Konuralp Journal of Mathematics
Volume 4 No. 2 pp. 183–192 (2016) ©KJM

$\mathcal{I}$-LIMIT SUPERIOR AND $\mathcal{I}$-LIMIT INFERIOR FOR SEQUENCES OF FUZZY NUMBERS

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Abstract. The statistical limit inferior and limit superior for sequences of fuzzy numbers have been introduced by Aytar, Pehlivan and Mammadov [Statistical limit inferior and limit superior for sequences of fuzzy numbers, Fuzzy Sets and Systems, 157(7) (2006) 976–985]. In this paper, we extend concepts of statistical limit superior and inferior to $\mathcal{I}$-limit superior and $\mathcal{I}$-inferior for a sequence of fuzzy numbers. Also, we prove some basic properties.

1. Introduction

The definition of convergence for sequences of fuzzy numbers has been firstly presented by Matloka [21] and the Cauchy Criterion for sequences of fuzzy numbers is defined by Nanda [22].

The notions of limit superior and limit inferior for a bounded sequence of fuzzy numbers is introduced by Aytar et al. [4]. Afterwards, some properties of these concepts have been obtained by Hong et al. [15], Talo and Çakan [29], Talo [30].

The notion of statistical convergence was defined by Nuray and Savaş [23] for sequences of of fuzzy numbers. Also, Aytar et al. [5] introduced the characterization of statistical limit superior and limit inferior for statistically bounded sequences of fuzzy numbers and proved some fuzzy-analogues of properties of statistical limit superior and limit inferior.

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [16]. Kostyrko et al. [17] and Aytar et al. [6] proved some of basic properties of $\mathcal{I}$-convergence. Also, Demirci [10] presented the notions of $\mathcal{I}$-limit superior and inferior of a real sequence and gave some properties.

Kumar and Kumar [18] studied the concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence and $\mathcal{I}$-Cauchy sequence for sequences of fuzzy numbers. Kumar et al. [19] introduced the concepts of $\mathcal{I}$-limit points and $\mathcal{I}$-cluster points for sequences of fuzzy numbers. Dündar and Talo [11] presented the notions of $\mathcal{I}_2$-convergence, $\mathcal{I}_2^*$-convergence

2010 *Mathematics Subject Classification.* Primary 03E72 ; Secondary 40A35.
Key words and phrases. Fuzzy numbers, sequences of fuzzy numbers, Ideal convergence, Ideal limit superior and inferior.

for double sequences of fuzzy numbers and proved their some properties and relations. Recently, various types of $\mathcal{I}$-convergence for sequences of fuzzy numbers have been studied by many authors [13, 14, 25, 27, 33]

In this paper, we extend the concepts of $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior to fuzzy numbers space and prove several basic properties.

2. Preliminaries, Background and Notation

First, we recall basics of fuzzy numbers.

Let E^1 denote the set of fuzzy subsets of the real line, if $u : \mathbb{R} \to [0,1]$, satisfying the following properties:

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, i.e.,
$u[\lambda x + (1-\lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$;

(iii) u is upper semi-continuous;

(iv) The set $[u]_0 := cl\{x \in \mathbb{R} : u(x) > 0\}$ is compact.

Then u is called a fuzzy number and E^1 is called fuzzy number space. λ-level set $[u]_\lambda$ of $u \in E^1$ is defined by

$$[u]_\lambda := \begin{cases} \{x \in \mathbb{R} : u(x) \geq \lambda\} & , \quad (0 < \lambda \leq 1), \\ \overline{\{x \in \mathbb{R} : u(x) > 0\}} & , \quad (\lambda = 0). \end{cases}$$

Obviously, $[u]_\lambda$ is closed, bounded and non-empty interval for each $\lambda \in [0,1]$ and denoted as $[u]_\lambda := [u^-(\lambda), u^+(\lambda)]$. For any $r \in \mathbb{R}$, define a fuzzy number $\hat{r}$ by

$$\widehat{r}(x) := \begin{cases} 1 & , \quad (x = r), \\ 0 & , \quad (x \neq r), \end{cases}$$

for any $x \in \mathbb{R}$.

