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QUADRATIC FORMULAS FOR GENERALIZED QUATERNIONS

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ABSTRACT

In this paper, we aim to find basic methods for calculation of th eroots of a generalized quaternionic quadratic polynomial.

Keywords: *Generalizedquaternion, quadratic form.*

GENELLEŞTİRİLMİŞ KUATERNİYONLAR İÇİN KUADRATİK FORMÜLLER

ÖZET

Bu makalede, bir genelleştirilmiş kuaterniyonik kuadratik polinomun köklerini bulmak için temel yöntemleri bulmayı amaçlamaktayız.

1. INTRODUCTION

Let ℝ be the set of real numbers and $K_{\alpha,\beta}$ be the set of generalized quaternions of the form $q = q_0 + q_1 i + q_2 j + q_3 k$ where set $q_0, q_1, q_2, q_3, \alpha, \beta \in \mathbb{R}$ and

$$i^2 = -\alpha,\ \ j^2 = -\beta,\ \ k^2 = -\alpha\beta$$
$$ij = -ji = k$$
$$jk = -kj = \beta i$$
$$ki = -ik = \alpha j.$$

For $q = q_0 + q_1 i + q_2 j + q_3 k$, the conjugate of q is $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$. Then norm, real part and imaginary part of q are defined as $N_q = q\overline{q} = q_0{}^2 + \alpha q_1{}^2 + \beta q_2{}^2 + \alpha\beta q_3{}^2$, $\operatorname{Re} q = \left(q + \overline{q}\right)/2 = q_0$ and $\operatorname{Im} q = q - \operatorname{Re} q = q_1 i + q_2 j + q_3 k$, respectively. For $q, p \in K_{\alpha,\beta}$, we say that q is similar to p if there is a nonzero $a \in K_{\alpha,\beta}$ such that $q = a^{-1}pa$ or equivalently $\operatorname{Re} q = \operatorname{Re} p$ and $|q| = |p|$. For the basics of generalized quaternions, see [1].

In this paper, we are interested in explicit formulas for computing the roots of a quadratic polynomial of the form

$$x^2 + bx + c$$

where $b, c \in K_{\alpha,\beta}$. Let $x = x_0 + x_1 i + x_2 j + x_3 k$, $b = b_0 + b_1 i + b_2 j + b_3 k$ and $c = c_0 + c_1 i + c_2 j + c_3 k$. Then

$x^2 + bx + c = 0$

becomes the real system of nonlinear equations

$$x_0{}^2 - \alpha x_1{}^2 - \beta x_2{}^2 - \alpha\beta x_3{}^2 + b_0 x_0 - \alpha b_1 x_1 - \beta b_2 x_2 - \alpha\beta b_3 x_3 + c_0 = 0$$
$$2x_0 x_1 + b_0 x_1 + b_1 x_0 + \beta b_2 x_3 - \beta b_3 x_2 + c_1 = 0$$

$$2x_0x_2 + b_0x_2 + b_2x_0 + \alpha b_1x_3 - \alpha b_3x_1 + c_2 = 0$$
$$2x_0x_3 + b_0x_3 + b_3x_0 + b_1x_2 - b_2x_1 + c_3 = 0.$$

It is not obvious at all that this nonlinear system will have an explicitsolution. By solving a real linear system, Zhangand Mu proposed to compute some roots of a quadratic polynomial in [2]. But, they did not discuss how to find all the roots. In [3], Porter reduced solving a quadratic polynomial to a linear polynomial of the form $px + xp + r$ provided a root of the given quadratic polynomial is already known. However, he did not discuss how to find such root. In [4], given determined how many roots a quadratic polynomial can have, but he did not give the explicit formulas for computing the roots. In Section 2, we adopt the idea in [5] of Huang and So to compute the roots of a quadratic polynomial using explicit formulas in terms of itscoe_cients.Then, we discuss some consequences and two applications of the generalized quaternionic quadratic formulas.

2. GENERALIZED QUATERNIONIC QUADRATIC FORMULAS

Firstly, we solve the monic standard quadratic equation

$$x^2 + bx + c = 0$$

where $b, c \in K_{\alpha,\beta}$. Now, we give two well-known lemmas about solutions of some special polynomials without their proofs.

