**Journal of Amasya University the Institute of Sciences and Technology**

https://dergipark.org.tr/en/pub/jauist

**Research Article**

**Open Access**


ISSN: 2717-8900 https://doi.org/10.54559/jauist.1589242

# Soft difference-product: A new product for soft sets with its decision-making

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**Abstract —** A thorough mathematical foundation for handling uncertainty is provided by the concept of soft sets. Soft set operations are key concepts in soft set theory since they offer novel approaches to problems requiring parametric data. The "soft difference-product" a new product operation for soft sets, is proposed in this study along with all of its algebraic properties concerning different types of soft equalities and subsets. Additionally, we explore the connections between this product and other soft set operations by investigating the distributions of soft difference-product over other soft set operations. Using the *uni-int* operator and the uni-int decision function for the soft-difference product, we apply the *uni-int* decision-making method, which selects a set of optimal elements from the alternatives by giving an example that shows how the approach may be conducted effectively in various areas. Since the theoretical underpinnings of soft computing techniques are drawn from purely mathematical concepts, this study is crucial to the literature on soft sets.

**Keywords:** *Soft set, soft difference-product, soft subset, soft equal relations*

**Subject Classification (2020):** 03E20, 03E72

## 1. Introduction

Numerous mathematicians have developed a variety of mathematical tools to solve and model complex problems involving ambiguity, vagueness, and uncertainty in a variety of domains, including the social sciences, engineering, economics, and the medical sciences. Molodtsov [1] showed that these theories have inherent challenges, which are related to the potential for identifying a membership function in the case of fuzzy set theory [2] and the need to investigate the existence of the mean by conducting a large number of trials in the case of probability theory.

Therefore, Molodtsov [1] proposed the soft set, a novel mathematical technique, and looked into its uses in several fields, including probability theory, operations research, and game theory. Molodstov's soft set theory differs greatly from traditional ideas since it does not impose any limitations on the approximate description. After Maji et al. [3] used soft set theory in a decision-making problem, several researchers [4–10] developed some innovative soft set-based decision-making solutions and methods such as parameterization reduction of soft sets, soft information based on the theory of soft sets, texture classification using a novel, soft-set theory based classification algorithm, soft decision making for patients suspected influenza, and soft set-based decision making for patients suspected influenza-like illness, respectively. The trumpeted soft set-based decision-making method known as "*uni-int* decision-making" was proposed by Çağman and Enginoğlu [11].

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Article History: Received: 21 Nov 2024 — Accepted: 22 Dec 2024 — Published: 31 Dec 2024

Additionally, the soft matrix was introduced by Çağman and Enginoğlu [12], who also developed decision-making techniques for the OR, AND, AND-NOT, and OR-NOT products of the soft matrices. They then applied these techniques to resolve uncertainties and other real-world problems. Soft set theory has been widely and successfully used to handle decision-making problems [13–24] via bijective soft set, exclusive disjunctive soft sets, generalized uni-int decision making schemes, soft discernibility matrix, soft approximations and uni-int decision making, the role of operators on soft set in decision making problems, reduced soft matrices and generalized products, cardinality inverse soft matrix theory, semantics of soft sets, the mean operators and generalized products of fuzzy soft matrices, and soft set-valued mappings, respectively.

In recent years, several researchers have investigated the underlying principles of soft set theory. A thorough theoretical study of soft sets, including soft subsets and supersets, equality of soft sets, and soft set operations like union, intersection, AND-product, and OR-product, was provided by Maji et al. [25]. Pei and Miao [26] investigated the relationship between soft sets and information systems and redefined intersection and soft set subsets. New soft set operations including the restricted union, restricted intersection, restricted difference, and extended intersection were proposed and studied by Ali et al. [27]. After that, the authors [28-41] examined the operations of the soft sets and the algebraic structures of the collection of the soft sets, proposed improved and novel methods, and identified several conceptual errors regarding the underlying assumptions of soft set theory that were presented in the published papers. Over the past several years, there has been a major advancement in the research of soft sets. Eren and Çalışıcı [42] defined a new kind of difference operation of soft sets. Stojanovic [43] described the extended symmetric difference of soft sets and investigated its fundamental properties. Many new types of soft set operations have been proposed and thoroughly examined in [44-49] such as soft binary piecewise difference operation, complementary soft binary piecewise theta, difference, union, intersection, and star operation of soft sets, respectively.

Two core concepts in soft set theory are soft equal relations and soft subsets. The first to use a somewhat accurate notion of soft subsets was Maji et al. [25]. One may consider the concept of soft subsets, which was established by Pei and Miao [26] and Feng et al. [29], to be an extension of Maji's earlier definitions [25]. Qin and Hong [50] introduced two new types of congruence relations and soft equal relations on soft sets. To modify Maji's soft distributive laws, Jun and Yang [51] used a wider range of soft subsets and extended soft equal relations, which we call J-soft equal relations for consistency's sake. Jun and Yang [51] conducted more research on the generalized soft distributive principles of soft product operations. Liu et al.[52] published a brief research note on soft L-subsets and soft L-equal relations, motivated by the novel ideas of Jun and Yang [51]. One significant result in [52] is that distributive rules do not hold for all of the soft equality described in the literature.

Thus, Feng et al. [53] extended the study reported in [52] by focusing on soft subsets and the soft products proposed in [24]. Feng et al. [53] focused on the different types of soft subsets and the algebraic properties of soft product operations. Along with commutative laws, association rules, and other crucial features, they also covered distributional laws, which were extensively researched by several academics. Besides, they provided theoretical research on the soft products, including the AND-product and OR-product using soft L-subsets, in addition to other relevant subjects. They completed some unfinished discoveries on soft product operations that had previously been reported in the literature and thoroughly investigated the algebraic characteristics of soft product operations in terms of J-equality and L-equality. Soft L-equal relations were shown to be congruent on free soft algebras and their associated quotient structures, which are commutative semigroups. For further information on soft equal connections such as generalized soft equality and soft lattice structure, generalized operations in soft set theory via relaxed conditions on parameters, g-soft equality and gf-soft equality relations, and T-soft equality relation, we refer to [54–58], respectively.

Çağman and Enginoğlu [11] revised the idea and workings of Molodtsov's soft sets to make them more useful. Moreover, they proposed four types of products in soft set theory: AND-product, OR-product, AND-NOT-product, OR-NOT-product, and uni-int decision function. Using these new definitions, they proposed a uniform decision-making procedure that chooses the best components from the range of options. Finally, they provided an example that demonstrates how the approach may be effectively used for a range of issues, including uncertainty. The AND-product of soft sets, which has long served as the foundation and a tool used by decision-makers in decision-making problems, was examined theoretically by Sezgin et al. [59]. Even though many scholars have studied the AND-product and its features concerning different types of soft equalities, such as soft L-equality and soft J-equality, the authors of [59] thoroughly examined the entire algebraic properties of the AND-product, including idempotent laws, commutative laws, associative laws, and other fundamental properties and compared them to previously obtained properties in terms of soft F-subsets, soft M-equality, soft L-equality, and soft J-equality. By establishing the distributive characteristics of AND-product over restricted, extended, and soft binary piecewise soft set operations, they also showed that the set of all soft sets over the universe is a commutative hemiring with identity in the sense of soft L-equality when combined with restricted/extended union and AND-product and that the set of all soft sets over the universe combined with restricted/extended symmetric difference and AND-product is also a commutative hemiring with identity in the sense of soft L-equality. Çağman and Enginoğlu [11] defined AND-product for soft sets, the domain of the approximation function of which is $ExE$, where $E$ is the set of parameters. Furthermore, they show that this product is not commutative and associative under M-equality, but holds De Morgan Laws.

In this study, we first propose a new product for soft sets, which we call the "soft difference-product", using Molodtsov's concept of soft sets. Unlike the AND-NOT-product for soft sets defined in [11], the domain of the approximation function of the soft difference-product is the cartesian product of the parameter sets of the soft sets. We give an example of a soft difference-product and study its algebraic properties in detail regarding several soft subsets and soft equality types, such as M-subset/equality, F-subset/equality, L-subset/equality, and J-subset/equality. Moreover, we derive the distributions of the soft difference-product over several types of certain soft set operations. Finally, we apply the *uni-int* decision-making method proposed by Çağman and Enginoğlu [11] on soft difference-product to choose the best elements from the possibilities and provide an example that demonstrates how the approach may be effectively applied for many areas. This study aims to add to the literature on soft sets, as soft sets are a useful mathematical tool for identifying uncertainty and the theoretical foundations of soft computing approaches are derived from purely mathematical principles. This paper is organized as follows. In Section 2, we remind the basic concepts of soft set theory. Section 3 proposes the soft difference-product and discusses its whole algebraic properties in terms of several types of soft equalities and soft subsets. In Section 4, we examine the distributions of the soft difference-product over several types of soft set operations. In Section 5, the *uni-int* decision operators and function for soft difference-product are applied to a decision-making problem. The conclusion section has a deduction.

## 2. Preliminaries

This section presents some basic concepts to be needed in the following sections.

**Definition 2.1.** [1] Let $U$ be the universal set, $E$ be the parameter, $P(U)$ be the power set of $U$ and $\mathcal{M} \subseteq E$. A pair $(\mho, \mathcal{M})$ is called a soft set over $U$ where $\mho$ is a set-valued function such that $\mho: \mathcal{M} \rightarrow P(U)$.

Although Çağman and Enginoğlu [11] modified Molodstov's concept of soft sets, we continue to use the original definition of the soft set in our study. Throughout this paper, the collection of all the soft sets defined over $U$ is designated as $S_E(U)$. Let $\mathcal{M}$ be a fixed subset of $E$ and $S_{\mathcal{M}}(U)$ be the collection of all those soft sets over $U$ with the fixed parameter set $\mathcal{M}$. That is, while in the set $S_{\mathcal{M}}(U)$, there are only soft sets whose parameter sets are $\mathcal{M}$; in the set $S_E(U)$, there are soft sets whose parameter sets may be any set. From now on, while soft sets will be designated by SS and parameter set by PS; soft sets will be designated by SSs and parameter sets by PSs for the sake of ease.

**Definition 2.2.** [27] Let $(\mho, \mathcal{M})$ be an SS over $U$. $(\mho, \mathcal{M})$ is called a relative null SS (with respect to the PS $\mathcal{M}$), denoted by $\emptyset_{\mathcal{M}}$, if $\mho(m) = \emptyset$ for all $m \in \mathcal{M}$ and $(\mho, \mathcal{M})$ is called a relative whole SS (with respect to the PS $\mathcal{M}$), denoted by $U_{\mathcal{M}}$ if $\mho(m) = U$ for all $m \in \mathcal{M}$. The relative whole SS $U_E$ with respect to the universe set of parameters E is called the absolute SS over $U$.

The empty SS over $U$ is the unique SS over $U$ with an empty PS, represented by $\emptyset_{\emptyset}$. Note $\emptyset_{\emptyset}$ and $\emptyset_{\mathcal{M}}$ are different [31]. In the following, we always consider SSs with non-empty PSs in the universe $U$, unless otherwise stated.

The concept of soft subset, which we refer to here as soft M-subset to prevent confusion, was initially defined by Maji et al. [25] in the following extremely strict way:

**Definition 2.3.** [25] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over $U$. $(\mho, \mathcal{M})$ is called a soft M-subset of $(\mathfrak{F}, \mathcal{D})$ denoted by $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$ if $\mathcal{M} \subseteq \mathcal{D}$ and $\mho(m) = \mathfrak{F}(m)$ for all $m \in \mathcal{M}$. Two SSs $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ are said to be soft M-equal, denoted by $(\mho, \mathcal{M}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$, if $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$ and $(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{M}} (\mho, \mathcal{M})$.

**Definition 2.4.** [26] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over $U$. $(\mho, \mathcal{M})$ is called a soft F-subset of $(\mathfrak{F}, \mathcal{D})$ denoted by $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F}, \mathcal{D})$ if $\mathcal{M} \subseteq \mathcal{D}$ and $\mho(m) \subseteq \mathfrak{F}(m)$ for all $m \in \mathcal{M}$. Two SSs $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ are said to be soft F-equal, denoted by $(\mho, \mathcal{M}) =_{\mathrm{F}} (\mathfrak{F}, \mathcal{D})$, if $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F}, \mathcal{D})$ and $(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{F}} (\mho, \mathcal{M})$.

It should be noted that the definitions of soft F-subset and soft F-equal were initially provided by Pei and Miao in [26]. However, some SS papers regarding soft subsets and soft equalities claimed that Feng et al. provided these definitions first in [29]. As a result, the letter "F" is used to denote this connection.

It was demonstrated in [52] that the soft equal relations $=_{\mathrm{M}}$ and $=_{\mathrm{F}}$ coincide. In other words, $(\mho, \mathcal{M}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D}) \Leftrightarrow (\mho, \mathcal{M}) =_{\mathrm{F}} (\mathfrak{F}, \mathcal{D})$. Since they share the same set of parameters and approximation function, two SSs that meet this soft equivalence are truly identical [52], hence $(\mho, \mathcal{M}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$ means, in fact, $(\mho, \mathcal{M}) = (\mathfrak{F}, \mathcal{D})$.