Let $u, v, w \in E^1$ and $k \in \mathbb{R}$, the addition, scalar multiplication and product are defined by

$$u + v = w \Longleftrightarrow [w]_\lambda = [u]_\lambda + [v]_\lambda \text{ for all } \lambda \in [0,1]$$

$$[ku]_\lambda = k[u]_\lambda \text{ for all } \lambda \in [0,1]$$

and

$$uv = w \Longleftrightarrow [w]_\lambda = [u]_\lambda [v]_\lambda \text{ for all } \lambda \in [0,1].$$

Let $W = \{A = [A^-, A^+] : A$ is closed bounded intervals on the real line $\mathbb{R}\}$. Define

$$d(A, B) := \max\{|A^- - B^-|, |A^+ - B^+|\}$$

as the metric on W.

Hausdorff metric D between fuzzy numbers defined by

$$D(u, v) = \sup_{\lambda \in [0,1]} d([u]_\lambda, [v]_\lambda) = \sup_{\lambda \in [0,1]} \max\{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\}.$$

The partial ordering relation on E^1 is defined as follows:

$$u \preceq v \Longleftrightarrow [u]_\lambda \preceq [v]_\lambda \Longleftrightarrow u^-(\lambda) \leq v^-(\lambda) \text{ and } u^+(\lambda) \leq v^+(\lambda) \text{ for all } \lambda \in [0,1].$$

$u \prec v$ means $u \preceq v$ and at least one of $u^-(\alpha) < v^-(\alpha)$ and $u^+(\alpha) < v^+(\alpha)$ holds for some $\alpha \in [0,1]$.

Two fuzzy numbers u and v are said to be incomparable if neither $u \preceq v$ nor $v \preceq u$ holds. In this case we write $u \not\sim v$.

Combining the results of Lemma 6 in [5], Lemma 5 in [3], Lemma 3.4, Theorem 4.9 in [20] and Lemma 14 in [31], following Lemma is obtained.

Lemma 2.1. *Let $u, v, w, e \in E^1$ and $\hat{\varepsilon} > 0$. The following statements hold:*

(i) $D(u,v) \leq \varepsilon$ *if and only if* $u - \hat{\varepsilon} \preceq v \preceq u + \hat{\varepsilon}$
(ii) *If* $u \preceq v + \hat{\varepsilon}$ *for every* $\varepsilon > 0$, *then* $u \preceq v$.
(iii) *If* $u \preceq v$ *and* $v \preceq w$, *then* $u \preceq w$
(iv) *If* $u \prec v$, $v \preceq w$, *then* $u \prec w$.
(v) *If* $u \preceq w$ *and* $v \preceq e$, *then* $u + v \preceq w + e$.
(vi) *if* $u \prec w$ *and* $v \preceq e$, *then* $u + v \prec w + e$.
(vii) *If* $u \succeq \overline{0}$ *and* $v \succ w$, *then* $uv \succeq uw$.
(viii) *If* $u + w \preceq v + w$ *then* $u \preceq v$.

Wu and Wu [28] defined boundness of a set of fuzzy numbers according to relation $\preceq$ and proved that if a set A of E^1 is bounded, then supremum and infimum of A exist.

We denote the set of all sequences of fuzzy numbers by $w(F)$.

A sequence $(u_n) \in w(F)$ is called convergent with limit $u \in E^1$, if and only if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$D(u_n, u) < \varepsilon \text{ for all } n \geq n_0.$$

A sequence (u_n) of fuzzy numbers is said to be bounded if there exists $M > 0$ such that $D(u_n, \hat{0}) \leq M$ for all $n \in \mathbb{N}$. By $\ell_\infty(F)$, we denote the set of all bounded sequences of fuzzy numbers.

The statistical convergence of sequences of fuzzy numbers defined as follows:

For a subset K of natural numbers $\mathbb{N}$, the natural density of K is given by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|$$

if this limit exists, where $|A|$ denotes the number of elements in A.

A sequence $u = (u_k)$ of fuzzy numbers is said to be statistically convergent to some fuzzy number μ_0, if for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{n} \big|\{k \leq n : D(u_k, \mu_0) \geq \varepsilon\}\big| = 0.$$

The statistical boundedness of a sequence of fuzzy numbers was introduced and studied by Aytar and Pehlivan [3]. The sequence $u = (u_k)$ is said to be statistically bounded if there exists a real number M such that the set $\{k \in \mathbb{N} : D(u_k, \overline{0}) > M\}$ has natural density zero.

Aytar et al. [5] defined the concepts of statistical limit superior and limit inferior of statistically bounded sequences of fuzzy numbers.

