{x \in \omega : I_{\mathcal{M}}(cx) < \infty \text{ for some c} > 0\right\},$$
$$h_{\mathcal{M}} = \left\{x \in \omega : I_{\mathcal{M}}(cx) < \infty \text{ for all c} > 0\right\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\left\{\rho > 0 : I_{\mathcal{M}}\left(\frac{x}{\rho}\right) \leq 1\right\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\left\{\frac{1}{\rho}\left(1 + I_{\mathcal{M}}\left(\rho x\right)\right) : \rho > 0\right\}.$$

The following inequality will be used throughout this paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < h = \inf p_n \leq p_n \leq H = \sup p_n < \infty$ and let $D = \max\left\{1, 2^{H-1}\right\}$. Then for $a_k, b_k \in \mathbb{C}$, the set of complex numbers for all $k \in \mathbb{N}$, we have

$$|a_k + b_k|^{p_k} \leq D\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\}. \tag{1.1}$$

Also, $|a|^{p_k} \leq \max\left\{1, |a|^H\right\}$ for all $a \in \mathbb{C}$.

The notion of paranormed space was introduced by Nakano [10] and Simons [11] and many others.

**Definition 1.7.** Let $X$ be a linear metric space. A function $g : X \to \mathbb{R}$ is called paranorm if

(1) $g(x) \geq 0$, for all $x \in X$,
(2) $g(-x) = g(x)$, for all $x \in X$,
(3) $g(x + y) \leq g(x) + g(y)$, for all $x, y \in X$,
(4) if $(\lambda_n)$ be a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and $(x_n)$ is a sequence of vectors with $g(x_n - x) \to 0$ as $n \to \infty$, then $g(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

**Definition 1.8.** A sequence space $X$ is solid (or normal) if $(\alpha_n x_n) \in X$ whenever $(x_n) \in X$ for all sequences $(\alpha_n)$ of scalars with $|\alpha_n| \leq 1$ for all $n \in \mathbb{N}$.

**Definition 1.9.** A sequence space $X$ is said to be monotone if it contains the canonical preimages of its step spaces.[19]

**Lemma 1.1.** *If a sequence space $X$ is solid, then $X$ is monotone.*[12]

**Definition 1.10.** A sequence space $X$ is sequence algebra if $xy = (x_n y_n) \in X$ whenever $x = (x_n), y = (y_n) \in X$.

We define the following sequence spaces in this article,

$$c^I(M, A, p) = \left\{x \in w : I - \lim_k \left[M_k\left(\frac{|A_k(x) - L|}{\rho}\right)\right]^{p_k} = 0 \quad \text{for some } L \text{ and } \rho > 0\right\},$$

$$c_0^I(\mathcal{M}, A, p) = \left\{x \in w : I - \lim_k \left[M_k\left(\frac{|A_k(x)|}{\rho}\right)\right]^{p_k} = 0 \quad \text{for some } \rho > 0\right\},$$

$$l_\infty(\mathcal{M}, A, p) = \left\{x \in w : \sup_k \left[M_k\left(\frac{|A_k(x)|}{\rho}\right)\right]^{p_k} < \infty \quad \text{for some } \rho > 0\right\}.$$

Also we write

$$m^I(\mathcal{M}, A, p) = c^I(\mathcal{M}, A, p) \cap l_\infty(\mathcal{M}, A, p)$$
$$m_0^I(\mathcal{M}, A, p) = c_0^I(\mathcal{M}, A, p) \cap l_\infty(\mathcal{M}, A, p).$$

If we take $A = \lambda$, these spaces are respectively reduced to the spaces $c_0^I(\mathcal{M}, \lambda, p)$, $c^I(\mathcal{M}, \lambda, p)$, $l_\infty(\mathcal{M}, \lambda, p)$, $m_0^I(\mathcal{M}, \lambda, p)$, $m^I(\mathcal{M}, \lambda, p)$ defined by Mursaleen and Sharma [19]. If we take $p_k = 1$ for all $k$, $\mathcal{M}(x) = M(x)$ and $A = I$, we get the spaces $c_0^I(\mathcal{M})$, $c^I(\mathcal{M})$, $l_\infty(\mathcal{M})$, $m_0^I(\mathcal{M})$, $m^I(\mathcal{M})$ which were studied by Tripathy and Hazarika [14].

Our aim is to define the paranormed space of ideal convergent sequence space with matrix transformation and Musielak-Orlicz function.