Jun and Yang [51] extended the ideas of F-soft subsets and soft F-equal relations by loosening the restrictions on PSs. We refer to them as soft J-subsets and soft J-equal relations, the initial letter of Jun, even though in [51] they are named generalized soft subset and generalized soft equal relation.

**Definition 2.5** [51] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over $U$. $(\mho, \mathcal{M})$ is called a soft J-subset of $(\mathfrak{F}, \mathcal{D})$ denoted by $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$ if for all $m \in \mathcal{M}$, there exists $d \in \mathcal{D}$ such that $\mho(m) \subseteq \mathfrak{F}(d)$. Two SSs $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ are said to be soft J-equal, denoted by $(\mho, \mathcal{M}) =_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$, if $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$ and $(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (\mho, \mathcal{M})$.

In [52] and [53], it was shown that $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{F}, \mathcal{D}) \Rightarrow (\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F}, \mathcal{D}) \Rightarrow (\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$, but the converse may not be true.

Besides, Liu et al. [52] presented the following new kind of soft subsets (henceforth referred to as soft L-subsets and soft L-equality) that generalize both soft M-subsets and ontology-based soft subsets, inspired by the ideas of soft J-subset [51] and ontology-based soft subsets [30]:

**Definition 2.6** [52] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over $U$. $(\mho, \mathcal{M})$ is called a soft L-subset of $(\mathfrak{F}, \mathcal{D})$ denoted by $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{L}} (\mathfrak{F}, \mathcal{D})$ if for all $m \in \mathcal{M}$, there exists $d \in \mathcal{D}$ such that $\mho(m) = \mathfrak{F}(d)$. Two SSs $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ are said to be soft L-equal, denoted by $(\mho, \mathcal{M}) =_{\mathrm{L}} (\mathfrak{F}, \mathcal{D})$, if $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{L}} (\mathfrak{F}, \mathcal{D})$ and $(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (\mho, \mathcal{M})$.

As regards the relations between certain types of soft subsets and soft equalities, $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{M}} (\mathfrak{F}, \mathcal{D}) \Rightarrow (\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{L}} (\mathfrak{F}, \mathcal{D}) \Rightarrow (\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$ and $(\mho, \mathcal{M}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D}) \Rightarrow (\mho, \mathcal{M}) =_{\mathrm{L}} (\mathfrak{F}, \mathcal{D}) \Rightarrow (\mho, \mathcal{M}) =_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$ [52]. However, the converses may not be true. Moreover, it is well-known that $(\mho, \mathcal{M}) =_{\mathrm{M}} (\mathfrak{F}, \mathcal{D})$ if and only if $(\mho, \mathcal{M}) =_{\mathrm{F}} (\mathfrak{F}, \mathcal{D})$.

We may thus deduce that soft M-equality (and so soft F-equality) is the strictest sense, whereas soft J-equality is the weakest soft equal connection. In the middle of these is the idea of the soft L-equal connection [52].

**Example 2.7.** Let $E = \{c_1, c_2, c_3, c_4, c_5\}$ be the PS, $\mathcal{M} = \{c_1, c_4\}$ and $\mathcal{D} = \{c_1, c_4, c_5\}$ be the subsets of $E$, and $U = \{z_1, z_2, z_3, z_4, z_5\}$ be the initial universe set. Let

$$(\mho, \mathcal{M}) = \{(c_1, \{z_1, z_3\}), (c_4, \{z_2, z_3, z_5\})\},$$

$$(\mathfrak{F}, \mathcal{D}) = \{(c_1, \{z_1, z_3\}), (c_4, \{z_2, z_3\}), (c_5, \{z_1, z_2, z_3, z_5\})\},$$

and

$$(\mathcal{V}, \mathcal{D}) = \{(c_1, \{z_2, z_3, z_5\}), (c_4, \{z_1, z_3\}), (c_5, \{z_1, z_2, z_3, z_5\})\}$$

Since $\mho(c_1) \subseteq \mathfrak{F}(c_1)$ (and also $\mho(c_1) \subseteq \mathfrak{F}(c_5)$) and $\mho(c_4) \subseteq \mathfrak{F}(c_5)$, it is obvious that $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$. However, since $\mho(c_4) \neq \mathfrak{F}(c_1)$, $\mho(c_4) \neq \mathfrak{F}(c_4)$, and $\mho(c_4) \neq \mathfrak{F}(c_5)$, we can deduce that $(\mho, \mathcal{M})$ is not a soft L-subset of $(\mathfrak{F}, \mathcal{D})$. Moreover, as $\mho(c_4) \neq \mathfrak{F}(c_4)$, $(\mho, \mathcal{M})$ is not a soft M-subset of $(\mathfrak{F}, \mathcal{D})$. Moreover as

$\mho(c_1) = \mathcal{V}(c_4)$ and $\mho(c_4) = \mathcal{V}(c_1)$, it is obvious that $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{L}} (\mathcal{V}, \mathcal{D})$. However, as $\mho(c_1) \neq \mathcal{V}(c_1)$, $\mho(c_4) \neq \mathcal{V}(c_4)$, $(\mho, \mathcal{M})$ is not again a soft M-subset of $(\mathcal{V}, \mathcal{D})$.

**Example 2.8.** Let $E = \{c_1, c_2, c_3, c_4, c_5\}$ be the PS, $\mathcal{M} = \{c_1, c_4\}$ and $\mathcal{D} = \{c_1, c_4, c_5\}$ be the subsets of $E$, and $U = \{z_1, z_2, z_3, z_4, z_5\}$ be the initial universe set. Let $(\mho, \mathcal{M}) = \{(c_1, \{z_1, z_3\}), (c_4, \{z_1, z_2, z_3, z_5\})\}$ and $(\mathfrak{F}, \mathcal{D}) = \{(c_1, \{z_1, z_2, z_3\}), (c_4, \{z_1, z_2, z_3, z_5\}), (c_5, \{z_1\})\}$. Since $\mho(c_1) \neq \mathfrak{F}(c_1)$, $\mho(c_1) \neq \mathfrak{F}(c_4)$, and $\mho(c_1) \neq \mathfrak{F}(c_5)$, it is obvious that $(\mho, \mathcal{M}) \neq_{\mathrm{L}} (\mathfrak{F}, \mathcal{D})$. However, since $\mho(c_1) \subseteq \mathfrak{F}(c_1)$ (moreover $\mho(c_1) \subseteq \mathfrak{F}(c_4)$) and $\mho(c_4) \subseteq \mathfrak{F}(c_4)$, we can deduce that $(\mho, \mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$. Moreover, since $\mathfrak{F}(c_1) \subseteq \mho(c_4)$ and $\mathfrak{F}(c_4) \subseteq \mho(c_4)$, and $\mathfrak{F}(c_5) \subseteq \mho(c_1)$, we can deduce that $(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (\mho, \mathcal{M})$. Therefore, $(\mho, \mathcal{M}) =_{\mathrm{J}} (\mathfrak{F}, \mathcal{D})$. As $\mho(c_1) \neq \mathfrak{F}(c_1)$ and $\mho(c_4) \neq \mathfrak{F}(c_4)$, it is obvious that $(\mho, \mathcal{M})$ is not a soft M-subset of $(\mathfrak{F}, \mathcal{D})$.

For more on soft F-equality, soft M-equality, soft J-equality, soft L-equality, and some other existing definitions of soft subsets and soft equal relations in the literature, we refer to [50-58].

**Definition 2.9.** [27] Let $(\mho, \mathcal{M})$ be an SS over $U$. The relative complement of an SS Let $(\mho, \mathcal{M})$, denoted by $(\mho, \mathcal{M})^{\mathrm{r}}$, is defined by $(\mho, \mathcal{M})^{\mathrm{r}} = (\mho^{\mathrm{r}}, \mathcal{M})$, where $\mho^{\mathrm{r}}: \mathcal{M} \to P(U)$ is a mapping given by $\mho^{\mathrm{r}}(m) = \mathrm{U} \setminus \mho(m)$, for all $m \in \mathcal{M}$. From now on, $\mathrm{U} \setminus \mho(m) = [\mho(m)]'$ is designated by $\mho'(m)$ for the sake of designation.

**Definition 2.10.** [25] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over $U$. The AND-product ($\wedge$-product) of $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ is denoted by $(\mho, \mathcal{M}) \wedge (\mathfrak{F}, \mathcal{D})$, and is defined by $(\mho, \mathcal{M}) \wedge (\mathfrak{F}, \mathcal{D}) = (\mho, \mathcal{M} \mathrm{x} \mathcal{D})$, where for all $(m, d) \in \mathcal{M} \mathrm{x} \mathcal{D}$, $\mho(m, d) = \mho(m) \cap \mathfrak{F}(d)$.

**Definition 2.11.** [25] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over $U$. The OR-product ($\vee$-product) of $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ is denoted by $(\mho, \mathcal{M}) \vee (\mathfrak{F}, \mathcal{D})$, and is defined by $(\mho, \mathcal{M}) \vee (\mathfrak{F}, \mathcal{D}) = (\mho, \mathcal{M} \mathrm{x} \mathcal{D})$, where for all $(m, d) \in \mathcal{M} \mathrm{x} \mathcal{D}$, $\mho(m, d) = \mho(m) \cup \mathfrak{F}(d)$.

Let "$\circledast$" to stand for set operations like $\cap, \cup, \setminus, \triangle$. The following definitions are for restricted, extended, and soft binary piecewise operations of soft sets.

**Definition 2.12.** [27] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over $U$. The restricted $\circledast$ operation of $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$, denoted by $(\mho, \mathcal{M}) \circledast_{\mathrm{R}} (\mathfrak{F}, \mathcal{D})$ is defined by $(\mho, \mathcal{M}) \circledast_{\mathrm{R}} (\mathfrak{F}, \mathcal{D}) = (\mho, \mathcal{J})$, where $\mathcal{J} = \mathcal{M} \cap \mathcal{D}$ and if $\mathcal{J} \neq \emptyset$, then for all $\mathrm{j} \in \mathcal{J}$, $\mho(\mathrm{j}) = \mho(\mathrm{j}) \circledast \mathfrak{F}(\mathrm{j})$; if $\mathcal{J} = \emptyset$, then $(\mho, \mathcal{M}) \circledast_{\mathrm{R}} (\mathfrak{F}, \mathcal{D}) = \emptyset_{\emptyset}$.

**Definition 2.13.** [27,43] Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be two SSs over $U$. The extended $\circledast$ operation of $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$, denoted by $(\mho, \mathcal{M}) \circledast_{\varepsilon} (\mathfrak{F}, \mathcal{D})$ is defined by $(\mho, \mathcal{M}) \circledast_{\varepsilon} (\mathfrak{F}, \mathcal{D}) = (\mho, \mathcal{J})$, where $\mathcal{J} = \mathcal{M} \cup \mathcal{D}$ and then for all $\mathrm{j} \in \mathcal{J}$,

$$\mho(\mathrm{j}) = \begin{cases} \mho(\mathrm{j}), & \mathrm{j} \in \mathcal{M} \setminus \mathcal{D} \\ \mathfrak{F}(\mathrm{j}), & \mathrm{j} \in \mathcal{D} \setminus \mathcal{M} \\ \mho(\mathrm{j}) \circledast \mathfrak{F}(\mathrm{j}), & \mathrm{j} \in \mathcal{M} \cap \mathcal{D} \end{cases}$$

**Definition 2.14.** [44] Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be two SSs over $U$. The soft binary piecewise $\circledast$ operation of $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$, denoted by $(\mho,\mathcal{M})\ \widetilde{\circledast}\ (\mathfrak{F},\mathcal{D})$, is defined by $(\mho,\mathcal{M})\ \widetilde{\circledast}\ (\mathfrak{F},\mathcal{D})=(\mho,\mathcal{M})$, where for all $\mathfrak{j}\in\mathcal{M}$,

$$\mho(\mathfrak{j})=\begin{cases}\mho(\mathfrak{j}), & \mathfrak{j}\in\mathcal{M}\backslash\mathcal{D},\\ \mho(\mathfrak{j})\circledast\mathfrak{F}(\mathfrak{j}), & \mathfrak{j}\in\mathcal{M}\cap\mathcal{D}\end{cases}$$

For more about soft sets and picture fuzzy soft sets, we refer to [60-81].

## 3. Soft Difference-Product and Its Algebraic Properties

Çağman and Enginoğlu [11] defined AND-NOT-product for soft sets as Definition 3.1. In this subsection, we introduce a new product for soft sets, called soft difference-product in a similar way to the AND-NOT-product for soft sets. We give its example and examine its algebraic properties in detail depth in terms of specific kinds of soft equalities and soft subsets.

**Definition 3.1.** [11] Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over $U$. The $\overline{\wedge}$-product (AND-NOT-product) of $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$, denoted by $(\mho,\mathcal{M})\ \overline{\wedge}\ (\mathfrak{F},\mathcal{D})$, is defined by $(\mho,\mathcal{M})\ \overline{\wedge}\ (\mathfrak{F},\mathcal{D})=(\mho,ExE)$, where for all $(m,d)\in$ ExE, $\mho(m,d)=\mho(m)\backslash\mathfrak{F}(d)=\mho(m)\cap\mathfrak{F}'(d)$.