Let $u = (u_k)$ be statistically bounded and let us define the following sets:

$$A_u = \left\{\mu \in E^1 : \delta\left(\{k \in \mathbb{N} : u_k \prec \mu\}\right) \neq 0\right\},$$

$$\overline{A}_u = \left\{\mu \in E^1 : \delta\left(\{k \in \mathbb{N} : u_k \succ \mu\}\right) = 1\right\},$$

$$B_u = \left\{\mu \in E^1 : \delta(\{k \in \mathbb{N} : u_k \succ \mu\}) \neq 0\right\},$$

$$\overline{B}_u = \left\{\mu \in E^1 : \delta(\{k \in \mathbb{N} : u_k \prec \mu\}) = 1\right\}.$$

The statistical limit superior and limit inferior are defined as follows:

$$\text{st}-\liminf u_k = \inf A_u = \sup \overline{A}_u,$$
$$\text{st}-\limsup u_k = \sup B_u = \inf \overline{B}_u.$$

For more result on sequences of fuzzy numbers we refer to [1, 2, 7, 9, 26, 32] and [8, Section 8].

Now, we recall the concept of ideal and ideal convergence of sequences of fuzzy numbers.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in \mathcal{I}$,
(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathcal{F}$,
(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Lemma 2.2. [16] *If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Lemma 2.3. [24, Lemma 2.5] *$K \in F(\mathcal{I})$ and $M \subseteq \mathbb{N}$. If $M \notin \mathcal{I}$ then $M \cap K \notin \mathcal{I}$.*

Throughout this paper we take $\mathcal{I}$ as a nontrivial admissible ideal in $\mathbb{N}$.

Definition 2.1. Let $u = (u_n)$ be a sequences of fuzzy numbers.

(i)[18] $u = (u_n)$ is said to be $\mathcal{I}$-convergent to a fuzzy number u_0, if for any $\varepsilon > 0$ we have

$$A(\varepsilon) = \{n \in \mathbb{N} : D(u_n, u_0) \geq \varepsilon\} \in \mathcal{I}.$$

In this case we say that u is $\mathcal{I}$-convergent and we write $\mathcal{I} - \lim_{n\to\infty} u_n = u_0$.

(ii)[19] The fuzzy number μ is said to be $\mathcal{I}$-limit point of $u = (u_n)$ if there exits a subset $K = \{k_1 < k_2 < k_3 < \cdots\} \notin \mathcal{I}$ such that $\lim_{n\to\infty} u_{k_n} = \mu$. The set of all $\mathcal{I}$-limit points of the sequence $u = (u_n)$ will be denoted by $\mathcal{I}(\Lambda_u)$.

(iii)[19] The fuzzy number μ is said to be the $\mathcal{I}$-cluster point of $u = (u_n)$ if for each $\varepsilon > 0$, $\{n \in \mathbb{N} : D(u_n, \mu) < \varepsilon\} \notin \mathcal{I}$. The set of all $\mathcal{I}$-cluster points of the fuzzy number sequence $u = (u_n)$ will be denoted by $\mathcal{I}(\Gamma_u)$.

The propose of this paper is to present the notions of ideal limit superior and inferior for a sequence of fuzzy numbers and give some ideal analogues of properties of the statistical limit superior and inferior of sequences of fuzzy numbers.

3. The Main Results

Definition 3.1. $u = (u_k) \in w(F)$ is said to be *$\mathcal{I}$-bounded above* if there exists a fuzzy number μ such that

$$\{k \in \mathbb{N} : u_k \succ \mu\} \cup \{k \in \mathbb{N} : u_k \not\sim \mu\} \in \mathcal{I}.$$

Similarly, $u = (u_k)$ is said to be *$\mathcal{I}$-bounded below* if there exists a fuzzy number ν such that

$$\{k \in \mathbb{N} : u_k \prec \nu\} \cup \{k \in \mathbb{N} : u_k \not\sim \nu\} \in \mathcal{I}.$$

If $u = (u_k)$ is both $\mathcal{I}$-bounded above and below, then it is said to be $\mathcal{I}$-bounded.

This definition can be stated as follows:

$u = (u_k) \in w(F)$ is said to be $\mathcal{I}$-bounded if there is a real number M such that

$$\{k \in \mathbb{N} : D(u_k, \hat{0}) > M\} \in \mathcal{I}.$$

Since $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$, if $u = (u_k)$ is bounded, then u is $\mathcal{I}$-bounded.