## 2. Main Results

**Theorem 2.1.** *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. Then, the spaces $c^I(\mathcal{M}, A, p)$, $c_0^I(\mathcal{M}, A, p)$, $m^I(\mathcal{M}, A, p)$ and $m_0^I(\mathcal{M}, A, p)$ are linear.*

*Proof.* Let $x, y \in c^I(\mathcal{M}, A, p)$ and $\alpha, \beta$ be scalars. So, there exist positive numbers $\rho_1, \rho_2$ and for given $\varepsilon > 0$, we have

$$A_1 = \left\{ k \in \mathbb{N} : \left[ M_k \left( \frac{|A_k(x) - L_1|}{\rho_1} \right) \right]^{p_k} \geq \frac{\varepsilon}{2D} \right\} \in I,$$

$$A_2 = \left\{ k \in \mathbb{N} : \left[ M_k \left( \frac{|A_k(x) - L_2|}{\rho_2} \right) \right]^{p_k} \geq \frac{\varepsilon}{2D} \right\} \in I.$$

Let $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since $\mathcal{M} = (M_k)$ is nondecreasing and convex function, we can obtain

$$M_k \left( \frac{|A_k(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3} \right) < M_k \left( \frac{|A_k(x) - L_1|}{\rho_1} \right) + M_k \left( \frac{|A_k(y) - L_2|}{\rho_2} \right).$$

So, we have

$$\left[ M_k \left( \frac{|A_k(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3} \right) \right]^{p_k} < D \left\{ \left[ M_k \left( \frac{|A_k(x) - L_1|}{\rho_1} \right) \right]^{p_k} + \left[ M_k \left( \frac{|A_k(y) - L_2|}{\rho_2} \right) \right]^{p_k} \right\}.$$

Suppose that $k \notin A_1 \cup A_2$. So, $\left[ M_k \left( \frac{|A_k(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3} \right) \right]^{p_k} < \varepsilon$ and hence

$$k \notin \left\{ k \in \mathbb{N} : \left[ M_k \left( \frac{|A_k(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3} \right) \right]^{p_k} \geq \varepsilon \right\} \subset A_1 \cup A_2.$$

Therefore, $I\text{–}\lim_k \left[ M_k \left( \frac{|A_k(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)|}{\rho_3} \right) \right]^{p_k} = 0$. Hence $\alpha x + \beta y \in c^I(\mathcal{M}, A, p)$ and so $c^I(\mathcal{M}, A, p)$ is a linear space. Similarly, we can prove that $c_0^I(\mathcal{M}, A, p)$, $m_0^I(\mathcal{M}, A, p)$ and $m^I(\mathcal{M}, A, p)$ are linear spaces. □

**Theorem 2.2.** *$l_\infty(\mathcal{M}, A, p)$ is a paranormed space with the paranorm $g$ defined by*

$$g(x) = \inf \left\{ \rho^{\frac{p_k}{S}} : \sup_k \left[ M_k \left( \frac{|A_k(x)|}{\rho} \right) \right]^{\frac{p_k}{S}} \leq 1,\ k = 1, 2, ... \right\},$$

*where $S = \max\{1, H\}$.*

*Proof.* It is clear that $g(x) = g(-x)$. Since $M_k(0) = 0$, we get $g(0) = 0$. Let us take $x = (x_k)$ and $y = (y_k)$ in $l_\infty(\mathcal{M}, A, p)$. We denote,

$$B(x) = \left\{ \rho_1 : \sup_k \left[ M_k \left( \frac{|A_k(x)|}{\rho_1} \right) \right]^{\frac{p_k}{S}} \leq 1 \right\}$$

$$B(y) = \left\{ \rho_2 : \sup_k \left[ M_k \left( \frac{|A_k(y)|}{\rho_2} \right) \right]^{\frac{p_k}{S}} \leq 1 \right\}.$$

Let $\rho = \rho_1 + \rho_2$. Then using the convexity of Mursielak-Orlicz function $\mathcal{M} = (M_k)$, we obtain

$$M_k \left( \frac{|A_k(x+y)|}{\rho} \right) \leq \frac{\rho_1}{\rho} M_k \left( \frac{|A_k(x)|}{\rho_1} \right) + \frac{\rho_2}{\rho} M_k \left( \frac{|A_k(x)|}{\rho_2} \right) \leq \frac{\rho_1}{\rho} + \frac{\rho_2}{\rho} = 1.$$

Therefore,

$$\sup_k \left[ M_k \left( \frac{|A_k(x+y)|}{\rho} \right) \right]^{\frac{p_k}{S}} \leq 1.$$