Lemma2.1. Let B, E, and$D$ be real numbers such that

i. $D \neq 0,$ and
ii. $B< 0$ implies $B^2 < 4E$.

Then the cubic equation

$$y^3 + 2By^2 + \left(B^2 - 4E\right)y - D^2 = 0$$

Has exactly one positive solution y.

Lemma2.2. Let$B$, E, and$D$ be real numbers such that

i. $E> 0$, and
ii. $B< 0$ implies $B^2 < 4E$.

Then the real system

$$N^2 - \left(B + T^2\right)N + E = 0$$
$$T^3 + \left(B - 2N\right)T + D = 0$$

has at most two solutions (T, N) satisfying $T \in \mathbb{R}$ and$N> 0$ as follows.

a. $T = 0, N = \left(B \pm \sqrt{B^2 - 4E}\right)/2$ provided that $D = 0, B^2 \geq 4E.$
b. $T = \pm\sqrt{2\sqrt{E} - B}, N = \sqrt{E}$ provided that $D = 0, B^2 < 4E.$
c. $T = \pm\sqrt{z}, N = \left(T^3 + BT + D\right)/2T$ provided that $D \neq 0$ and z is the unique positive root of the real polynomial $z^3 + 2Bz^2 + \left(B^2 - 4E\right)z - D^2.$

Theorem2.1. The solutions of the quadratic equation $x^2 + bx + c = 0$ can be obtained by formulas according to the following cases:

Case 1. If $b, c \in \mathbb{R}$ and $b^2 < 4c,$ then

$$x = \frac{1}{2}\left(-b + x_1i + x_2j + x_3k\right)$$

where $\alpha x_1{}^2 + \beta x_2{}^2 + \alpha\beta x_3{}^2 = \alpha\left(4c - b^2\right)$ and $x_1, x_2, x_3, \alpha, \beta \in \mathbb{R}$.

Case 2. If $b, c \in \mathbb{R}$ and $b^2 \geq 4c$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Case 3. If $b \in \mathbb{R}$, $c \notin \mathbb{R}$ then

$$x = -\frac{b}{2} \pm \frac{\rho}{2} \mp \frac{c_1}{\rho} i \mp \frac{c_2}{\rho} j \mp \frac{c_3}{\rho} k$$

where $c = c_0 + c_1 i + c_2 j + c_3 k$ and $\rho = \sqrt{\left(b^2 - 4c_0 \pm \sqrt{\left(b^2 - 4c_0\right)^2 + 16\left(\alpha c_1^{\,2} + \beta c_2^{\,2} + \alpha\beta c_3^{\,2}\right)}\right) / 2}.$

Case 4. If $b \notin \mathbb{R}$ then

$$x = \frac{-\operatorname{Re} b}{2} - \left(b' + T\right)^{-1}\left(c' - N\right),$$

where $b' = \operatorname{Im} b, c' = c - \frac{\operatorname{Re} b}{2}\left(b - \frac{\operatorname{Re} b}{2}\right)$ and (T, N) is chosen as follows.

1. $T = 0, N = \left(B \pm \sqrt{B^2 - 4E}\right) / 2$ provided that $D = 0, B^2 \geq 4E.$
2. $T = \pm\sqrt{2\sqrt{E} - B}, N = \sqrt{E}$ provided that $D = 0, B^2 < 4E.$
3. $T = \pm\sqrt{z}, N = \left(T^3 + BT + D\right) / 2T$ provided that $D \neq 0$ and z is the unique positive root of the real polynomial $z^3 + 2Bz^2 + \left(B^2 - 4E\right)z - D^2$,

where $B = b'\overline{b'} + \operatorname{Re} c'$, $E = c'\overline{c'}$ and $D = 2\operatorname{Re}\overline{b'}c'$.