**Definition 3.2.** Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over $U$. The soft difference-product of $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$, denoted by $(\mho,\mathcal{M})\wedge_{\backslash}(\mathfrak{F},\mathcal{D})$, is defined by $(\mho,\mathcal{M})\wedge_{\backslash}(\mathfrak{F},\mathcal{D})=(\mho,\mathcal{M}\mathrm{x}\mathcal{D})$, where for all $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, $\mho(m,d)=\mho(m)\backslash\mathfrak{F}(d)=\mho(m)\cap\mathfrak{F}'(d)$.

It is observed that while the domain of the approximation function of AND-NOT-product of $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ defined by Çağman and Enginoğlu [11] is $ExE$, the domain of the approximation function of soft difference-product of $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ is $\mathcal{M}\mathrm{x}\mathcal{D}$, leading to $(\mho,\mathcal{M})\wedge_{\backslash}(\mathfrak{F},\mathcal{D})\neq(\mho,\mathcal{M})\ \overline{\wedge}\ (\mathfrak{F},\mathcal{D})$. Since every input value (from the domain) must be associated with exactly one output value (in the range) in order for a function to be defined, this case giving rise to the resulting soft sets of AND-NOT-product and soft difference-product differ from each other as seen in the following example.

**Example 3.3.** Assume that $E=\{e_1,e_2,e_3,e_4,e_5\}$ be the PS, $\mathcal{M}=\{e_1,e_2,e_3\}$ and $\mathcal{D}=\{e_1,e_4,e_5\}$ be the subsets of $E$, $U=\{h_1,h_2,h_3,h_4,h_5,h_6\}$ be the universal set, the SSs $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be over $U$ such that

$$(\mho,\mathcal{M})=\{(e_1,\{h_1,h_2,h_3,h_5\}),(e_2,\{h_1,h_2,h_3\}),(e_3,\{h_4,h_5,h_6\})\}$$

and

$$(\mathfrak{F},\mathcal{D})=\{(e_1,\{h_6\}),(e_4,\{h_2,h_3,h_5\}),(e_5,\{h_2\})\}$$

Let $(\mho,\mathcal{M})\wedge_{\backslash}(\mathfrak{F},\mathcal{D})=(\mho,\mathcal{M}\mathrm{x}\mathcal{D})$. Then,

$$(\mho,\mathcal{M}\mathrm{x}\mathcal{D})=\{((e_1,e_1),\{h_1,h_2,h_3,h_5\}),((e_1,e_4),\{h_1\}),((e_1,e_5),\{h_1,h_3,h_5\}),((e_2,e_1),\{h_1,h_2,h_3\}),((e_2,e_4),\{h_1\}),$$
$$((e_2,e_5),\{h_1,h_3\}),((e_3,e_1),\{h_4,h_5\}),((e_3,e_4),\{h_4,h_6\}),((e_3,e_5),\{h_4,h_5,h_6\})\}$$

Assume that $(\mho,\mathcal{M})\ \overline{\wedge}\ (\mathfrak{F},\mathcal{D})=(\daleth,ExE)$. Then,

$$(\daleth,ExE)=\{((e_1,e_1),\{h_1,h_2,h_3,h_5\}),((e_1,e_2),\{h_1,h_2,h_3,h_5\}),((e_1,e_3),\{h_1,h_2,h_3,h_5\}),((e_1,e_4),\{h_1\}),$$
$$((e_1,e_5),\{h_1,h_3,h_5\}),((e_2,e_1),\{h_1,h_2,h_3\}),((e_2,e_2),\{h_1,h_2,h_3\}),((e_2,e_3),\{h_1,h_2,h_3\}),((e_2,e_4),\{h_1\}),((e_2,e_5),\{h_1,h_3\}),$$
$$((e_3,e_1),\{h_4,h_5\}),((e_3,e_2),\{h_4,h_5,h_6\}),((e_3,e_3),\{h_4,h_5,h_6\}),((e_3,e_4),\{h_4,h_6\}),((e_3,e_5),\{h_4,h_5,h_6\})\}$$

Here, note that since $\daleth(e_4,e_1)=\daleth(e_4,e_2)=\daleth(e_4,e_3)=\daleth(e_4,e_4)=\daleth(e_4,e_5)=\daleth(e_5,e_1)=\daleth(e_5,e_2)=\daleth(e_5,e_3)=\daleth(e_5,e_4)=\daleth(e_5,e_5)=\emptyset$, they are not designated in the soft set $(\daleth,ExE)$. It can be easily observed that $(\mho,\mathcal{M}\mathrm{x}\mathcal{D})\neq(\daleth,ExE)$.

It is more convenient to use the table method to write the result of the soft difference-product than writing it in the list method.

**Table 1.** The table designation of the soft difference-product's result of the soft sets in Example 3.3

| $(\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{D})$ | $e_1$ | $e_4$ | $e_5$ |
|---|---|---|---|
| $e_1$ | $\{h_1,h_2,h_3,h_5\}$ | $\{h_1\}$ | $\{h_1,h_3,h_5\}$ |
| $e_2$ | $\{h_1,h_2,h_3\}$ | $\{h_1\}$ | $\{h_1,h_3\}$ |
| $e_3$ | $\{h_4,h_5\}$ | $\{h_4,h_6\}$ | $\{h_4,h_5,h_6\}$ |

**Proposition 3.4.** $\wedge_{\setminus}$-product is closed in $S_E(U)$.

Proof. It is obvious that $\wedge_{\setminus}$-product is a binary operation in $S_E(U)$. In fact, let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over $U$. Then,

$$\wedge_{\setminus}: S_E(U)\ x\ S_E(U) \longrightarrow S_E(U)$$

$$\big((\mho,\mathcal{M}),(\mathfrak{F},\mathcal{D})\big) \longrightarrow (\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{D}) = (\mathcal{V},\mathcal{M}\text{x}\mathcal{D}) = (\mathcal{V},\mathcal{J})$$

Since the set $S_E(U)$ contains all the SS over $U$, $(\mathcal{V},\mathcal{J}) \in S_E(U)$. Here, note that the set $S_{\mathcal{M}}(U)$ is not closed under $\wedge_{\setminus}$-product. That is, when $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{M})$ are the elements of $S_{\mathcal{M}}(U)$, $(\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{M})$ is an element of $S_{\mathcal{M}x\mathcal{M}}(U)$ not $S_{\mathcal{M}}(U)$. □

**Proposition 3.5.** Let $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$, and $(\mathcal{V},\mathcal{J})$ be SSs over $U$. Then,

$$(\mho,\mathcal{M})\wedge_{\setminus}[(\mathfrak{F},\mathcal{D})\wedge_{\setminus}(\mathcal{V},\mathcal{J})] \neq_{\mathrm{M}} [(\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{D})]\wedge_{\setminus}(\mathcal{V},\mathcal{J})$$

Thus, $\wedge_{\setminus}$ -product is not associative in $S_E(U)$.

Proof. In order to show that $\wedge_{\setminus}$-product is not associative in $S_E(U)$, we provided an example: Let $E = \{e_1,e_2,e_3,e_4\}$ be the PS, $\mathcal{M} = \{e_2,e_3\}$, $\mathcal{D} = \{e_1\}$, and $\mathcal{J} = \{e_4\}$ be the subsets of $E$ , $U = \{h_1,h_2,h_3,h_4,h_5\}$ be the universal set, and $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$ ve $(\mathcal{V},\mathcal{J})$ be SSs over $U$ such that $(\mho,\mathcal{M}) = \{(e_2,\{h_3,h_4\}),(e_3,\{h_1\})\}$, $(\mathfrak{F},\mathcal{D}) = \{(e_1,\emptyset)\}$, and $(\mathcal{V},\mathcal{J}) = \{(e_4,\{h_1,h_3,h_5\})\}$. We show that

$$(\mho,\mathcal{M})\wedge_{\setminus}[(\mathfrak{F},\mathcal{D})\wedge_{\setminus}(\mathcal{V},\mathcal{J})] \neq_{\mathrm{M}} [(\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{D})]\wedge_{\setminus}(\mathcal{V},\mathcal{J})$$

Let $(\mathfrak{F},\mathcal{D})\wedge_{\setminus}(\mathcal{V},\mathcal{J}) = (\mho,\mathcal{D}\text{x}\mathcal{J})$, Then,

$$(\mathfrak{F},\mathcal{D})\wedge_{\setminus}(\mathcal{V},\mathcal{J}) = (\mho,\mathcal{D}\text{x}\mathcal{J}) = \{((e_1,e_4),\emptyset)\}$$

and let $(\mho,\mathcal{M})\wedge_{\setminus}(\mho,\mathcal{D}\text{x}\mathcal{J}) = \big(\mathfrak{X},\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})\big)$. Thus,

$$(\mho,\mathcal{M})\wedge_{\setminus}(\mho,\mathcal{D}\text{x}\mathcal{J}) = \big(\mathfrak{X},\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})\big) = \{((e_2,(e_1,e_4)),\{h_3,h_4\}),((e_3,(e_1,e_4)),\{h_1\})\}$$

Assume that $(\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{D}) = (\mathcal{Z},\mathcal{M}\text{x}\mathcal{D})$. Thereby,

$$(\mho,\mathcal{M})\wedge_{\setminus}(\mathfrak{F},\mathcal{D}) = (\mathcal{Z},\mathcal{M}\text{x}\mathcal{D}) = \{\big((e_2,e_1),\{h_3,h_4\}\big),((e_3,e_1),\{h_1\})\}$$

Suppose that $(\mathcal{Z},\mathcal{M}\text{x}\mathcal{D})\wedge_{\setminus}(\mathcal{V},\mathcal{J}) = \big(\mathcal{W},(\mathcal{M}\text{x}\mathcal{D})\text{x}\mathcal{J}\big)$. Therefore,

$$(\mathcal{Z},\mathcal{M}\text{x}\mathcal{D})\wedge_{\setminus}(\mathcal{V},\mathcal{J}) = \big(\mathcal{W},(\mathcal{M}\text{x}\mathcal{D})\text{x}\mathcal{J}\big) = \{(((e_2,e_1),e_4),\{h_4\}),(((e_3,e_1),e_4),\emptyset)\}$$

It is seen that $\big(\mathfrak{X},\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})\big) \neq_{\mathrm{M}} \big(\mathcal{W},(\mathcal{M}\text{x}\mathcal{D})\text{x}\mathcal{J}\big)$. It is also seen that $\big(\mathfrak{X},\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})\big) \neq_{\mathrm{L}} \big(\mathcal{W},(\mathcal{M}\text{x}\mathcal{D})\text{x}\mathcal{J}\big)$ and $\big(\mathfrak{X},\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})\big) \neq_{\mathrm{J}} \big(\mathcal{W},(\mathcal{M}\text{x}\mathcal{D})\text{x}\mathcal{J}\big)$. □

**Proposition 3.6.** Let $(\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})$ be SSs over $U$. Then, $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) \neq_{\mathrm{M}} (\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})$. That is, $\Lambda_{\backslash}$-product is not commutative in $S_E(U)$.

Proof. Let $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) = (\Omega,\mathcal{D}\mathrm{x}\mathcal{J})$ and $(\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})=(\mathfrak{X},\mathcal{J}\mathrm{x}\mathcal{D})$. Since $\mathcal{D}\mathrm{x}\mathcal{J} \neq \mathcal{J}\mathrm{x}\mathcal{D}$, the rest of the proof is obvious. □

**Proposition 3.7.** Let $(\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})$ be SSs over $U$. Then, $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) \neq_{\mathrm{J}} (\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})$. That is, $\Lambda_{\backslash}$-product is not commutative in $S_E(U)$ under J-equlity.

Proof. In order to show that $\Lambda_{\backslash}$-product is not commutative in $S_E(U)$ under J-equality, we provide an example. Let $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the PS, $\mathcal{D} = \{e_3, e_5\}$ and $\mathcal{J} = \{e_1\}$ be the subsets of $E$, $U = \{h_1, h_2, h_3, h_4, h_5\}$ be the universal set, $(\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})$ be SSs over $U$ such that $(\mathfrak{F},\mathcal{D}) = \{(e_3,\{h_1,h_2\}),(e_5,U)\}$ and $(\mathcal{V},\mathcal{J}) = \{(e_1,\{h_5\})\}$. We show that $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) \neq_{\mathrm{J}} (\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})$. Let $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) = (\Omega,\mathcal{D}\mathrm{x}\mathcal{J})$. Then,

$$(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) = (\Omega,\mathcal{D}\mathrm{x}\mathcal{J}) = \{((e_3,e_1),\{h_1,h_2\}),((e_5,e_1),\{h_1,h_2,h_3,h_4\})\}$$

Suppose that $(\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\mathfrak{z},\mathcal{J}\mathrm{x}\mathcal{D})$. Then,

$$(\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\mathfrak{z},\mathcal{J}\mathrm{x}\mathcal{D}) = \{\big((e_1,e_3),\{h_5\}\big),((e_1,e_5),\emptyset)\}$$

Thus, $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) \neq_{\mathrm{J}} (\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})$.

Moreover, it is obvious that $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) \neq_{\mathrm{L}} (\mathcal{V},\mathcal{J})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})$. □

**Proposition 3.8.** Let $(\mathfrak{F},\mathcal{D})$ be an SS over $U$. Then, $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}\emptyset_{\emptyset} =_{\mathrm{M}} \emptyset_{\emptyset}\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) =_{\mathrm{M}} \emptyset_{\emptyset}$. That is, $\emptyset_{\emptyset}$ (the empty SS) is the absorbing element of $\Lambda_{\backslash}$-product in $S_E(U)$ under M-equality.