We give a generalization of notions of $st-\liminf u$ and $st-\limsup u$ of a sequence $u = (u_k)$ of [5]. Given $\mathcal{I}$-bounded sequence $u = (u_k) \in w(F)$, we define the following sets:

$$\begin{aligned} A_u &= \left\{\mu \in E^1 : \{k \in \mathbb{N} : u_k \prec \mu\} \notin \mathcal{I}\right\}, \\ \overline{A}_u &= \left\{\mu \in E^1 : \{k \in \mathbb{N} : u_k \succ \mu\} \in \mathcal{F}(\mathcal{I})\right\}, \\ B_u &= \left\{\mu \in E^1 : \{k \in \mathbb{N} : u_k \succ \mu\} \notin \mathcal{I}\right\}, \\ \overline{B}_u &= \left\{\mu \in E^1 : \{k \in \mathbb{N} : u_k \prec \mu\} \in \mathcal{F}(\mathcal{I})\right\}. \end{aligned}$$

It is evident that if the sequence $u = (u_k)$ is $\mathcal{I}$-bounded, then the sets $A_u, \overline{A}_u, B_u$ and $\overline{B}_u$ are non-empty. It is also evident that the sets A_u and $\overline{B}_u$ have lower bounds, and the sets $\overline{A}_u$ and B_u have upper bounds. Hence, we obtain that $\inf A_u$, $\sup \overline{A}_u$, $\sup B_u$ and $\inf \overline{B}_u$ exist.

Now, we prove the main results in line of Theorem 2, Theorem 3, Theorem 5 and Theorem 7 in [5]. Our proofs are similar to those in [5].

Theorem 3.1. *If $u = (u_k) \in w(F)$ is $\mathcal{I}$-bounded, then $\inf A_u = \sup \overline{A}_u$ and $\sup B_u = \inf \overline{B}_u$.*

Proof. We prove only for $\inf A_u = \sup \overline{A}_u$. Denote $\nu := \inf A_u$ and $\mu := \sup \overline{A}_u$. Then, we have $\nu \preceq \widetilde{\nu}$ for all $\widetilde{\nu} \in A_u$, and $\mu \succeq \widetilde{\mu}$ for all $\widetilde{\mu} \in \overline{A}_u$. Since $\widetilde{\nu} \in A_u$, $\{k \in \mathbb{N} : u_k \prec \widetilde{\nu}\} \notin \mathcal{I}$. On the other hand, from $\widetilde{\mu} \in \overline{A}_u$, we have $\{k \in \mathbb{N} : u_k \succ \widetilde{\mu}\} \in \mathcal{F}(\mathcal{I})$. Therefore,

$$\{k \in \mathbb{N} : u_k \prec \widetilde{\nu}\} \cap \{k \in \mathbb{N} : u_k \succ \widetilde{\mu}\} \notin \mathcal{I}$$

that is, $\{k \in \mathbb{N} : u_k \prec \widetilde{\nu}\} \cap \{k \in \mathbb{N} : u_k \succ \widetilde{\mu}\} \neq \emptyset$. Then, there is a number $k \in \mathbb{N}$ such that $\widetilde{\mu} \prec u_k \prec \widetilde{\nu}$. This implies that

$$\widetilde{\mu} \prec \widetilde{\nu} \ \text{ for all } \ \widetilde{\nu} \in A_u, \ \widetilde{\mu} \in \overline{A}_u. \tag{3.1}$$

From (3.1), it is immediate that $\widetilde{\mu}$ is a lower bound of the set A_u. Then, we have $\widetilde{\mu} \preceq \nu = \inf A_u$. This inequality is valid for all $\widetilde{\mu} \in \overline{A}_u$. Then, we get $\mu \preceq \nu$. Now, we show that the case $\mu \prec \nu$ is impossible.

To the contrary, assume that $\mu \prec \nu$. This means that, there is a number $\alpha \in [0, 1]$ such that

$$\mu^-(\alpha) < \nu^-(\alpha) \ \text{ or } \ \mu^+(\alpha) < \nu^+(\alpha).$$

Without of loss of generality, we take into account the case $\mu^-(\alpha) < \nu^-(\alpha)$ and show that it leads to a contradiction.