Proof:

Case 1. $b, c \in \mathbb{R}$ and $b^2 < 4c$. Note that x_0 is a solution if and only if $q^{-1} x_0 q$ is also a solution for $q \neq 0$, and there are at least two complex solutions

$$\frac{-b \pm \sqrt{4c - b^2}\, i}{2}.$$

Hence, thesolution set is

$$\left\{ q^{-1} \frac{-b \pm \sqrt{4c - b^2}\, i}{2} q : q \neq 0 \right\} = \left\{ \frac{1}{2}\left(-b + x_1 i + x_2 j + x_3 k\right) : \alpha x_1^{\,2} + \beta x_2^{\,2} + \alpha\beta x_3^{\,2} = \alpha\left(4c - b^2\right) \right\}.$$

Case 2. $b, c \in \mathbb{R}$ and $b^2 \geq 4c$. Note that x_0 is a solution if and onlyif $q^{-1} x_0 q$ is also a solution for $q \neq 0$, and hence, there are at most two solutions, both are real

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Case 3. $b \in \mathbb{R}$, $c \notin \mathbb{R}$. Let $x = x_0 + x_1 i + x_2 j + x_3 k$ and $c = c_0 + c_1 i + c_2 j + c_3 k$. Then $x^2 + bx + c = 0$ becomes the real system

$$x_0^{\,2} - \alpha x_1^{\,2} - \beta x_2^{\,2} - \alpha\beta x_3^{\,2} + b x_0 + c_0 = 0$$

$$\left(2x_0 + b\right) x_1 = -c_1$$

$$\left(2x_0 + b\right) x_2 = -c_2$$

$$(2x_0+b)x_3=-c_3.$$

Since $c\notin\mathbb{R}$, $2x_0+b\neq 0$ andso x_1,x_2,x_3 can be expressed in terms of x_0 and be substituted into the first equation to obtain

$$(2x_0+b)^4+(4c_0-b^2)(2x_0+b)^2-4(\alpha c_1{}^2+\beta c_2{}^2+\alpha\beta c_3{}^2)=0.$$

It follows that $2x_0+b=\pm\sqrt{\left(b^2-4c_0\pm\sqrt{(b^2-4c_0)^2+16(\alpha c_1{}^2+\beta c_2{}^2+\alpha\beta c_3{}^2)}\right)/2}$ and

therefore $x_0=(-b\pm\rho)/2$ where $\rho=\sqrt{\left(b^2-4c_0\pm\sqrt{(b^2-4c_0)^2+16(\alpha c_1{}^2+\beta c_2{}^2+\alpha\beta c_3{}^2)}\right)/2}\neq 0$ since $c\notin\mathbb{R}$. Finally

$$x=x_0-\frac{c_1}{2x_0+b}i-\frac{c_2}{2x_0+b}j-\frac{c_3}{2x_0+b}k$$
$$=-\frac{b}{2}\pm\frac{\rho}{2}\mp\frac{c_1}{\rho}i\mp\frac{c_2}{\rho}j\mp\frac{c_3}{\rho}k.$$

Case 4. $b\notin\mathbb{R}$. Rewrite the equation $x^2+bx+c=0$ as

$$y^2+b'y+c'=0,$$

where $y=x+\frac{\operatorname{Re}b}{2}, b'=\operatorname{Im}b\notin\mathbb{R}$ and $c'=c-\frac{\operatorname{Re}b}{2}\left(b-\frac{\operatorname{Re}b}{2}\right).$

Following the idea of [4], we observe that the solution of the quadratic equation $y^2+b'y+c'=0$ also satisfies

$$y^2-Ty+N=0$$

where $N=\overline{y}y\geq 0$ and $T=y+\overline{y}\in\mathbb{R}$. Hence $(b'+T)(c'-N)=0,$ and so

$$y=(b'+T)^{-1}(c'-N)$$

because $T\in\mathbb{R}$ and $b'\notin\mathbb{R}$ implies that $b'+T\neq 0.$. To solve for *T* and *N*,we substitute *y* back into definitions $T=y+\overline{y}$ and $N=\overline{y}y$ and simplify to obtain the real system

$$N^2-(B+T^2)N+E=0$$
$$T^3+(B-2N)T+D=0$$

where $B=b'\overline{b'}+c'+\overline{c'}=b'\overline{b'}+\operatorname{Re}c'$, $E=c'\overline{c'}, D=\overline{b'}c'+\overline{c'}b'=2\operatorname{Re}\overline{b'}c'$, $D=2\operatorname{Re}\overline{b'}c'$ are real numbers. Note that $E=c'\overline{c'}\geq 0.$