Proof. Let $\emptyset_{\emptyset} = (\Omega,\emptyset)$ and $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}\emptyset_{\emptyset} = (\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\Omega,\emptyset) = \big(\mathfrak{w},\mathcal{D}\mathrm{x}\emptyset\big) = (\mathfrak{w},\emptyset)$. Since the only SS whose PS is $\emptyset_{\emptyset}$, $\big(\mathfrak{w},\emptyset\big) = \emptyset_{\emptyset}$. One can similarly show that $\emptyset_{\emptyset}\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) =_{\mathrm{M}} \emptyset_{\emptyset}$. □

**Proposition 3.9.** Let $(\mho,\mathcal{M})$ be an SS over $U$. Then, $\emptyset_{\mathcal{M}}\Lambda_{\backslash}(\mho,\mathcal{M}) =_{\mathrm{L}} \emptyset_{\mathcal{M}}$. That is, $\emptyset_{\mathcal{M}}$ is the left absorbing element of $\Lambda_{\backslash}$-product in $S_{\mathcal{M}}(U)$ under L-equality.

Proof. Let $\emptyset_{\mathcal{M}} = (\mathcal{V},\mathcal{M})$ and $(\mathcal{V},\mathcal{M})\Lambda_{\backslash}(\mho,\mathcal{M}) = (\mathfrak{z},\mathcal{M}\mathrm{x}\mathcal{M})$. Then, for all $m \in \mathcal{M}$, $\mathcal{V}(m) = \emptyset$ and for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, $\mathfrak{z}(m,d) = \mathcal{V}(m) \cap \mho'(d) = \emptyset \cap \mho'(d) = \emptyset$. Since, for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, there exists $m \in \mathcal{M}$ such that $\mathfrak{z}(m,d) = \emptyset = \mathcal{V}(m)$, $\emptyset_{\mathcal{M}}\Lambda_{\backslash}(\mho,\mathcal{M}) \tilde{\subseteq}_{\mathrm{L}} \emptyset_{\mathcal{M}}$. Moreover, for all $m \in \mathcal{M}$, there exists $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$ such that $\mathcal{V}(m) = \emptyset = \mathfrak{z}(m,d)$, implying that $\emptyset_{\mathcal{M}} \tilde{\subseteq}_{\mathrm{L}} \emptyset_{\mathcal{M}}\Lambda_{\backslash}(\mho,\mathcal{M})$ Thereby, $\emptyset_{\mathcal{M}}\Lambda_{\backslash}(\mho,\mathcal{M}) =_{\mathrm{L}} \emptyset_{\mathcal{M}}$. □

**Proposition 3.10.** Let $(\mho,\mathcal{M})$ be an SS over $U$. Then, $(\mho,\mathcal{M})\Lambda_{\backslash}\emptyset_{\mathcal{M}} =_{\mathrm{L}} (\mho,\mathcal{M})$. That is, $\emptyset_{\mathcal{M}}$ is the right identity element of $\Lambda_{\backslash}$-product in $S_{\mathcal{M}}(U)$ under L-equality.

Proof. Let $\emptyset_{\mathcal{M}} = (\mathcal{V},\mathcal{M})$ and $(\mho,\mathcal{M})\Lambda_{\backslash}(\mathcal{V},\mathcal{M}) = (\mathfrak{z},\mathcal{M}\mathrm{x}\mathcal{M})$. Then, for all $m \in \mathcal{M}$, $\mathcal{V}(m) = \emptyset$ and for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, $\mathfrak{z}(m,d) = \mho(m) \cap \emptyset' = \mho(d) \cap U = \mho(d)$. Since, for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, there exists $d \in \mathcal{M}$ such that $\mathfrak{z}(m,d) = \mho(d)$ , $(\mho,\mathcal{M})\Lambda_{\backslash}\emptyset_{\mathcal{M}} \tilde{\subseteq}_{\mathrm{L}} (\mho,\mathcal{M})$. Moreover, for all $d \in \mathcal{M}$, there exists $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$ such that $\mho(d) = \mathfrak{z}(m,d)$, implying that $(\mho,\mathcal{M})_{\mathrm{L}} \tilde{\subseteq}_{\mathrm{L}} (\mho,\mathcal{M})\Lambda_{\backslash}\emptyset_{\mathcal{M}}$. Thereby, $(\mho,\mathcal{M})\Lambda_{\backslash}\emptyset_{\mathcal{M}} =_{\mathrm{L}} (\mho,\mathcal{M})$. □

**Proposition 3.11.** Let $(\mathfrak{F},\mathcal{D})$ be an SS over $U$. Then, $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}\emptyset_{\mathcal{D}} =_{\mathrm{M}} (\mathfrak{F},\mathcal{D}\mathrm{x}\mathcal{D})$ and $\emptyset_{\mathcal{D}}\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) =_{\mathrm{M}} \emptyset_{\mathcal{D}\mathrm{x}\mathcal{D}}$.

Proof. Let $\emptyset_{\mathcal{D}} = (\Omega,\mathcal{D})$. Then, for all $d \in \mathcal{D}, \Omega(d) = \emptyset$. Let $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}\emptyset_{\mathcal{D}} = (\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\Omega,\mathcal{D}) = \big(\mathfrak{w},\mathcal{D}\mathrm{x}\mathcal{D}\big)$. Thus, for all $(d,m) \in \mathcal{D}\mathrm{x}\mathcal{D}, \mathfrak{w}(d,m) = \mathfrak{F}(d) \cap \Omega'(m) = \mathfrak{F}(d) \cap \emptyset' = \mathfrak{F}(d) \cap \mathrm{U} = \mathfrak{F}(d)$, implying that $\big(\mathfrak{w},\mathcal{D}\mathrm{x}\mathcal{D}\big) = (\mathfrak{F},\mathcal{D}\mathrm{x}\mathcal{D})$. Let $\emptyset_{\mathcal{D}}\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\Omega,\mathcal{D})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\mathcal{V},\mathcal{D}\mathrm{x}\mathcal{D})$. Then, for all $(d,m) \in \mathcal{D}\mathrm{x}\mathcal{D}, \mathcal{V}(d,m) = \Omega(d) \cap \mathfrak{F}'(m) = \emptyset \cap \mathfrak{F}'(m) = \emptyset$, hence $(\mathcal{V},\mathcal{D}\mathrm{x}\mathcal{D}) = \emptyset_{\mathcal{D}\mathrm{x}\mathcal{D}}$. □

**Proposition 3.12.** Let $(\mathfrak{F},\mathcal{D})$ be an SS over $U$. Then, $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}\mathrm{U}_{\mathcal{D}} =_{\mathrm{M}} \emptyset_{\mathcal{D}\mathrm{x}\mathcal{D}}$ and $\mathrm{U}_{\mathcal{D}}\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) =_{\mathrm{M}} (\mathfrak{F},\mathcal{D}\mathrm{x}\mathcal{D})^{\mathrm{r}}$.

Proof. Let $\mathrm{U}_{\mathcal{D}} = (\mathcal{V},\mathcal{D})$. Then, for all $d \in \mathcal{D}$, $\mathcal{V}(d) = \mathrm{U}$. Let $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}\mathrm{U}_{\mathcal{D}} = (\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{D}) = (\mathfrak{X},\mathcal{D}\mathrm{x}\mathcal{D})$. Thus, for all $(d,m) \in \mathcal{D}\mathrm{x}\mathcal{D}$, $\mathfrak{X}(d,m) = \mathfrak{F}(d) \cap \mathcal{V}'(m) = \mathfrak{F}(d) \cap \mathrm{U}' = \mathfrak{F}(d) \cap \emptyset = \emptyset$, implying that $(\mathfrak{X},\mathcal{D}\mathrm{x}\mathcal{D}) = \emptyset_{\mathcal{D}\mathrm{x}\mathcal{D}}$. Let $\mathrm{U}_{\mathcal{D}}\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\mathcal{V},\mathcal{D})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\wp,\mathcal{D}\mathrm{x}\mathcal{D})$. Then, for all $(d,m) \in \mathcal{D}\mathrm{x}\mathcal{D}$, $\wp(d,m) = \mathcal{V}(d) \cap \mathfrak{F}'(m) = \mathrm{U} \cap \mathfrak{F}'(m) = \mathfrak{F}'(m)$, implying that $(\wp,\mathcal{D}\mathrm{x}\mathcal{D}) = (\mathfrak{F},\mathcal{D}\mathrm{x}\mathcal{D})^{\mathrm{r}}$. □

**Proposition 3.13.** Let $(\mho,\mathcal{M})$ be an SS over $U$. Then, $(\mho,\mathcal{M})\Lambda_{\backslash}(\mho,\mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mho,\mathcal{M})$. That is, $\Lambda_{\backslash}$-product is not idempotent in $S_E(U)$ under J-equality.

Proof. Let $(\mho,\mathcal{M})\Lambda_{\backslash}(\mho,\mathcal{M}) = (\mathfrak{F},\mathcal{M}\mathrm{x}\mathcal{M})$. Then, for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, $\mathfrak{F}(m,d) = \mho(m) \cap \mho'(d)$. Since for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{M}$, there exists $m \in \mathcal{M}$ such that $\mathfrak{F}(m,d) = \mho(m) \cap \mho'(d) \subseteq \mho(m)$, $(\mathfrak{F},\mathcal{M}\mathrm{x}\mathcal{M}) \tilde{\subseteq}_{\mathrm{J}} (\mho,\mathcal{M})$ is obtained. □

**Proposition 3.14.** Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over $U$, Then, $(\mho,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F},\mathcal{D})^{\mathrm{r}}$ and $(\mho,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (\mho,\mathcal{M})$. □

Proof. Let $(\mho,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\mathcal{V},\mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m,d) = \mho(m) \cap \mathfrak{F}'(d)$. Since for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$ , there exists $d \in \mathcal{D}$ such that $\mho(m) \cap \mathfrak{F}'(d) \subseteq \mathfrak{F}'(d)$, $(\mho,\mathcal{M})\Lambda_{\theta}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (\mathfrak{F},\mathcal{D})^{\mathrm{r}}$. Similarly, since for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, there exists $m \in \mathcal{M}$ such that $\mho(m) \cap \mathfrak{F}'(d) \subseteq \mho(m)$, $(\mho,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{\mathrm{J}} (\mho,\mathcal{M})$ is obtained.

**Proposition 3.15.** Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over $U$. Then, $[(\mho,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})]^{\mathrm{r}} = (\mho,\mathcal{M})^{\mathrm{r}} \underline{\vee} (\mathfrak{F},\mathcal{D})^{\mathrm{r}}$.

Proof. Let $(\mho,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\mathcal{V},\mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m,d) = \mho(m) \cap \mathfrak{F}'(d)$. Thus, $\mathcal{V}'(m,d) = \mho'(m) \cup \mathfrak{F}(d) = \mho'(m) \cup (\mathfrak{F}')'(d)$. Hence, $(\mathcal{V}',\mathcal{M}\mathrm{X}\mathcal{D}) = (\mho,\mathcal{M})^{\mathrm{r}} \underline{\vee} (\mathfrak{F},\mathcal{D})^{\mathrm{r}}$. (For $\underline{\vee}$-product, please see [11]). □

**Proposition 3.16.** Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over $U$. Then, $(\mho,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) \tilde{\subseteq}_{\mathrm{F}} (\mho,\mathcal{M}) \underline{\vee} (\mathfrak{F},\mathcal{D})$.

Proof. Let $(\mho,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D}) = (\mathcal{V},\mathcal{M}\mathrm{x}\mathcal{D})$ and $(\mho,\mathcal{M}) \underline{\vee} (\mathfrak{F},\mathcal{D}) = (\sigma,\mathcal{M}\mathrm{x}\mathcal{D})$. Then, for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m,d) = \mho(m) \cap \mathfrak{F}'(d)$ and for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\sigma(m,d) = \mho(m) \cup \mathfrak{F}'(d)$. Thus, for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $\mathcal{V}(m,d) = \mho(m) \cap \mathfrak{F}'(d) \subseteq \mho(m) \cup \mathfrak{F}'(d) = \sigma(m,d)$. □

**Proposition 3.17.** Let $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$, and $(\mathcal{V},\mathcal{J})$ be SSs over $U$. If $(\mho,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D})$, then $(\mho,\mathcal{M})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mathcal{V},\mathcal{J})$.

Proof. Let $(\mho,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D})$. Then, $\mathcal{M} \subseteq \mathcal{D}$ and for all $m \in \mathcal{M}$, $\mho(m) \subseteq \mathfrak{F}(m)$. Thus, $\mathcal{M} \mathrm{x} \mathcal{J} \subseteq \mathcal{D} \mathrm{x} \mathcal{J}$ and for all $(m,\mathrm{j}) \in \mathcal{M} \mathrm{x} \mathcal{J}$, $\mho(m) \cap \mathcal{V}'(\mathrm{j}) \subseteq \mathfrak{F}(m) \cap \mathcal{V}'(\mathrm{j})$. □

**Proposition 3.18.** Let $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$, $(\mathcal{V},\mathcal{J})$, and $(\sigma,\mathfrak{X})$ be SSs over $U$. If $(\mho,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})^{\mathrm{r}} \tilde{\subseteq}_{\mathrm{F}} (\sigma,\mathfrak{X})^{\mathrm{r}}$, then $(\mho,\mathcal{M})\Lambda_{\backslash}(\mathcal{V},\mathcal{J}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\sigma,\mathfrak{X})$.