Denote $b := \nu(\mu^-(\alpha))$. It is obvious that $b < \alpha$ (b may be zero). Furthermore, the inequality $\mu^-(\lambda) < \nu^-(\lambda)$ holds, for all $\lambda \in (b, \alpha]$. Since the functions $\mu(x)$ and

$\nu(x)$ are upper semi-continuous, there is a point (z,β) such that $z\in(\mu^-(\alpha),\nu^-(\alpha))$, $\beta\in(b,\alpha)$ and

$$\mu^-(\lambda)<z,\ \nu^-(\lambda)>z \text{ for all } \lambda\in[\beta,\alpha]. \tag{3.2}$$

We define the numbers $\gamma_1,\gamma_2\in E^1$ by

$$\gamma_1(t):=\begin{cases}0 & , \quad t<t^-(0),\\ \beta & , \quad t\in[t^-(0),z],\\ 1 & , \quad t=z,\\ 0 & , \quad t>z,\end{cases} \quad\text{and}\quad \gamma_2(t):=\begin{cases}0 & , \quad t<z,\\ \beta & , \quad t\in[z,t^+(0)],\\ 1 & , \quad t=t^+(0),\\ 0 & , \quad t>t^+(0),\end{cases}$$

where the numbers $t^-(0)=\mathcal{I}-\liminf u_k^-(0)-1$ and $t^+(0)=\mathcal{I}-\limsup u_k^+(0)+1$ are finite.

From (3.2), it is easily seen that

$$\begin{aligned}\mu^-(\beta) &\geq \mathcal{I}-\liminf u_k^-(\beta)\geq\mathcal{I}-\liminf u_k^-(0)>t^-(0)=\gamma_1^-(\beta),\\ &\mu^-(\alpha)<z=\gamma_1^-(\alpha)\end{aligned}$$

and

$$\nu^-(b)\leq\mu^-(\alpha)<z=\gamma_2^-(b),\ \nu^-(\beta)>z=\gamma_2^-(\beta).$$

This means that

$$\mu\not\succ\gamma_1 \text{ and } \nu\not\prec\gamma_2. \tag{3.3}$$

Let

$$C_1:=\left\{k\in\mathbb{N}:u_k^-(\lambda)\leq z \text{ for some } \lambda\in(\beta,\alpha]\right\},$$

$$C_2:=\left\{k\in\mathbb{N}:u_k^-(\lambda)\geq z \text{ for some } \lambda\in(\beta,\alpha]\right\}.$$

Clearly, we have

$$C_1\cup C_2=\mathbb{N}. \tag{3.4}$$

First we assume that $C_1\notin\mathcal{I}$. Considering γ_2 and $t^+(0)$, we have

$$u_k\prec\gamma_2, \quad \text{for all} \quad k\in C_1\backslash K_1,$$

where $K_1:=\left\{k\in\mathbb{N}:u_k^+(\lambda)>t^+(0), \text{ for some } \lambda\in[0,1]\right\}$. This means that

$$\{k\in\mathbb{N}:u_k\prec\gamma_2\}\supseteq C_1\setminus K_1.$$

It is evident that $K_1\in\mathcal{I}$ and $C_1\setminus K_1\notin\mathcal{I}$. For this reason, $\{k\in\mathbb{N}:u_k\prec\gamma_2\}\notin\mathcal{I}$. This means that $\gamma_2\in A_u$ and therefore, from the definition of $\inf A_u$ we get $\gamma_2\succeq\nu=\inf A_u$. This contradicts to (3.3), that is, $\nu\not\prec\gamma_2$.

Hence, we have shown that $C_1\in\mathcal{I}$. In this case, from (3.4), it follows that the set $C_2\in\mathcal{F}(\mathcal{I})$. Considering γ_1 and $t^-(0)$, we have

$$u_k\succ\gamma_1 \text{ for all } k\in C_2\backslash(C_1\cup K_2),$$

where $K_2:=\left\{k\in\mathbb{N}:u_k^-(\lambda)<t^-(0) \text{ for some } \lambda\in[0,\beta]\right\}$. This means that

$$\{k\in\mathbb{N}:u_k\succ\gamma_1\}\supseteq C_2\backslash(C_1\cup K_2).$$

It is obvious that the set $K_2\in\mathcal{I}$ and consequently we have $C_2\backslash(C_1\cup K_2)\in\mathcal{F}(\mathcal{I})$. Therefore

$$\{k\in\mathbb{N}:u_k\succ\gamma_1\}\in\mathcal{F}(\mathcal{I}).$$

This implies that $\gamma_1\in\overline{A}_u$. Thus, $\gamma_1\preceq\mu=\sup\overline{A}_u$. This contradicts to (3.3), that is, $\mu\not\succ\gamma_1$. This completes the proof. □

Definition 3.2. If $u = (u_k)$ is a $\mathcal{I}$-bounded sequence of fuzzy numbers, then

$$\mathcal{I} - \liminf u_k := \inf A_u,$$

and

$$\mathcal{I} - \limsup u_k := \sup B_u.$$

Example 3.1. We will give some example of ideals.

1. Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$ limit superior and inferior coincides with the ordinary limit superior and inferior of sequences of fuzzy numbers [4],[15].
2. Let $\mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$ where $\delta(A)$ denotes the natural density of the set A. Then $\mathcal{I}_\delta$ is a non-trivial admissible ideal and $\mathcal{I}_\delta$ limit superior and inferior coincides with the statistical limit superior and inferior of sequences of fuzzy numbers [5].
3. A set $K \subset \mathbb{N}$ has C-density if $\delta_C(K) := \lim_{n\to\infty} \sum_{k\in K} c_{nk}$ exists, where $C = (c_{nk})$ is a non-negative regular matrix [12]. If $\mathcal{I}_{\delta_C} = \{A \subset \mathbb{N} : \delta_C(A) = 0\}$, then $\mathcal{I}_{\delta_C}$ is a non-trivial admissible ideal and $\mathcal{I}_{\delta_C}$ limit superior and inferior coincides with the C-statistical limit superior and inferior of sequences of fuzzy numbers, which is also mentioned in [5].

Theorem 3.2. *For any $\mathcal{I}$-bounded sequence of fuzzy numbers $u = (u_k)$,*

$$\mathcal{I} - \liminf u \preceq \mathcal{I} - \limsup u.$$

Proof. Let $\mu \in \overline{A}_u$. Then $\{k : u_k \succ \mu\} \in \mathcal{F}(\mathcal{I})$. Since $\mathcal{I}$ is a nontirvial ideal of $\mathbb{N}$, we get $\{k : u_k \succ \mu\} \notin \mathcal{I}$. Therefore $\mu \in B_u$. This implies $\overline{A}_u \subseteq B_u$. Hence $\sup \overline{A}_u \preceq \sup B_u$. This means that $\mathcal{I} - \liminf u \preceq \mathcal{I} - \limsup u$. □

Since $\mathcal{I}$ is an admissible ideal, the inclusion $\mathcal{I}_f \subset \mathcal{I}$ holds. Therefore, the inequalities

$$\operatorname{Lim\,inf} u \preceq \mathcal{I} - \liminf u \preceq \mathcal{I} - \limsup u \preceq \operatorname{Lim\,sup} u$$

hold for every bounded sequence (u_k) of fuzzy numbers.

Theorem 3.3. *Let $u = (u_k)$ be a $\mathcal{I}$-bounded sequence of fuzzy numbers.*

(i) If $\nu := \mathcal{I} - \liminf u_k$, then

$$\{k \in \mathbb{N} : u_k \prec \nu - \hat{\varepsilon}\} \in \mathcal{I},\ \{k \in \mathbb{N} : u_k \prec \nu + \hat{\varepsilon}\} \cup \{k \in \mathbb{N} : u_k \not\sim \nu + \hat{\varepsilon}\} \notin \mathcal{I} \tag{3.5}$$

for every $\varepsilon > 0$.

(ii) If $\mu := \mathcal{I} - \limsup u_k$, then

$$\{k \in \mathbb{N} : u_k \succ \mu + \hat{\varepsilon}\} \in \mathcal{I} \text{ and } \{k \in \mathbb{N} : u_k \succ \mu - \hat{\varepsilon}\} \cup \{k \in \mathbb{N} : u_k \not\sim \mu - \hat{\varepsilon}\} \notin \mathcal{I}$$

for every $\varepsilon > 0$.

Proof. We prove (i). To the contrary, we assume that there exists $\varepsilon > 0$ such that $\{k \in \mathbb{N} : u_k \prec \nu - \hat{\varepsilon}\} \notin \mathcal{I}$. This means that $\nu - \hat{\varepsilon} \in A_u$. Since $\nu = \inf A_u$, we get $\nu \preceq \nu - \hat{\varepsilon}$ which is a contradiction.