If*B*< 0 then $c'+\overline{c'}<0$ and $B^2-4E=b'\overline{b'}B+b'\overline{b'}(c'+\overline{c'})+(c'-\overline{c'})^2\leq 0,$ that is

because $(c'-\overline{c'})^2\leq 0.$ Then $B^2-4E<0,$ otherwise $B^2-4E=0$ andtherefore $b'\overline{b'}B=b'\overline{b'}(c'+\overline{c'})=(c'-\overline{c'})^2=0$ i.e , $b'=0\in\mathbb{R}$, , a contradiction. Hence by Lemma 2.2 suchsystem can be solved explicitly as claimed. Consequently

$$x=\frac{-\operatorname{Re}b}{2}-(b'+T)^{-1}(c'-N).$$

Corollary2.1. The quadratic equation $x^2+bx+c=0$ has infinitely many solutions if and only if $b,c\in\mathbb{R}$ and $b^2<4c.$

Example2.1. For the quadratic equation $x^2+4=0,$ i.e, $b=0$ and $c=4.$ This is the Case 1 in Theorem 2.1. Then $x=(x'_1 i+x'_2 j+x'_3 k)/2$ where $\alpha x'^2_1+\beta x'^2_2+\alpha\beta x'^2_3=16\alpha.$

Corollary2.2. The quadratic equation $x^2+bx+c=0$ has a unique solution if and only if either

i. $b, c \in \mathbb{R}$ and $b^2 - 4c = 0$, or
ii. $b \notin \mathbb{R}$ and $D = 0 = B^2 - 4E$.

Example2.2. Consider the quadratic equation $x^2 - x + \frac{1}{4} = 0$, i.e, $b = -1$ and $c = 1/4$. This is the Case 2 in Theorem 2.1. Then the unique solution is $x = 1/2$.

Example2.3. Consider the quadratic equation $x^2 + 2ix - \alpha = 0$, i.e, $b = 2i$ and $c = -\alpha$. This is the Case 4 in Theorem 2.1. Then $b' = 2i$ and $c' = -\alpha$. Moreover, $B = 2\alpha, E = \alpha^2$ and $D = 0$. It is Subcase 1 in Case 4. Hence$T = 0$, $N = \alpha$.
Consequently$x = i$.

Corollary2.3. The quadratic equation $x^2 + bx + c = 0$ has exactly two solutions if and only if either
i. $b, c \in \mathbb{R}$ and $b^2 - 4c > 0$, or
ii. $b \in \mathbb{R}$ and $c \notin \mathbb{R}$, or
iii. $b \notin \mathbb{R}$ and $D = 0, B^2 - 4E \neq 0$, or
iv. $b \notin \mathbb{R}$ and $D \neq 0$.

Example2.4. Consider the quadratic equation $x^2 + 3x - 4 = 0$, i.e, $b = 3$ and $c = -4$. This is the Case 2 in Theorem 2.1. Then the two solutions are$x = -4$ are$x = 1$.

Example2.5. Consider the quadratic equation $x^2 - x + i = 0$, i.e, $b = -1$ and $c = i$. This is the Case 3 in Theorem 2.1. Then $c_0 = c_2 = c_3 = 0, c_1 = 1$, and $\rho = \sqrt{\left(1 \pm \sqrt{1 + 16\alpha}\right)/2}$. Hence the two solutions are $x = (1+\rho)/2 - i/\rho$ are $x = (1+\rho)/2 + i/\rho$.

Example2.6. Consider the quadratic equation $x^2 + \alpha\beta x - \alpha^2\beta^2 = 0$, i.e, $b = \alpha\beta k$ and $c = -\alpha^2\beta^2$. This is the Case 4 in Theorem 2.1. Then $b' = \alpha\beta k$ and $c' = -\alpha^2\beta^2$. Moreover, $B = -\alpha^2\beta^2, E = \alpha^4\beta^4$ and $D = 0$. It is Subcase 2 in Case 4. Hence $N = \alpha^2\beta^2, T = \pm\alpha\beta\sqrt{3}$. Hence two solutions are $x = -2\alpha^2\beta^2\left(\alpha\beta k + \alpha\beta\sqrt{3}\right)^{-1}$ and
$x = -2\alpha^2\beta^2\left(\alpha\beta k - \alpha\beta\sqrt{3}\right)^{-1}$