Proof. Let $(\mho,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D})$ and $(\mathcal{V},\mathcal{J})^{\mathrm{r}} \tilde{\subseteq}_{\mathrm{F}} (\sigma,\mathfrak{X})^{\mathrm{r}}$. Then, $\mathcal{M} \subseteq \mathcal{D}$ , $\mathcal{J} \subseteq \mathfrak{X}$, for all $m \in \mathcal{M}$, $\mho(m) \subseteq \mathfrak{F}(m)$ and for all $\mathrm{j} \in \mathcal{J}$, $\mathcal{V}'(\mathrm{j}) \subseteq \sigma'(\mathrm{j})$. Thus, $\mathcal{M}\mathrm{x}\mathcal{J} \subseteq \mathcal{D}\mathrm{x}\mathfrak{X}$, for all $(m,\mathrm{j}) \in \mathcal{M}\mathrm{x}\mathcal{J}$, $\mho(m) \cap \mathcal{V}'(\mathrm{j}) \subseteq \mathfrak{F}(m) \cap \sigma'(\mathrm{j})$. □

**Proposition 3.19.** Let $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{M})$, $(\sigma,\mathcal{M})$, and $(\mathcal{V},\mathcal{M})$ be SSs over $U$. If $(\mho,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{M})$ and $(\mathcal{V},\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\sigma,\mathcal{M})$, then $(\mho,\mathcal{M})\Lambda_{\backslash}(\sigma,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{M})\Lambda_{\backslash}(\mathcal{V},\mathcal{M})$.

Proof. Let $(\mho,\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{M})$ and $(\mathcal{V},\mathcal{M}) \tilde{\subseteq}_{\mathrm{F}} (\sigma,\mathcal{M})$. Thus, for all $m \in \mathcal{M}$, $\mho(m) \subseteq \mathfrak{F}(m)$ and for all $\mathrm{j} \in \mathcal{M}$, $\mathcal{V}(\mathrm{j}) \subseteq \sigma(\mathrm{j})$. Hence, for all $(m,\mathrm{j}) \in \mathcal{M}\mathrm{x}\mathcal{M}$, $\mho(m) \cap \sigma'(\mathrm{j}) \subseteq \mathfrak{F}(m) \cap \mathcal{V}'(\mathrm{j})$. □

**Proposition 3.20.** Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SSs over $U$. Then, $\emptyset_{\mathcal{M}\mathrm{x}\mathcal{D}} \tilde{\subseteq}_{\mathrm{F}} (\mho,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})$ and $\emptyset_{\mathcal{D}\mathrm{x}\mathcal{M}} \tilde{\subseteq}_{\mathrm{F}} (\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(\mho,\mathcal{M})$.

Proof. Let $\emptyset_{\mathcal{M}x\mathcal{D}} = (\mathcal{V}, \mathcal{M}x\mathcal{D})$ and $(\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D}) = (\mathcal{Z}, \mathcal{M}x\mathcal{D})$. Then, for $(m, d) \in \mathcal{M}x\mathcal{D}$, $\mathcal{V}(m, d) = \emptyset$ and for all $(m, d) \in \mathcal{M}x\mathcal{D}$, $\mathcal{Z}(m, d) = \mho(m) \cap \mathfrak{F}'(d)$. Since $\mathcal{M}x\mathcal{D} \subseteq \mathcal{M}x\mathcal{D}$ and for all $(m, d) \in \mathcal{M}x\mathcal{D}$, $\mathcal{V}(m, d) = \emptyset \subseteq \mho(m) \cap \mathfrak{F}'(d) = \mathcal{Z}(m, d)$, $\emptyset_{\mathcal{M}x\mathcal{D}} \tilde{\subseteq}_{F} (\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D})$ is obtained. Similarly, $\emptyset_{\mathcal{D}x\mathcal{M}} \tilde{\subseteq}_{F} (\mathfrak{F}, \mathcal{D})\Lambda_{\setminus}(\mho, \mathcal{M})$ can be illustrated. □

**Proposition 3.21.** Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be SSs over $U$. Then, $\emptyset_{\mathcal{M}} \tilde{\subseteq}_{J} (\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D})$, $\emptyset_{\mathcal{D}} \tilde{\subseteq}_{J} (\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D})$ and $\emptyset_{E} \tilde{\subseteq}_{J} (\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D})$.

Proof. Let $\emptyset_{\mathcal{M}} = (\mathcal{V}, \mathcal{M})$ and $(\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D}) = (\mathcal{Z}, \mathcal{M}x\mathcal{D})$. Then, for all $m \in \mathcal{M}$, $\mathcal{V}(m) = \emptyset$ and for all $(m, d) \in \mathcal{M}x\mathcal{D}$, $\mathcal{Z}(m, d) = \mho(m) \cap \mathfrak{F}'(d)$. Since for all $m \in \mathcal{M}$, there exist $(m, d) \in \mathcal{M}x\mathcal{D}$ such that $\mathcal{V}(m) = \emptyset \subseteq \mho(m) \cap \mathfrak{F}'(d) = \mathcal{Z}(m, d)$, $\emptyset_{\mathcal{M}} \tilde{\subseteq}_{J} (\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D})$ is obtained. One can similarly show that $\emptyset_{\mathcal{D}} \tilde{\subseteq}_{J} (\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D})$ and $\emptyset_{E} \tilde{\subseteq}_{J} (\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D})$. □

**Proposition 3.22.** Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be SSs over $U$. Then, $(\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{F} U_{\mathcal{M}x\mathcal{D}}$ and $(\mathfrak{F}, \mathcal{D})\Lambda_{\setminus}(\mho, \mathcal{M}) \tilde{\subseteq}_{F} U_{\mathcal{D}x\mathcal{M}}$.

Proof. Let $U_{\mathcal{M}x\mathcal{D}} = (\mathcal{V}, \mathcal{M}x\mathcal{D})$ and $(\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D}) = (\mathcal{Z}, \mathcal{M}x\mathcal{D})$. Then, for all $(m, d) \in \mathcal{M}x\mathcal{D}$, $\mathcal{V}(m, d) = U$ and for all $(m, d) \in \mathcal{M}x\mathcal{D}$, $\mathcal{Z}(m, d) = \mho(m) \cap \mathfrak{F}'(d)$. Since $\mathcal{M}x\mathcal{D} \subseteq \mathcal{M}x\mathcal{D}$ and for all $(m, d) \in \mathcal{M}x\mathcal{D}$, $\mathcal{Z}(m, d) = \mho(m) \cap \mathfrak{F}'(d) \subseteq U$, $(\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{F} U_{\mathcal{M}x\mathcal{D}}$ is obtained. One can similarly show that $(\mathfrak{F}, \mathcal{D})\Lambda_{\setminus}(\mho, \mathcal{M}) \tilde{\subseteq}_{F} U_{\mathcal{D}x\mathcal{M}}$. □

**Proposition 3.23.** Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be SSs over $U$. Then, $(\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{J} U_{\mathcal{M}}$, $(\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{J} U_{\mathcal{D}}$.

Proof. Let $U_{\mathcal{M}} = (\mathcal{V}, \mathcal{M})$ and $(\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D}) = (\mathcal{Z}, \mathcal{M}x\mathcal{D})$. Then, for all $m \in \mathcal{M}$, $\mathcal{V}(m) = U$ and for all $(m, d) \in \mathcal{M}x\mathcal{D}$, $\mathcal{Z}(m, d) = \mho(m) \cap \mathfrak{F}'(d)$. Since for all $(m, d) \in \mathcal{M}x\mathcal{D}$, there exist $m \in \mathcal{M}$ such that $\mathcal{Z}(m, d) = \mho(m) \cap \mathfrak{F}'(d) \subseteq U = \mathcal{V}(m)$, $(\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{J} U_{\mathcal{M}}$ is obtained. Similarly, $(\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D}) \tilde{\subseteq}_{J} U_{\mathcal{D}}$ can be observed. □

**Proposition 3.24.** Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be SSs over $U$. Then, $(\mho, \mathcal{M})\Lambda_{\theta}(\mathfrak{F}, \mathcal{D}) =_{M} U_{\mathcal{M}x\mathcal{D}}$ if and only if $(\mho, \mathcal{M}) =_{M} U_{\mathcal{M}}$ and $(\mathfrak{F}, \mathcal{D}) =_{M} \emptyset_{\mathcal{D}}$.

Proof. Let $U_{\mathcal{M}x\mathcal{D}} = (\mathcal{U}, \mathcal{M}x\mathcal{D})$ and $(\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D}) = (\mathcal{V}, \mathcal{M}x\mathcal{D})$. Then, for all $(m, d) \in \mathcal{M}x\mathcal{D}$, $\mathcal{U}(m, d) = U$ and for all $(m, d) \in \mathcal{M}x\mathcal{D}$, $\mathcal{V}(m, d) = \mho(m) \cap \mathfrak{F}'(d)$. Let $(\mathcal{U}, \mathcal{M}x\mathcal{D}) = (\mathcal{V}, \mathcal{M}x\mathcal{D})$. Then, for all $(m, d) \in \mathcal{M}x\mathcal{D}$, $\mho(m) \cap \mathfrak{F}'(d) = U$. Thus, for all $m \in \mathcal{M}$, $\mho(m) = U$ and for all $d \in \mathcal{D}$, $\mathfrak{F}'(d) = U$. Thereby, $(\mho, \mathcal{M}) = U_{\mathcal{M}}$ and $(\mathfrak{F}, \mathcal{D}) = \emptyset_{\mathcal{D}}$.

Conversely, let $(\mho, \mathcal{M}) =_{M} U_{\mathcal{M}}$ and $(\mathfrak{F}, \mathcal{D}) =_{M} \emptyset_{\mathcal{D}}$. Then, for all $m \in \mathcal{M}$, $\mho(m) = U$ and for all $d \in \mathcal{D}$, $\mathfrak{F}(d) = \emptyset$. Thus, for all $(m, d) \in \mathcal{M}x\mathcal{D}$, $\mathcal{V}(m, d) = \mho(m) \cap \mathfrak{F}'(d) = U \cap U = U$, implying that $(\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D}) =_{M} U_{\mathcal{M}x\mathcal{D}}$. □

**Proposition 3.25.** Let $(\mho, \mathcal{M})$ and $(\mathfrak{F}, \mathcal{D})$ be SSs over $U$. Then, $(\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D}) =_{M} \emptyset_{\emptyset}$ if and only if $(\mho, \mathcal{M}) =_{M} \emptyset_{\emptyset}$ or $(\mathfrak{F}, \mathcal{D}) =_{M} \emptyset_{\emptyset}$.

Proof. Let $(\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D}) =_{M} \emptyset_{\emptyset}$. Then, $\mathcal{M}x\mathcal{D} = \emptyset$, and so $\mathcal{M} = \emptyset$ or $\mathcal{D} = \emptyset$ Since $\emptyset_{\emptyset}$ is the only SS with the empty PS, $(\mho, \mathcal{M}) =_{M} \emptyset_{\emptyset}$ or $(\mathfrak{F}, \mathcal{D}) =_{M} \emptyset_{\emptyset}$ .

Conversely, let $(\mho, \mathcal{M}) =_{M} \emptyset_{\emptyset}$ or $(\mathfrak{F}, \mathcal{D}) =_{M} \emptyset_{\emptyset}$. Thus, $\mathcal{M} = \emptyset$ or $\mathcal{D} = \emptyset$, implying that $\mathcal{M}x\mathcal{D} = \emptyset$ and $(\mho, \mathcal{M})\Lambda_{\setminus}(\mathfrak{F}, \mathcal{D}) =_{M} \emptyset_{\emptyset}$. □

## 4. Distributions of Soft Difference-Product over Certain Types of Soft Set Operations

In this section, we explore the distributions of soft difference-product over restricted, extended, soft binary piecewise intersection and union operations, AND-product and OR-product.

**Theorem 4.1.** Let $(Ʊ,\mathcal{M})$, $(𝔉,\mathcal{D})$, and $(Ꮙ,\mathcal{J})$ be SSs over $U$. Then, we have the following distributions of soft difference-product over restricted intersection and union operations:

*i.* $(Ʊ,\mathcal{M})\Lambda_{\backslash}[(𝔉,\mathcal{D}) \cup_R (Ꮙ,\mathcal{J})] =_M [(Ʊ,\mathcal{M})\Lambda_{\backslash}(𝔉,\mathcal{D})] \cap_R [(Ʊ,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$
*ii.* $(Ʊ,\mathcal{M})\Lambda_{\backslash}[(𝔉,\mathcal{D}) \cap_R (Ꮙ,\mathcal{J})] =_M [(Ʊ,\mathcal{M})\Lambda_{\backslash}(𝔉,\mathcal{D})] \cup_R [(Ʊ,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$
*iii.* $[(Ʊ,\mathcal{M}) \cap_R (𝔉,\mathcal{D})]\Lambda_{\backslash}(Ꮙ,\mathcal{J}) =_M [(Ʊ,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})] \cap_R [(𝔉,\mathcal{D})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$
*iv.* $[(Ʊ,\mathcal{M}) \cup_R (𝔉,\mathcal{D})]\Lambda_{\backslash}(Ꮙ,\mathcal{J}) =_M [(Ʊ,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})] \cup_R [(𝔉,\mathcal{D})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$

PROOF.