Now, let us show that (3.5) holds. Suppose that it is not true, that is, there exists $\varepsilon > 0$ such that

$$\{k \in \mathbb{N} : u_k \prec \nu + \hat{\varepsilon}\} \in \mathcal{I} \text{ and } \{k \in \mathbb{N} : u_k \not\sim \nu + \hat{\varepsilon}\} \in \mathcal{I}.$$

For each $k \in \mathbb{N}$, only the following three cases are possible: $u_k \prec \nu + \hat{\varepsilon}$, $u_k \not\sim \nu + \hat{\varepsilon}$ and $u_k \succeq \nu + \hat{\varepsilon}$. Then,

$$\{k \in \mathbb{N} : u_k \prec \nu + \hat{\varepsilon}\} \cup \{k \in \mathbb{N} : u_k \not\sim \nu + \hat{\varepsilon}\} \cup \{k \in \mathbb{N} : u_k \succeq \nu + \hat{\varepsilon}\} = \mathbb{N}.$$

Thus, from (3.6), we have $\{k \in \mathbb{N} : u_k \succeq \nu+\hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I})$. This means that $\nu+\hat{\varepsilon} \in \overline{A}_u$. Hence, we can write $\nu + \hat{\varepsilon} \preceq \sup \overline{A}_u = \nu$, which is a contradiction. □

Theorem 3.4. *If $u = (u_k) \in w(F)$ is $\mathcal{I}$ convergent to μ, then*

$$\mathcal{I} - \liminf u_k = \mathcal{I} - \limsup u_k = \mu.$$

Proof. First suppose that $\mathcal{I} - \lim u_k = \mu$ and $\varepsilon > 0$. Then, $\{k \in \mathbb{N} : D(x_k, \mu) \geq \varepsilon\} \in \mathcal{I}$, so we have $\{k \in \mathbb{N} : D(x_k, \mu) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$. By Lemma 2.1, we get $\{k \in \mathbb{N} : \mu - \hat{\varepsilon} \prec u_k \prec \mu + \hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I})$,
$\{k \in \mathbb{N} : \mu - \hat{\varepsilon} \prec u_k\} \cap \{k \in \mathbb{N} : u_k \prec \mu + \hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I})$. Therefore,

1) $\{k \in \mathbb{N} : \mu - \hat{\varepsilon} \prec u_k\} \in \mathcal{F}(\mathcal{I})$. This means that $\mu - \hat{\varepsilon} \in \overline{A}_u$. Then, $\mathcal{I} - \liminf u_k = \sup \overline{A}_u \succeq \mu - \hat{\varepsilon}$.
2) $\{k \in \mathbb{N} : u_k \prec \mu + \hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I})$. This means that $\mu + \hat{\varepsilon} \in \overline{B}_u$. Then, $\mathcal{I} - \limsup u_k = \inf \overline{B}_u \preceq \mu + \hat{\varepsilon}$.

By these inequalities and Theorem 3.4, we obtain

$$\mu - \hat{\varepsilon} \preceq \mathcal{I} - \liminf u_k \preceq \mathcal{I} - \limsup u_k \preceq \mu + \hat{\varepsilon}. \tag{3.6}$$

Since $\varepsilon > 0$ is an arbitrary, we obtain $\mathcal{I} - \liminf u_k = \mathcal{I} - \limsup u_k = \mu$. □

Example 3.2. We decompose the set $\mathbb{N}$ into countably many disjoint sets

$$N_p = \{2^{p-1}(2k-1) : k \in \mathbb{N}\}, \ (j = 1, 2, 3, ...).$$

It is obvious that $\mathbb{N} = \bigcup_{p=1}^{\infty} N_p$ and $N_i \cap N_j = \emptyset$ for $i \neq j$. Denote by $\mathcal{I}$ the class of all $A \subseteq \mathbb{N}$ such that A intersects only a finite number of N_p. It is easy to see that $\mathcal{I}$ is an admissible ideal. Define (u_n) as follows: for $n \in N_p$ we put $u_n = v_p$ $(p = 1, 2, 3, ...)$, where

$$v_p(x) := \begin{cases} 1 - px & , \ \text{if } 0 \leq x \leq \frac{1}{p}, \\ 0 & , \ \text{otherwise.} \end{cases}$$

Then, for $n \in N_p$, $D(u_n, \hat{0}) = 1/p$ $(p = 1, 2, 3, \dots)$. Then, obviously $\mathcal{I}-\lim D(u_n, \hat{0}) = 0$ that is $\mathcal{I} - \lim u_n = \hat{0}$.

Now, consider the ideal $\mathcal{I}_\delta$. It can be easily shown that the natural density of N_p is $\delta(N_p) = 1/2^p$ $(p = 1, 2, 3, ...)$. Then, it is clear that $a \in \overline{A_u}$ for each $a \in E^1$ with $a \preceq \hat{0}$ and $b \in \overline{B_u}$ for each with $b \in E^1$ with $b \succ v_1$. So, we obtain

$$\mathcal{I}_\delta - \liminf u = \hat{0} \text{ and } \mathcal{I}_\delta - \limsup u = v_1.$$

The converse of Theorem 3.4 is not valid in general as shown Example 2 in [5]. The following theorem gives a sufficient condition for a sequence of fuzzy numbers to be $\mathcal{I}-$onvergent.