We can see that

$$\begin{aligned} g(x+y) &= \inf \left\{ (\rho_1+\rho_2)^{\frac{p_k}{S}} : \rho_1 \in B(x), \rho_2 \in B(y) \right\} \\ &\leq \inf \left\{ (\rho_1)^{\frac{p_k}{S}} : \rho_1 \in B(x) \right\} + \inf \left\{ (\rho_2)^{\frac{p_k}{S}} : \rho_2 \in B(y) \right\} = g(x) + g(y). \end{aligned}$$

Let $B(x^n) = \left\{ \rho : \sup_k \left[ M_k \left( \frac{|A_k(x^n)|}{\rho} \right) \right]^{\frac{p_k}{S}} \leq 1 \right\}$, $B(x^n - x) = \left\{ \rho : \sup_k \left[ M_k \left( \frac{|A_k(x^n-x)|}{\rho} \right) \right]^{\frac{p_k}{S}} \leq 1 \right\}$ and $\rho_n \in B(x^n)$, $\rho'_n \in B(x^n - x)$. We can obtain,

$$\begin{aligned} M_k \left( \frac{|A_k(\gamma_n x^n - \gamma x)|}{\rho_n |\gamma_n - \gamma| + \rho'_n |\gamma|} \right) &\leq \frac{|\gamma_n - \gamma| \rho_n}{\rho_n |\gamma_n - \gamma| + \rho'_n |\gamma|} M_k \left( \frac{|A_k(x^n)|}{\rho_n} \right) + \frac{|\gamma| \rho'_n}{\rho_n |\gamma_n - \gamma| + \rho'_n |\gamma|} M_k \left( \frac{|A_k(x^n - x)|}{\rho'_n} \right) \\ &\leq \frac{|\gamma_n - \gamma| \rho_n}{\rho_n |\gamma_n - \gamma| + \rho'_n |\gamma|} + \frac{|\gamma| \rho'_n}{\rho_n |\gamma_n - \gamma| + \rho'_n |\gamma|} = 1. \end{aligned}$$

Taking supremum over $k$ on both sides,

$$\sup_k \left[ M_k \left( \frac{|A_k(\gamma_n x^n - \gamma x)|}{\rho_n |\gamma_n - \gamma| + \rho'_n |\gamma|} \right) \right]^{\frac{p_k}{S}} \leq 1$$

and so,

$$\{\rho_n |\gamma_n - \gamma| + \rho'_n |\gamma| : \rho_n \in B(x^n),\ \rho'_n \in B(x^n - x)\} \subset \left\{ \rho > 0 : \sup_k \left[ M_k \left( \frac{|A_k(\gamma_n x^n - \gamma x)|}{\rho} \right) \right]^{p_k} \leq 1 \right\}.$$

Therefore,

$$\begin{aligned} g(\gamma_n x^n - \gamma x) &= \inf \left\{ (\rho_n |\gamma_n - \gamma| + \rho'_n |\gamma|)^{\frac{p_k}{S}} : \rho_n \in B(x^n),\ \rho'_n \in B(x^n - x) \right\} \\ &\leq |\gamma_n - \gamma|^{\frac{p_k}{S}} \inf \left\{ (p_n)^{\frac{p_k}{S}} : \rho_n \in B(x^n),\ k = 1, 2, ... \right\} \\ &\quad + \max\{1, |\gamma|^s\} \inf \left\{ (\rho'_n)^{\frac{p_k}{S}} : \rho'_n \in B(x^n - x),\ k = 1, 2, ... \right\} \end{aligned}$$

where $s = \sup_k \left( \frac{p_k}{S} \right) = \min\{1, H\}$. Since $|\gamma_n - \gamma| \to 0$ and $g(x^n - x) \to 0$ as $n \to \infty$, we obtain that $g(\gamma_n x^n - \gamma x) \to 0$ as $n \to \infty$. □

**Theorem 2.3.** *Let $(M_k)$ and $(M'_k)$ be Musielak-Orlicz functions that $\Delta_2-$condition satisfies. Then,*
*(i)* $W(M_k, A, p) \subseteq W(M'_k \circ M_k, A, p)$
*(ii)* $W(M_k, A, p) \cap W(M'_k, A, p) \subseteq W(M_k + M'_k, A, p)$
*where* $W = c_0^I, c^I, m_0^I, m^I$.