Theorem2.2. If the quadratic equation $x^2 + bx + c = 0$ has exactly two distinct solutions x_1 and x_2, then $x_1 + b/2$ and $-\left(x_2 + b/2\right)$ are similar. Indeed, there exists nonzero $q \in K_{\alpha,\beta}$ such that $bq = qb$ and
$q\left(x_1 + b/2\right)q^{-1} = -\left(x_2 + b/2\right)$.

Proof: ByCorollary 3, wehaveseveralcasestodealwith.
i. . If$b, c \in \mathbb{R}$ and $b^2 > 4c$, by Case 2 in Theorem 1, it is clearthat $x_1 + b/2 = -\left(x_2 + b/2\right)$.
ii. If$b \in \mathbb{R}, c \notin \mathbb{R}$ by Case 3 in Theorem 1, it is clearthat $x_1 + b/2 = -\left(x_2 + b/2\right)$.
iii. a) If$b \notin \mathbb{R}$, $D = 0$ and $B^2 - 4E > 0$, then by Subcase 1 in Case 4 of Theorem 1, wehave

$$x_{1,2} = \frac{-\operatorname{Re} b}{2} - \left(b'\right)^{-1}\left(c' - \frac{B \pm \sqrt{B^2 - 4E}}{2}\right).$$

Thus, it is easy to see that

$$x_1 + \frac{b}{2} = -\left(b'\right)^{-1}\left(\operatorname{Im} c' - \frac{\sqrt{B^2 - 4E}}{2}\right) = \frac{b'}{b'\overline{b'}}\left(\operatorname{Im} c' - \frac{\sqrt{B^2 - 4E}}{2}\right)$$

and

$$x_2 + \frac{b}{2} = -\left(b'\right)^{-1}\left(\operatorname{Im} c' + \frac{\sqrt{B^2 - 4E}}{2}\right) = \frac{b'}{b'\overline{b'}}\left(\operatorname{Im} c' + \frac{\sqrt{B^2 - 4E}}{2}\right).$$

Clearly, $\operatorname{Re}(x_1+b/2)=\operatorname{Re}(-(x_2+b/2))=0$ and $|x_1+b/2|^2=|-(x_2+b/2)|^2$, thus $x_1+b/2$ and $x_2+b/2$ are also similar. Then it is easy to see that

$$b'\left(x_1+\frac{b}{2}\right)(b')^{-1}=-\left(x_2+\frac{b}{2}\right).$$

b) If $b\notin$, $D=0$ and $B^2-4E<0$, then by Subcase2 in Case 4 of Theorem 1, we have

$$x_{1,2}=\frac{-\operatorname{Re}b}{2}-\left(b'\pm\sqrt{2\sqrt{E}-B}\right)^{-1}\left(c'-\sqrt{E}\right).$$

Thus,

$$\begin{aligned}x_1+\frac{b}{2}&=\frac{b'}{2}-\frac{-b'+\sqrt{2\sqrt{E}-B}}{2\left(\sqrt{E}-\operatorname{Re}c'\right)}^{-1}\left(c'-\sqrt{E}\right)\\&=\frac{b'}{2}-\frac{-b'+\sqrt{2\sqrt{E}-B}}{2}^{-1}\left(1-\frac{\operatorname{Im}c'}{\sqrt{E}-\operatorname{Re}c'}\right)\\&=\frac{\sqrt{2\sqrt{E}-B}}{2}-\frac{\left(-b'+\sqrt{2\sqrt{E}-B}\right)\operatorname{Im}c'}{2\left(\sqrt{E}-\operatorname{Re}c'\right)}\end{aligned}$$

Similarly, we have

$$-\left(x_2+\frac{b}{2}\right)=\frac{\sqrt{2\sqrt{E}-B}}{2}+\frac{\left(-b'-\sqrt{2\sqrt{E}-B}\right)\operatorname{Im}c'}{2\left(\sqrt{E}-\operatorname{Re}c'\right)}.$$