Theorem 3.5. *Assume that $\mathcal{I} - \limsup u_k = \mathcal{I} - \liminf u_k = \mu$ and there is a number $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ the sets $\{k \in \mathbb{N} : u_k \not\sim \mu + \hat{\varepsilon}\}$ and $\{k \in \mathbb{N} : u_k \not\sim \mu - \hat{\varepsilon}\}$ belong to $\mathcal{I}$. Then, we have $\mathcal{I} - \lim u_k = \mu$.*

Proof. Take any number $\varepsilon \in (0, \varepsilon_0)$. Since $\mathcal{I} - \liminf x_k = \mathcal{I} - \limsup x_k = \mu$, by Theorem 3.3 we have

$$\{k \in \mathbb{N} : u_k \prec \mu - \hat{\varepsilon}\} \in \mathcal{I} \text{ and } \{k \in \mathbb{N} : u_k \succ \mu + \hat{\varepsilon}\} \in \mathcal{I},$$

for all $\varepsilon > 0$. From $\{k \in \mathbb{N} : u_k \not\prec \mu - \hat{\varepsilon}\} \in \mathcal{I}$ and $\{k \in \mathbb{N} : u_k \not\prec \mu + \hat{\varepsilon}\} \in \mathcal{I}$, we conclude that

$$\{k \in \mathbb{N} : u_k \preceq \mu + \hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I}) \text{ and } \{k \in \mathbb{N} : u_k \succeq \mu - \hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I}).$$

By Lemma 2.1, we obtain $\{k \in \mathbb{N} : u_k \preceq \mu + \hat{\varepsilon}\} \cap \{k \in \mathbb{N} : u_k \succeq \mu - \hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I})$, $\{k \in \mathbb{N} : \mu - \hat{\varepsilon} \preceq u_k \preceq \mu + \hat{\varepsilon}\} \in \mathcal{F}(\mathcal{I})$, $\{k \in \mathbb{N} : D(u_k, \mu) \leq \varepsilon\} \in \mathcal{F}(\mathcal{I})$. Therefore, $\{k \in \mathbb{N} : D(u_k, \mu) > \varepsilon\} \in \mathcal{I}$. Since $\varepsilon > 0$ is an arbitrary number, we conclude that $\mathcal{I} - \lim u_k = \mu$. □

The proofs of following theorems are clear and omitted.

Theorem 3.6. *If $u = (u_k)$ and $v = (v_k)$ are $\mathcal{I}$-bounded sequences of fuzzy numbers such that $\{k \in \mathbb{N} : u_k \neq v_k\} \in \mathcal{I}$, then we have:*

(i) $\mathcal{I} - \limsup u_k = \mathcal{I} - \limsup v_k$,
(ii) $\mathcal{I} - \liminf u_k = \mathcal{I} - \liminf v_k$.

Theorem 3.7. *Let $u = (u_k) \in w(F)$ be $\mathcal{I}$-bounded from above. Assume that $\mathcal{I} - \limsup u_k = \mu$ and there is a number $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, the sets*

$$\{k \in \mathbb{N} : u_k \not\prec \mu + \hat{\varepsilon}\} \text{ and } \{k \in \mathbb{N} : u_k \not\prec \mu - \hat{\varepsilon}\}$$

belong to $\mathcal{I}$. Then, $\mu \in \mathcal{I}(\Gamma_u)$.

Theorem 3.8. *Let $u = (u_k) \in w(F)$ be $\mathcal{I}$-bounded from below. Assume that $\mathcal{I} - \liminf u_k = \nu$ and there exists a number $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, the sets*

$$\{k \in \mathbb{N} : u_k \not\prec \nu + \hat{\varepsilon}\} \text{ and } \{k \in \mathbb{N} : u_k \not\prec \nu - \hat{\varepsilon}\}$$

belong to $\mathcal{I}$. Then, $\nu \in \mathcal{I}(\Gamma_u)$.

Theorem 3.9. *Let $u = (u_k) \in w(F)$ be $\mathcal{I}$-bounded. If $\gamma \in \mathcal{I}(\Gamma_u)$, then $\mathcal{I}$-$\liminf u \preceq \gamma \preceq \mathcal{I} - \limsup u$.*

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