*Proof.* $(i)$ Since $W \in \{c^I, m_0^I, m^I\}$ can be proved similarly, we give the prove only for $W = c_0^I$. Let $x \in c_0^I(\mathcal{M}, A, p)$. So, we have $\rho > 0$ for every $\varepsilon > 0$,

$$B = \left\{ k \in \mathbb{N} : \left( M_k \left( \frac{|A_k(x)|}{\rho} \right) \right)^{p_k} \geq \varepsilon \right\} \in I.$$

Since $(M'_k)$ is continuous, given for $\varepsilon > 0$ chosen $\delta$ with $0 < \delta < 1$ such that $M'_k(t) < \varepsilon$ for $0 \leq t \leq \delta$. We define $y_k = M_k \left( \frac{|A_k(x)|}{\rho} \right)$. For $y_k > \delta$,

$$y_k < \frac{y_k}{\delta} < 1 + \frac{y_k}{\delta}$$

Therefore;

$$M'_k(y_k) < M'_k \left( 1 + \frac{y_k}{\delta} \right) = M'_k \left( \frac{1}{2} 2 + \frac{1}{2} \frac{y_k}{\delta} 2 \right) \leq \frac{1}{2} M'_k(2) + \frac{1}{2} M'_k \left( \frac{y_k}{\delta} 2 \right) \tag{2.1}$$

Since $(M_k')$ satisfies $\Delta_2 - condition$, we can write that

$$M_k'\left(\frac{y_k}{\delta}2\right) \le K\frac{y_k}{\delta}M_k'(2)\, for K \ge 1. \tag{2.2}$$

From (2.1) and (2.2), we have

$$\begin{aligned} M_k'(y_k) &< \tfrac{1}{2}M_k'(2) + \tfrac{1}{2}K\tfrac{y_k}{\delta}M_k'(2) \\ &\le \tfrac{1}{2}K\tfrac{y_k}{\delta}M_k'(2) + \tfrac{1}{2}K\tfrac{y_k}{\delta}M_k'(2) \\ &= K\tfrac{y_k}{\delta}M_k'(2). \end{aligned}$$

Hence;$[M_k'(y_k)]^{p_k} < \left[K\frac{1}{\delta}M_k'(2)\right]^{p_k}(y_k)^{p_k} \le \max\left\{1, \left(K\frac{1}{\delta}M_k'(2)\right)^H\right\}(y_k)^{p_k}$. Since $y_k = M_k\left(\frac{|A_k(x)|}{\rho}\right)$, we have $I - \lim_k (y_k)^{p_k} = 0$. So,

$$C = \left\{k : (y_k)^{p_k} \ge \frac{\varepsilon}{\max\left\{1, \left(K\frac{y_k}{\delta}M_k'(2)\right)^H\right\}}\right\} \in I.$$

Suppose that $k \notin C$. Then, $(y_k)^{p_k} < \frac{\varepsilon}{\max\left\{1, \left(K\frac{y_k}{\delta}M_k'(2)\right)^H\right\}}$. Hence,

$$(M_k'(y_k))^{p_k} < \max\left\{1, \left(K\frac{y_k}{\delta}M_k'(2)\right)^H\right\}\frac{\varepsilon}{\max\left\{1, \left(K\frac{y_k}{\delta}M_k'(2)\right)^H\right\}} = \varepsilon.$$

Therefore, $k \notin \left\{k : (M_k'(y_k))^{p_k} \ge \varepsilon,\ y_k > \delta\right\} = D$. Thus $D \subseteq C$ and $D \in I$. Since $M_k'(y_k) < \varepsilon$ for $y_k \le \delta$, we have

$$[M_k(y_k)]^{p_k} < \varepsilon^{p_k} \le \max\left\{\varepsilon^h, \varepsilon^H\right\}.$$

From this inequality, we have $I - \lim [M_k'(y_k)]^{p_k} = 0$ for $y_k \le \delta$. Therefore $E = \left\{k : (M_k'(y_k))^{p_k} \ge \varepsilon,\ y_k \le \delta\right\} \in I$. So $D \cup E \in I$ and $x \in c_0^I(M_k' \circ M_k, A, p)$.

$(ii)$ Let $x \in c_0^I(M_k, A, p) \cap c_0^I(M_k', A, p)$. So, there exists $\rho > 0$ such that

$$B = \left\{k \in \mathbb{N} : \left(M_k\left(\frac{|A_k(x)|}{\rho}\right)\right)^{p_k} \ge \frac{\varepsilon}{2D}\right\} \in I,$$

$$C = \left\{k \in: \left(M_k'\left(\frac{|A_k(x)|}{\rho}\right)\right)^{p_k} \ge \frac{\varepsilon}{2D}\right\} \in I.$$

Let $k \notin B \cup C$. Hence $k \notin \left\{k : \left((M_k + M_k')\left(\frac{|A_k(x)|}{\rho}\right)\right)^{p_k} \ge \varepsilon\right\}$. Therefore $\left\{k : \left((M_k + M_k')\left(\frac{|A_k(x)|}{\rho}\right)\right)^{p_k} \ge \varepsilon\right\} \in I$. This completes the proof. □

**Corollary 2.1.** *Let* $\mathcal{M} = (M_k)$ *be a Musielak-Orlicz functions which satisfies* $\Delta_2 -$ *condition. Then* $W(A, p) \subseteq W(\mathcal{M}, A, p)$ *where* $W = c_0^I, c^I, m_0^I, m^I$.