Thus, it is clear that

$$\operatorname{Re}\left(x_1+\frac{b}{2}\right)=\operatorname{Re}\left(-\left(x_2+\frac{b}{2}\right)\right)=\frac{\sqrt{2\sqrt{E}-B}}{2}$$

and

$$\left|x_1+\frac{b}{2}\right|^2=\left|-\left(x_2+\frac{b}{2}\right)\right|^2,$$

Thus, $x_1+b/2$ and $-(x_2+b/2)$ are similar. Since

$$\operatorname{Im}\left(x_1+\frac{b}{2}\right)=-\left(b'+\sqrt{2\sqrt{E}-B}\right)^{-1}\operatorname{Im}c'$$

and

$$\operatorname{Im}\left(-\left(x_2+\frac{b}{2}\right)\right)=\left(\operatorname{Im}c'\right)\left(b'+\sqrt{2\sqrt{E}-B}\right)^{-1}.$$

Note that $\operatorname{Im}\left[-(x_2+b/2)\right]=\operatorname{Im}(x_2+b/2)$, it is easy to prove that

$$\left(b'+\sqrt{2\sqrt{E}-B}\right)\operatorname{Im}\left(x_1+\frac{b}{2}\right)=\operatorname{Im}\left(-\left(x_2+\frac{b}{2}\right)\right)\left(b'+\sqrt{2\sqrt{E}-B}\right).$$

Thus, we have

$$\left(b'+\sqrt{2\sqrt{E}-B}\right)\left(x_1+\frac{b}{2}\right)\left(b'+\sqrt{2\sqrt{E}-B}\right)^{-1}=-\left(x_2+\frac{b}{2}\right).$$

iv. If $b\notin\mathbb{R}$ and $D\neq 0$, from Theorem2.1, Case 4, Subcase 3, we have

$$x_1=-\frac{\operatorname{Re}b}{2}-\left(b'+T\right)^{-1}\left(c'-\frac{T^3+BT+D}{2T}\right)$$

and

$$x_2=-\frac{\operatorname{Re}b}{2}-\left(b'+T\right)^{-1}\left(c'-\frac{T^3+BT-D}{2T}\right)$$

where $T=\sqrt{z}$ and z is the unique positive solution of the cubic equation $z^3+2Bz^2+\left(B^2-4E\right)z-D^2=0$. By using $b'=\operatorname{Im}b$ and $B=|b'|^2+2\operatorname{Re}c'$, we have

$$x_1+\frac{b}{2}=\frac{T}{2}-\frac{T-b'}{T^2+|b'|^2}\left(\operatorname{Im}c'-\frac{D}{2T}\right)$$

And also the fact that $D=2\operatorname{Re}\overline{b'}c'$, we have

$$\operatorname{Re}\left\{\left(T-b'\right)\left(\operatorname{Im}c'-\frac{D}{2T}\right)\right\}=0.$$

Hence, $\operatorname{Re}\left(x_1+b/2\right)=T/2$ and

$$\operatorname{Im}\left(x_1+\frac{b}{2}\right)=-\frac{1}{T^2+|b'|^2}\left\{\left(T-b'\right)\left(\operatorname{Im}c'-\frac{D}{2T}\right)\right\}.$$

Similarly, we have

$$x_2+\frac{b}{2}=\frac{T}{2}-\frac{T-b'}{T^2+|b'|^2}\left(\operatorname{Im}c'+\frac{D}{2T}\right)$$

$\operatorname{Re}\left(-\left(x_2+b/2\right)\right)=T/2$ and

$$\operatorname{Im}\left(-\left(x_2+\frac{b}{2}\right)\right)=-\frac{1}{T^2+|b'|^2}\left\{\left(T-b'\right)\left(\operatorname{Im}c'+\frac{D}{2T}\right)\right\}.$$

3. CONCLUSION

The results obtained from quadratic formulas of generalized quaternions; in particular

i. For $\alpha=\beta=1$, , are reduced to the results obtained from [5] for quadratic formulas of quaternions.

ii. For $\alpha=-1,\beta=1$, are reduced to the results obtained from quadratic formulas of split quaternions (see [4] and [6]).

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