*i.* The PS of the (left-hand side) LHS is $\mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J})$, and the PS of the right-hand side (RHS) is $(\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J})$. Since $\mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J}) = (\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J})$, the first condition of the M-equality is satisfied. Let $(𝔉,\mathcal{D}) \cup_R (Ꮙ,\mathcal{J}) = (𝔛,\mathcal{D} \cap \mathcal{J})$, where for all $\varphi \in \mathcal{D} \cap \mathcal{J}$, $𝔛(\varphi) = 𝔉(\varphi) \cup Ꮙ(\varphi)$. Let $(𝔉,\mathcal{D}) \cup_R (Ꮙ,\mathcal{J}) = (𝔛,\mathcal{D} \cap \mathcal{J})$, where for all $\varphi \in \mathcal{D} \cap \mathcal{J}$, $𝔛(\varphi) = 𝔉(\varphi) \cup Ꮙ(\varphi)$ and $(Ʊ,\mathcal{M})\Lambda_{\backslash}(𝔛,\mathcal{D} \cap \mathcal{J}) = (ƻ,\mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J}))$, where for all $(m,\varphi) \in \mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J})$, $ƻ(m,\varphi) = Ʊ(m) \cap 𝔛'(\varphi)$. Thus

$$ƻ(m,\varphi) = Ʊ(m) \cap [𝔉(\varphi) \cup Ꮙ(\varphi)]' = Ʊ(m) \cap [𝔉'(\varphi) \cap Ꮙ'(\varphi)]$$

Let $(Ʊ,\mathcal{M})\Lambda_{\backslash}(𝔉,\mathcal{D}) = (Ŧ,\mathcal{M}\mathrm{x}\mathcal{D})$ and $(Ʊ,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J}) = (ღ,\mathcal{M}\mathrm{x}\mathcal{J})$, where for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$, $Ŧ(m,d) = Ʊ(m) \cap 𝔉'(d)$ and for all $(m,\mathrm{j}) \in \mathcal{M}\mathrm{x}\mathcal{J}$, $ღ(m,\mathrm{j}) = Ʊ(m) \cap Ꮙ'(\mathrm{j})$. Suppose that $(Ŧ,\mathcal{M}\mathrm{x}\mathcal{D}) \cap_R (ღ,\mathcal{M}\mathrm{x}\mathcal{J}) = (☼,(\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J}))$, where for all $(m,\varphi) \in (\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J}) = \mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J})$,

$$☼(m,\varphi) = Ŧ(m,\varphi) \cap ღ(m,\varphi) = [Ʊ(m) \cap 𝔉'(\varphi)] \cap [Ʊ(m) \cap Ꮙ'(\varphi)]$$

Thereby, $(Ʊ,\mathcal{M})\Lambda_{\backslash}[(𝔉,\mathcal{D}) \cup_R (Ꮙ,\mathcal{J})] =_M [(Ʊ,\mathcal{M})\Lambda_{\backslash}(𝔉,\mathcal{D})] \cap_R [(Ʊ,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$.

Here, if $\mathcal{D} \cap \mathcal{J} = \emptyset$, then $\mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J}) = (\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J}) = \emptyset$. Since the only soft set with an empty PS is $\emptyset_\emptyset$, then both sides are $\emptyset_\emptyset$. Since $(\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J}) = \mathcal{M}\mathrm{x}(\mathcal{D} \cap \mathcal{J})$, if $(\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J})=\emptyset$, then $\mathcal{M}=\emptyset$ or $\mathcal{D} \cap \mathcal{J} = \emptyset$. By assumption, $\mathcal{M} \neq \emptyset$. Thus, $(\mathcal{M}\mathrm{x}\mathcal{D}) \cap (\mathcal{M}\mathrm{x}\mathcal{J})=\emptyset$ implies that $\mathcal{D} \cap \mathcal{J} = \emptyset$. Therefore, under this condition, both sides are again $\emptyset_\emptyset$. □

*iii.* The PS of the LHS is $(\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J}$, the PS of the RHS is $(\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J})$, and since $(\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J} = (\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J})$, the first condition of M-equality is satisfied. Let $(Ʊ,\mathcal{M}) \cap_R (𝔉,\mathcal{D}) = (𝔛,\mathcal{M} \cap \mathcal{D})$, where for all $\varphi \in \mathcal{M} \cap \mathcal{D}$, $𝔛(\varphi) = Ʊ(\varphi) \cap 𝔉(\varphi)$ and $(𝔛,\mathcal{M} \cap \mathcal{D})\Lambda_{\backslash}(Ꮙ,\mathcal{J}) = (ƻ,(\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J})$, where for all $(\varphi,\mathrm{j}) \in (\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J}$, $ƻ(\varphi,\mathrm{j}) = 𝔛(\varphi) \cap Ꮙ'(\mathrm{j})$. Thus,

$$ƻ(\varphi,\mathrm{j}) = [Ʊ(\varphi) \cap 𝔉(\varphi)] \cap Ꮙ'(\mathrm{j})$$

Let $(Ʊ,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J}) = (Ŧ,\mathcal{M}\mathrm{x}\mathcal{J})$ and $(𝔉,\mathcal{D})\Lambda_{\backslash}(Ꮙ,\mathcal{J}) = (ღ,\mathcal{D}\mathrm{x}\mathcal{J})$, where for all $(m,\mathrm{j}) \in \mathcal{M}\mathrm{x}\mathcal{J}$, $Ŧ(m,\mathrm{j}) = Ʊ(m) \cap Ꮙ'(\mathrm{j})$ and for all $(d,\mathrm{j}) \in \mathcal{D}\mathrm{x}\mathcal{J}$, $ღ(d,\mathrm{j}) = 𝔉(d) \cap Ꮙ'(\mathrm{j})$. Assume that $(Ŧ,\mathcal{M}\mathrm{x}\mathcal{J}) \cap_R (ღ,\mathcal{D}\mathrm{x}\mathcal{J}) = (☼,(\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J}))$, where for all $(\varphi,\mathrm{j}) \in (\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J}) = (\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J}$,

$$☼(\varphi,\mathrm{j}) = Ŧ(\varphi,\mathrm{j}) \cap ღ(\varphi,\mathrm{j}) = [Ʊ(\varphi) \cap Ꮙ'(\mathrm{j})] \cap [𝔉(\varphi) \cap Ꮙ'(\mathrm{j})]$$

Thus, $[(Ʊ,\mathcal{M}) \cap_R (𝔉,\mathcal{D})]\Lambda_{\backslash}(Ꮙ,\mathcal{J}) =_M [(Ʊ,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})] \cap_R [(𝔉,\mathcal{D})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$.

Here, if $\mathcal{M} \cap \mathcal{D} = \emptyset$, then $(\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J} = (\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J}) = \emptyset$. Since the only soft set with the empty parameter set is $\emptyset_\emptyset$, both sides of the equality are $\emptyset_\emptyset$. Moreover, since $(\mathcal{M}\mathrm{x}\mathcal{J}) \cap (\mathcal{D}\mathrm{x}\mathcal{J}) = (\mathcal{M} \cap \mathcal{D})\mathrm{x}\mathcal{J}$, if

$(\mathcal{M}\text{x}\mathcal{J}) \cap (\mathcal{D}\text{x}\mathcal{J})=\emptyset$, then $\mathcal{M} \cap \mathcal{D} = \emptyset$ or $\mathcal{J}=\emptyset$. By assumption, $\mathcal{J} \neq \emptyset$. Thus, $(\mathcal{M}\text{x}\mathcal{J}) \cap (\mathcal{D}\text{x}\mathcal{J}) = \emptyset$ implies that $\mathcal{M} \cap \mathcal{D} = \emptyset$. Hence, under this condition, both sides of the equality are again $\emptyset_\emptyset$. □

**Note 4.2.** The restricted soft set operation can not distribute over soft difference-product as the intersection does not distribute over cartesian product and it is compulsory for two SSs to be M-equal that their PS should be the same.

**Theorem 4.3.** Let $(Ʊ,\mathcal{M})$, $(𝔉,\mathcal{D})$, and $(Ꮙ,\mathcal{J})$ be SSs over $U$. Then, we have the following distributions of soft difference-product over extended intersection and union operations:

*i.* $(Ʊ,\mathcal{M})\Lambda_\backslash[(𝔉,\mathcal{D}) \cap_\varepsilon (Ꮙ,\mathcal{J})] =_{\text{M}} [(Ʊ,\mathcal{M})\Lambda_\backslash(𝔉,\mathcal{D})] \cup_\varepsilon [(Ʊ,\mathcal{M})\Lambda_\backslash(Ꮙ,\mathcal{J})]$
*ii.* $(Ʊ,\mathcal{M})\Lambda_\backslash[(𝔉,\mathcal{D}) \cup_\varepsilon (Ꮙ,\mathcal{J})] =_{\text{M}} [(Ʊ,\mathcal{M})\Lambda_\backslash(𝔉,\mathcal{D})] \cap_\varepsilon [(Ʊ,\mathcal{M})\Lambda_\backslash(Ꮙ,\mathcal{J})]$
*iii.* $[(Ʊ,\mathcal{M}) \cup_\varepsilon (𝔉,\mathcal{D})]\Lambda_\backslash(Ꮙ,\mathcal{J}) =_{\text{M}} [(Ʊ,\mathcal{M})\Lambda_\backslash(Ꮙ,\mathcal{J})] \cup_\varepsilon [(𝔉,\mathcal{D})\Lambda_\backslash(Ꮙ,\mathcal{J})]$
*iv.* $[(Ʊ,\mathcal{M}) \cap_\varepsilon (𝔉,\mathcal{D})]\Lambda_\backslash(Ꮙ,\mathcal{J}) =_{\text{M}} [(Ʊ,\mathcal{M})\Lambda_\backslash(Ꮙ,\mathcal{J})] \cap_\varepsilon [(𝔉,\mathcal{D})\Lambda_\backslash(Ꮙ,\mathcal{J})]$

Proof.

*i.* The PS of the LHS is $\mathcal{M}\text{x}(\mathcal{D} \cup \mathcal{J})$, and the PS of the RHS is $(\mathcal{M}\text{x}\mathcal{D}) \cup (\mathcal{M}\text{x}\mathcal{J})$. Since $\mathcal{M}\text{x}(\mathcal{D} \cup \mathcal{J}) = (\mathcal{M}\text{x}\mathcal{D}) \cup (\mathcal{M}\text{x}\mathcal{J})$, the first condition of the M-equality is satisfied. As $\mathcal{M} \neq \emptyset$, $\mathcal{D} \neq \emptyset$ and $\mathcal{J} \neq \emptyset$, $\mathcal{M}\text{x}(\mathcal{D} \cup \mathcal{J}) \neq \emptyset$ and $(\mathcal{M}\text{x}\mathcal{D}) \cup (\mathcal{M}\text{x}\mathcal{J}) \neq \emptyset$. Thus, no side may be equal to an empty soft set. Let $(𝔉,\mathcal{D}) \cap_\varepsilon (Ꮙ,\mathcal{J}) = (𝔛,\mathcal{D} \cup \mathcal{J})$, where for all $\varphi \in \mathcal{D} \cup \mathcal{J}$,

$$𝔛(\varphi)=\begin{cases} 𝔉(\varphi), & \varphi\in\mathcal{D}\text{-}\mathcal{J} \\ Ꮙ(\varphi), & \varphi\in\mathcal{J}\text{-}\mathcal{D} \\ 𝔉(\varphi) \cap Ꮙ(\varphi), & \varphi\in\mathcal{D} \cap \mathcal{J} \end{cases}$$

Let $(Ʊ,\mathcal{M})\Lambda_\backslash(𝔛,\mathcal{D} \cup \mathcal{J}) = \big(ƻ,\mathcal{M}\text{x}(\mathcal{D} \cup \mathcal{J})\big)$, where for all $(m,\varphi) \in \mathcal{M}\text{x}(\mathcal{D} \cup \mathcal{J})$, $ƻ(m,\varphi) = Ʊ(m) \cap 𝔛'(\varphi)$. Thus, for all $(m,\varphi) \in \mathcal{M}\text{x}(\mathcal{D} \cup \mathcal{J})$,

$$ƻ(m,\varphi)=\begin{cases} Ʊ(m) \cap 𝔉'(\varphi), & (m,\varphi)\in\mathcal{M}\text{x}(\mathcal{D}\text{-}\mathcal{J}) \\ Ʊ(m) \cap Ꮙ'(\varphi), & (m,\varphi)\in\mathcal{M}\text{x}(\mathcal{J}\text{-}\mathcal{D}) \\ Ʊ(m) \cap [𝔉'(\varphi) \cup Ꮙ'(\varphi)], & (m,\varphi)\in\mathcal{M}\text{x}(\mathcal{D} \cap \mathcal{J}) \end{cases}$$