*Proof.* We can obtain $W(A, p) \subseteq W(\mathcal{M}, A, p)$ from Theorem 2.3 by taking $M_k(x) = x$ and $M_k'(x) = M_k(x)$ for all $x \in [0, \infty)$. □

**Theorem 2.4.** *The spaces* $c_0^I(\mathcal{M}, A, p)$ *and* $m_0^I(\mathcal{M}, A, p)$ *are solid for* $A = I$.

*Proof.* We will prove for the space $c_0^I(\mathcal{M}, A, p)$.
Let $x \in c_0^I(\mathcal{M}, A, p)$. So, for every $\varepsilon > 0$

$$B = \left\{ k \in \mathbb{N} : \left( M_k \left( \frac{|A_k(x)|}{\rho} \right) \right)^{p_k} \geq \varepsilon \right\} \in I(\rho > 0).$$

Let $\alpha = (\alpha_k)$ be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Suppose that $k \notin B$. Therefore, we obtain

$$\begin{aligned} \left[ M_k \left( \tfrac{|A_k(\alpha x)|}{\rho} \right) \right]^{p_k} &= \left[ M_k \left( \tfrac{|I_k(\alpha x)|}{\rho} \right) \right]^{p_k} = \left[ M_k \left( \tfrac{|\alpha_k x_k|}{\rho} \right) \right]^{p_k} \\ &\leq \left[ M_k \left( \tfrac{|x_k|}{\rho} \right) \right]^{p_k} = \left[ M_k \left( \tfrac{|I_k(x)|}{\rho} \right) \right]^{p_k} = \left[ M_k \left( \tfrac{|A_k(x)|}{\rho} \right) \right]^{p_k}. \end{aligned}$$

Hence, $k \notin \left\{ k \in \mathbb{N} : \left( M_k \left( \frac{|A_k(\alpha x)|}{\rho} \right) \right)^{p_k} \geq \varepsilon \right\}$. Therefore, we obtain
$I - \lim_k \left( M_k \left( \frac{|A_k(\alpha x)|}{\rho} \right) \right)^{p_k} = 0.$ □

**Corollary 2.2.** *The spaces $c_0^I(\mathcal{M}, A, p)$ and $m_0^I(\mathcal{M}, A, p)$ are monotone for $A = I$.*

*Proof.* This is clear from Lemma 1.1. □

**Theorem 2.5.** *The spaces $c_0^I(\mathcal{M}, A, p)$ and $c^I(\mathcal{M}, A, p)$ are sequence algebra for $A = I$.*

*Proof.* Let $x, y \in c_0^I(\mathcal{M}, A, p)$ . Then there exists $\rho_1, \rho_2 > 0$ such that for every $\varepsilon > 0$, we have

$$A_1 = \left\{ k \in \mathbb{N} : \left[ M_k \left( \frac{|x_k|}{\rho_1} \right) \right]^{p_k} \geq \frac{\varepsilon}{2D} \right\} \in I,$$

$$A_2 = \left\{ k \in \mathbb{N} : \left[ M_k \left( \frac{|y_k|}{\rho_2} \right) \right]^{p_k} \geq \frac{\varepsilon}{2D} \right\} \in I.$$

Let $\rho = \rho_2 |x_k| + \rho_1 |y_k| > 0$. By using this fact one can see that

$$M_k \left( \frac{|x_k y_k|}{\rho} \right) \leq \frac{\rho_2 |x_k|}{2\rho} M_k \left( \frac{|y_k|}{\rho_2} \right) + \frac{\rho_1 |y_k|}{2\rho} M_k \left( \frac{|y_k|}{\rho_2} \right) < M_k \left( \frac{|y_k|}{\rho_2} \right) + M_k \left( \frac{|y_k|}{\rho_2} \right),$$

which shows that $A_3 = \left\{ k \in \mathbb{N} : \left[ M_k \left( \frac{|x_k y_k|}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I$.
Thus $(x_k y_k) \in c_0^I(M, A, p)$ for $A = I$. □

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