Assume that $(Ʊ,\mathcal{M})\Lambda_\backslash(𝔉,\mathcal{D}) = (Ŧ,\mathcal{M}\text{x}\mathcal{D})$ and $(Ʊ,\mathcal{M})\Lambda_\backslash(Ꮙ,\mathcal{J}) = (ღ,\mathcal{M}\text{x}\mathcal{J})$, where for all $(m,d) \in \mathcal{M}\text{x}\mathcal{D}$, $Ŧ(m,d) = Ʊ(m) \cap 𝔉'(d)$ and for all $(m,\mathrm{j}) \in \mathcal{M}\text{x}\mathcal{J}$, $ღ(m,\mathrm{j}) = Ʊ(m) \cap Ꮙ'(\mathrm{j})$. Suppose that $(Ŧ,\mathcal{M}\text{x}\mathcal{D}) \cup_\varepsilon \big(ღ,\mathcal{M}\text{x}\mathcal{J}\big) = (☼,(\mathcal{M}\text{x}\mathcal{D}) \cup (\mathcal{M}\text{x}\mathcal{J}))$, where for al $(m,\varphi) \in (\mathcal{M}\text{x}\mathcal{D}) \cup (\mathcal{M}\text{x}\mathcal{J}) = \mathcal{M}\text{x}(\mathcal{D} \cup \mathcal{J})$,

$$☼(m,\varphi)=\begin{cases} Ŧ(m,\varphi), & (m,\varphi)\in(\mathcal{M}\text{x}\mathcal{D})\text{-}(\mathcal{M}\text{x}\mathcal{J}) = \mathcal{M}\text{x}(\mathcal{D}\text{-}\mathcal{J}) \\ ღ(m,\varphi), & (m,\varphi)\in(\mathcal{M}\text{x}\mathcal{J})\text{-}(\mathcal{M}\text{x}\mathcal{D}) = \mathcal{M}\text{x}(\mathcal{J}\text{-}\mathcal{D}) \\ Ŧ(m,\varphi) \cup ღ(m,\varphi), & (m,\varphi)\in(\mathcal{M}\text{x}\mathcal{D}) \cap (\mathcal{M}\text{x}\mathcal{J}) = \mathcal{M}\text{x}(\mathcal{D} \cap \mathcal{J}) \end{cases}$$

Thereby,

$$☼(m,\varphi)=\begin{cases} Ʊ(m) \cap 𝔉'(\varphi), & (m,\varphi)\in(\mathcal{M}\text{x}\mathcal{D})\text{-}(\mathcal{M}\text{x}\mathcal{J}) = \mathcal{M}\text{x}(\mathcal{D}\text{-}\mathcal{J}) \\ Ʊ(m) \cap Ꮙ'(\varphi), & (m,\varphi)\in(\mathcal{M}\text{x}\mathcal{J})\text{-}(\mathcal{M}\text{x}\mathcal{D}) = \mathcal{M}\text{x}(\mathcal{J}\text{-}\mathcal{D}) \\ [Ʊ(m) \cap 𝔉'(\varphi)] \cup [Ʊ(m) \cap Ꮙ'(\varphi)], & (m,\varphi)\in(\mathcal{M}\text{x}\mathcal{D}) \cap (\mathcal{M}\text{x}\mathcal{J}) = \mathcal{M}\text{x}(\mathcal{D} \cap \mathcal{J}) \end{cases}$$

Hence, $(Ʊ,\mathcal{M})\Lambda_\backslash[(𝔉,\mathcal{D}) \cap_\varepsilon (Ꮙ,\mathcal{J})] =_{\text{M}} [(Ʊ,\mathcal{M})\Lambda_\backslash(𝔉,\mathcal{D})] \cup_\varepsilon [(Ʊ,\mathcal{M})\Lambda_\backslash(Ꮙ,\mathcal{J})]$. □

*iii.* The PS of the LHS is $(\mathcal{M} \cup \mathcal{D})\text{x}\mathcal{J}$, and the PS of the RHS is $(\mathcal{M}\text{x}\mathcal{J}) \cup (\mathcal{D}\text{x}\mathcal{J})$. Since $(\mathcal{M} \cup \mathcal{D})\text{x}\mathcal{J} = (\mathcal{M}\text{x}\mathcal{J}) \cup (\mathcal{D}\text{x}\mathcal{J})$, the first condition of the M-equality is satisfied. By assumption, $\mathcal{M} \neq \emptyset$, $\mathcal{D} \neq \emptyset$, and $\mathcal{J} \neq$

$\emptyset$. Thus, $(\mathcal{M} \cup \mathcal{D})\text{x}\mathcal{J} \neq \emptyset$ and $(\mathcal{M}\text{x}\mathcal{J}) \cup (\mathcal{D}\text{x}\mathcal{J}) \neq \emptyset$. Thereby, no side may be equal to an empty soft set. Let $(Ʊ,\mathcal{M}) \cup_\varepsilon (\mathfrak{F},\mathcal{D}) = (\mathfrak{X},\mathcal{M} \cup \mathcal{D})$, where for all $\varphi \in \mathcal{M} \cup \mathcal{D}$,

$$\mathfrak{X}(\varphi)=\begin{cases} Ʊ(\varphi), & \varphi\in\mathcal{M}\text{-}\mathcal{D} \\ \mathfrak{F}(\varphi), & \varphi\in\mathcal{D}\text{-}\mathcal{M} \\ Ʊ(\varphi) \cup \mathfrak{F}(\varphi), & \varphi\in\mathcal{M} \cap \mathcal{D} \end{cases}$$

Let $(\mathfrak{X},\mathcal{M} \cup \mathcal{D})\Lambda_\backslash(Ꮙ,\mathcal{J}) = (₴,(\mathcal{M} \cup \mathcal{D})\text{x}\mathcal{J})$, where for all $(\varphi,ȷ) \in (\mathcal{M} \cup \mathcal{D})\text{x}\mathcal{J}$, $₴(\varphi,ȷ) = \mathfrak{X}(\varphi) \cap Ꮙ'(ȷ)$. Thus, for all $(\varphi,ȷ) \in (\mathcal{M} \cup \mathcal{D})\text{x}\mathcal{J}$,

$$₴(\varphi,ȷ)=\begin{cases} Ʊ(\varphi) \cap Ꮙ'(ȷ), & (\varphi,ȷ)\in(\mathcal{M}\text{-}\mathcal{D})\text{x}\mathcal{J} \\ \mathfrak{F}(\varphi) \cap Ꮙ'(ȷ), & (\varphi,ȷ)\in(\mathcal{D}\text{-}\mathcal{M})\text{x}\mathcal{J} \\ [Ʊ(\varphi) \cup \mathfrak{F}(\varphi)] \cap Ꮙ'(ȷ), & (\varphi,ȷ)\in(\mathcal{M} \cap \mathcal{D})\text{x}\mathcal{J} \end{cases}$$

Suppose that $(Ʊ,\mathcal{M})\Lambda_\backslash(Ꮙ,\mathcal{J}) = (₮,\mathcal{M}\text{x}\mathcal{J})$ and $(\mathfrak{F},\mathcal{D})\Lambda_\backslash(Ꮙ,\mathcal{J}) = (ლ,\mathcal{D}\text{x}\mathcal{J})$, where for all $(m,ȷ) \in \mathcal{M}\text{x}\mathcal{J}$, $₮(m,ȷ) = Ʊ(m) \cap Ꮙ'(ȷ)$ and for all $(d,ȷ) \in \mathcal{D}\text{x}\mathcal{J}$, $ლ(d,ȷ) = \mathfrak{F}(d) \cap Ꮙ'(ȷ)$. Let $(₮,\mathcal{M}\text{x}\mathcal{J}) \cup_\varepsilon (ლ,\mathcal{D}\text{x}\mathcal{J}) = (☼,(\mathcal{M}\text{x}\mathcal{J}) \cup (\mathcal{D}\text{x}\mathcal{J}))$, where for all $(\varphi,ȷ) \in (\mathcal{M}\text{x}\mathcal{J}) \cup (\mathcal{D}\text{x}\mathcal{J}) = (\mathcal{M} \cup \mathcal{D})\text{x}\mathcal{J}$,

$$☼(\varphi,ȷ)=\begin{cases} ₮(\varphi,ȷ), & (\varphi,ȷ)\in(\mathcal{M}\text{x}\mathcal{J})\text{-}(\mathcal{D}\text{x}\mathcal{J}) = (\mathcal{M}\text{-}\mathcal{D})\text{x}\mathcal{J} \\ ლ(\varphi,ȷ), & (\varphi,ȷ)\in(\mathcal{D}\text{x}\mathcal{J})\text{-}(\mathcal{M}\text{x}\mathcal{J}) = (\mathcal{D}\text{-}\mathcal{M})\text{x}\mathcal{J} \\ ₮(\varphi,ȷ) \cup ლ(\varphi,ȷ), & (\varphi,ȷ)\in(\mathcal{M}\text{x}\mathcal{J}) \cap (\mathcal{D}\text{x}\mathcal{J}) = (\mathcal{M} \cap \mathcal{D})\text{x}\mathcal{J} \end{cases}$$

Thereby,

$$☼(\varphi,ȷ)=\begin{cases} Ʊ(\varphi) \cap Ꮙ'(ȷ), & (\varphi,ȷ)\in(\mathcal{M}\text{x}\mathcal{J})\text{-}(\mathcal{D}\text{x}\mathcal{J}) = (\mathcal{M}\text{-}\mathcal{D})\text{x}\mathcal{J} \\ \mathfrak{F}(\varphi) \cap Ꮙ'(ȷ), & (\varphi,ȷ)\in(\mathcal{D}\text{x}\mathcal{J})\text{-}(\mathcal{M}\text{x}\mathcal{J}) = (\mathcal{D}\text{-}\mathcal{M})\text{x}\mathcal{J} \\ [Ʊ(\varphi) \cap Ꮙ'(ȷ)] \cup [\mathfrak{F}(\varphi) \cap Ꮙ'(ȷ)], & (\varphi,ȷ)\in(\mathcal{M}\text{x}\mathcal{J}) \cap (\mathcal{D}\text{x}\mathcal{J}) = (\mathcal{M} \cap \mathcal{D})\text{x}\mathcal{J} \end{cases}$$

Hence, $[(Ʊ,\mathcal{M}) \cup_\varepsilon (\mathfrak{F},\mathcal{D})]\Lambda_\backslash(Ꮙ,\mathcal{J}) =_M [(Ʊ,\mathcal{M})\Lambda_\backslash(Ꮙ,\mathcal{J})] \cup_\varepsilon [(\mathfrak{F},\mathcal{D})\Lambda_\backslash(Ꮙ,\mathcal{J})]$. □

**Note 4.4.** The extended soft set operation can not distribute over soft difference-product as the union operation does not distribute over cartesian product and it is compulsory for two SSs to be M-equal that their PS should be the same.

**Theorem 4.5.** Let $(Ʊ,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$, and $(Ꮙ,\mathcal{J})$ be SSs over $U$. Then, we have the following distributions of soft difference-product over soft binary piecewise intersection and union operations:

*i.* $(Ʊ,\mathcal{M})\Lambda_\backslash[(\mathfrak{F},\mathcal{D}) \tilde{\cap} (Ꮙ,\mathcal{J})] =_M [(Ʊ,\mathcal{M})\Lambda_\backslash(\mathfrak{F},\mathcal{D})] \tilde{\cup} [(Ʊ,\mathcal{M})\Lambda_\backslash(Ꮙ,\mathcal{J})]$
*ii.* $(Ʊ,\mathcal{M})\Lambda_\backslash[(\mathfrak{F},\mathcal{D}) \tilde{\cup} (Ꮙ,\mathcal{J})] =_M [(Ʊ,\mathcal{M})\Lambda_\backslash(\mathfrak{F},\mathcal{D})] \tilde{\cap} [(Ʊ,\mathcal{M})\Lambda_\backslash(Ꮙ,\mathcal{J})]$
*iii.* $[(Ʊ,\mathcal{M}) \tilde{\cup} (\mathfrak{F},\mathcal{D})]\Lambda_\backslash(Ꮙ,\mathcal{J}) =_M [(Ʊ,\mathcal{M})\Lambda_\backslash(Ꮙ,\mathcal{J})] \tilde{\cup} [(\mathfrak{F},\mathcal{D})\Lambda_\backslash(Ꮙ,\mathcal{J})]$
*iv.* $[(Ʊ,\mathcal{M}) \tilde{\cap} (\mathfrak{F},\mathcal{D})]\Lambda_\backslash(Ꮙ,\mathcal{J}) =_M [(Ʊ,\mathcal{M})\Lambda_\backslash(Ꮙ,\mathcal{J})] \tilde{\cap} [(\mathfrak{F},\mathcal{D})\Lambda_\backslash(Ꮙ,\mathcal{J})]$

PROOF.

*i.* Since the PS of the SSs of both sides is $\mathcal{M}\text{x}\mathcal{D}$, and the first condition of the M-equality is satisfied. Moreover since $\mathcal{M} \neq \emptyset$ and $\mathcal{D} \neq \emptyset$ by assumption, $\mathcal{M}\text{x}\mathcal{D} \neq \emptyset$. Thus, no side may be equal to an empty soft set. Let $(\mathfrak{F},\mathcal{D}) \tilde{\cap} (Ꮙ,\mathcal{J}) = (\mathfrak{X},\mathcal{D})$, where for all $d \in \mathcal{D}$,

$$\mathfrak{X}(d)=\begin{cases} \mathfrak{F}(d), & d\in\mathcal{D}\text{-}\mathcal{J} \\ \mathfrak{F}(d) \cap Ꮙ(d), & d\in\mathcal{D} \cap \mathcal{J} \end{cases}$$

Let $(Ʊ,\mathcal{M})\Lambda_\backslash(\mathfrak{X},\mathcal{D}) = (ლ,\mathcal{M}\text{x}\mathcal{D})$, where for all $(m,d) \in \mathcal{M}\text{x}\mathcal{D}$, $ლ(m,d) = Ʊ(m) \cap \mathfrak{X}'(d)$. Thus,

$$ლ(m,d)=\begin{cases}\mho(m)\cap\mathfrak{F}'(d), & (m,d)\in\mathcal{M}\mathrm{x}(\mathcal{D}\text{-}\mathcal{J})\\ \mho(m)\cap[\mathfrak{F}'(d)\cup Ꮙ'(d)], & (m,d)\in\mathcal{M}\mathrm{x}(\mathcal{D}\cap\mathcal{J})\end{cases}$$

Suppose that $(\mho,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})=(Ŧ,\mathcal{M}\mathrm{x}\mathcal{D})$ and $(\mho,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})=(₴,\mathcal{M}\mathrm{x}\mathcal{J})$, where for all $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, $Ŧ(m,d)=\mho(m)\cap\mathfrak{F}'(d)$ and for all $(m,j)\in\mathcal{M}\mathrm{x}\mathcal{J}$, $₴(m,j)=\mho(m)\cap Ꮙ'(j)$. Let $(Ŧ,\mathcal{M}\mathrm{x}\mathcal{D})\,\tilde{\cup}\,(₴,\mathcal{M}\mathrm{x}\mathcal{J})=(☼,\mathcal{M}\mathrm{x}\mathcal{D})$, where for all $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$,

$$☼(m,d)=\begin{cases}Ŧ(m,d), & (m,d)\in(\mathcal{M}\mathrm{x}\mathcal{D})\text{-}(\mathcal{M}\mathrm{x}\mathcal{J})=\mathcal{M}\mathrm{x}(\mathcal{D}\text{-}\mathcal{J})\\ Ŧ(m,d)\cup ₴(m,d), & (m,d)\in(\mathcal{M}\mathrm{x}\mathcal{D})\cap(\mathcal{M}\mathrm{x}\mathcal{J})=\mathcal{M}\mathrm{x}(\mathcal{D}\cap\mathcal{J})\end{cases}$$

Hence,

$$☼(m,d)=\begin{cases}\mho(m)\cap\mathfrak{F}'(d), & (m,d)\in(\mathcal{M}\mathrm{x}\mathcal{D})\text{-}(\mathcal{M}\mathrm{x}\mathcal{J})=\mathcal{M}\mathrm{x}(\mathcal{D}\text{-}\mathcal{J})\\ [\mho(m)\cap\mathfrak{F}'(d)]\cup[\mho(m)\cap Ꮙ'(d)], & (m,d)\in(\mathcal{M}\mathrm{x}\mathcal{D})\cap(\mathcal{M}\mathrm{x}\mathcal{J})=\mathcal{M}\mathrm{x}(\mathcal{D}\cap\mathcal{J})\end{cases}$$

Thus, $(\mho,\mathcal{M})\Lambda_{\backslash}[(\mathfrak{F},\mathcal{D})\,\tilde{\cap}\,(Ꮙ,\mathcal{J})]=_{\mathrm{M}}[(\mho,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})]\,\tilde{\cup}\,[(\mho,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$. Since $\mathcal{M}\neq\mathcal{M}\mathrm{x}\mathcal{M}$, the soft binary piecewise operations do not distribute over soft plus-product operations. □

*iii.* Since the PS of the SSs of both sides is $\mathcal{M}\mathrm{x}\mathcal{J}$, the first condition of the M-equality is satisfied. Moreover since $\mathcal{M}\neq\emptyset$ and $\mathcal{J}\neq\emptyset$ by assumption, $\mathcal{M}\mathrm{x}\mathcal{J}\neq\emptyset$. Thus, it is impossible that any side is equal to empty soft set. Let $(\mho,\mathcal{M})\,\tilde{\cup}\,(\mathfrak{F},\mathcal{D})=(Ӿ,\mathcal{M})$, where for all $m\in\mathcal{M}$,

$$Ӿ(m)=\begin{cases}\mho(m), & m\in\mathcal{M}\text{-}\mathcal{D}\\ \mho(m)\cup\mathfrak{F}(m), & m\in\mathcal{M}\cap\mathcal{D}\end{cases}$$

Let $(Ӿ,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})=(ლ,\mathcal{M}\mathrm{x}\mathcal{J})$, where for all $(m,j)\in\mathcal{M}\mathrm{x}\mathcal{J}$, $ლ(m,j)=Ӿ(m)\cap Ꮙ'(j)$. Thus,

$$Ӿ(m,j)=\begin{cases}\mho(m)\cap Ꮙ'(j), & (m,j)\in(\mathcal{M}\text{-}\mathcal{D})\mathrm{x}\mathcal{J}\\ [\mho(m)\cup\mathfrak{F}(m)]\cap Ꮙ'(j), & (m,j)\in(\mathcal{M}\cap\mathcal{D})\mathrm{x}\mathcal{J}\end{cases}$$

Assume that $(\mho,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})=(Ŧ,\mathcal{M}\mathrm{x}\mathcal{J})$ and $(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(Ꮙ,\mathcal{J})=(₴,\mathcal{D}\mathrm{x}\mathcal{J})$, where for all $(m,j)\in\mathcal{M}\mathrm{x}\mathcal{J}$ $Ŧ(m,j)=\mho(m)\cap Ꮙ'(j)$ and for all $(d,j)\in\mathcal{D}\mathrm{x}\mathcal{J}$, $₴(d,j)=\mathfrak{F}(d)\cap Ꮙ'(j)$. Let $(Ŧ,\mathcal{M}\mathrm{x}\mathcal{J})\,\tilde{\cup}\,(₴,\mathcal{D}\mathrm{x}\mathcal{J})=(☼,\mathcal{M}\mathrm{x}\mathcal{J})$, where for all $(m,j)\in\mathcal{M}\mathrm{x}\mathcal{J}$,

$$☼(m,j)=\begin{cases}Ŧ(m,j), & (m,j)\in(\mathcal{M}\mathrm{x}\mathcal{J})\text{-}(\mathcal{D}\mathrm{x}\mathcal{J})=(\mathcal{M}\text{-}\mathcal{D})\mathrm{x}\mathcal{J}\\ Ŧ(m,j)\cup ₴(m,j), & (m,j)\in(\mathcal{M}\mathrm{x}\mathcal{J})\cap(\mathcal{D}\mathrm{x}\mathcal{J})=(\mathcal{M}\cap\mathcal{D})\mathrm{x}\mathcal{J}\end{cases}$$

Thus,

$$☼(m,j)=\begin{cases}\mho(m)\cap Ꮙ'(j), & (m,j)\in(\mathcal{M}\mathrm{x}\mathcal{J})\text{-}(\mathcal{D}\mathrm{x}\mathcal{J})=(\mathcal{M}\text{-}\mathcal{D})\mathrm{x}\mathcal{J}\\ [\mho(m)\cap Ꮙ'(j)]\cup[\mathfrak{F}(m)\cap Ꮙ'(j)], & (m,j)\in(\mathcal{M}\mathrm{x}\mathcal{J})\cap(\mathcal{D}\mathrm{x}\mathcal{J})=(\mathcal{M}\cap\mathcal{D})\mathrm{x}\mathcal{J}\end{cases}$$

Thereby, $[(\mho,\mathcal{M})\,\tilde{\cup}\,(\mathfrak{F},\mathcal{D})]\Lambda_{\backslash}(Ꮙ,\mathcal{J})=_{\mathrm{M}}[(\mho,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]\,\tilde{\cup}\,[(\mathfrak{F},\mathcal{D})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$. □

**Proposition 4.6.** Let $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$ and $(Ꮙ,\mathcal{J})$ be SSs over $U$. Then,

*i.* $(\mho,\mathcal{M})\Lambda_{\backslash}[(\mathfrak{F},\mathcal{D})\Lambda(Ꮙ,\mathcal{J})]\subseteq_{\mathrm{L}}[(\mho,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})]\mathrm{V}[(\mho,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$
*ii.* $(\mho,\mathcal{M})\Lambda_{\backslash}[(\mathfrak{F},\mathcal{D})\mathrm{V}(Ꮙ,\mathcal{J})]\subseteq_{\mathrm{L}}[(\mho,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})]\Lambda[(\mho,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})]$

Proof.

*i.* Let $(\mathfrak{F},\mathcal{D})\Lambda(Ꮙ,\mathcal{J})=(\varsigma,\mathcal{D}\mathrm{x}\mathcal{J})$, where for all $(d,j)\in\mathcal{D}\mathrm{x}\mathcal{J}$, $\varsigma(d,j)=\mathfrak{F}(d)\cap Ꮙ(j)$ and $(\mho,\mathcal{M})\Lambda_{\backslash}(\varsigma,\mathcal{D}\mathrm{x}\mathcal{J})=(Ӿ,\mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J}))$, where for all $(m,(d,j))\in\mathcal{M}\mathrm{x}(\mathcal{D}\mathrm{x}\mathcal{J})$,

$$Ӿ(m,(d,j))=\mho(m)\cap[\mathfrak{F}(d)\cap Ꮙ(j)]'=\mho(m)\cap[\mathfrak{F}'(d)\cup Ꮙ'(j)]$$

Suppose that $(\mho,\mathcal{M})\Lambda_{\backslash}(\mathfrak{F},\mathcal{D})=(☼,\mathcal{M}\mathrm{x}\mathcal{D})$ and $(\mho,\mathcal{M})\Lambda_{\backslash}(Ꮙ,\mathcal{J})=(₴,\mathcal{M}\mathrm{x}\mathcal{J})$, where for all $(m,d)\in\mathcal{M}\mathrm{x}\mathcal{D}$, $☼(m,d)=\mho(m)\cap\mathfrak{F}'(d)$ and for all $(m,j)\in\mathcal{M}\mathrm{x}\mathcal{J}$, $₴(m,j)=\mho(m)\cap Ꮙ'(j)$. Suppose that $(☼,\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{V}(₴,\mathcal{M}\mathrm{x}\mathcal{J})=(ლ,(\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}(\mathcal{M}\mathrm{x}\mathcal{J}))$, where for all $((m,d),(m,j))\in(\mathcal{M}\mathrm{x}\mathcal{D})\mathrm{x}(\mathcal{M}\mathrm{x}\mathcal{J})$,

$$\text{ღ}((m,d),(m,\text{j})) = [\mho(m) \cap \mathfrak{F}'(d)] \cup [\mho(m) \cap \text{ꝟ}'(\text{j})]$$

Thus, for all $(m,(d,\text{j})) \in \mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})$, there exists $((m,d),(m,\text{j})) \in (\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J})$ such that

$\mathfrak{X}(m,(d,\text{j})) = \mho(m) \cap [\mathfrak{F}'(d) \cup \text{ꝟ}'(\text{j})] = [\mho(m) \cap \mathfrak{F}'(d)] \cup [\mho(m) \cap \text{ꝟ}'(\text{j})] = \text{ღ}((m,d),(m,\text{j}))$ □

It is obvious that the L-subset in Proposition 4.6. cannot be L-equality with the following example:

**Example 4.7.** Let $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the parameter set, $\mathcal{M} = \{e_1, e_5\}$, $\mathcal{D} = \{e_3\}$ and $\mathcal{J} = \{e_2\}$ be the subsets of $E$, $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be the universal set and $(\mho,\mathcal{M})$, $(\mathfrak{F},\mathcal{D})$, and $(\text{ꝟ},\mathcal{J})$ be SSs over $U$ as follows:

$$(\mho,\mathcal{M}) = \{(e_1,\{h_1,h_6\}),(e_5,\{h_2,h_4,h_5\})\},$$
$$(\mathfrak{F},\mathcal{D}) = \{(e_3,\{h_1,h_3,h_4\})\},$$

and

$$(\text{ꝟ},\mathcal{J}) = \{(e_2,\{h_1,h_4,h_5\})\}$$

We show that

$$(\mho,\mathcal{M})\wedge_{\backslash}[(\mathfrak{F},\mathcal{D})\wedge(\text{ꝟ},\mathcal{J})] \neq_{\mathrm{L}} [(\mho,\mathcal{M})\wedge_{\backslash}(\mathfrak{F},\mathcal{D})]\vee[(\mho,\mathcal{M})\wedge_{\backslash}(\text{ꝟ},\mathcal{J})]$$

Let $(\mathfrak{F},\mathcal{D})\wedge(\text{ꝟ},\mathcal{J}) = (\text{Ʊ},\mathcal{D}\text{x}\mathcal{J})$, where

$$(\mathfrak{F},\mathcal{D})\wedge(\text{ꝟ},\mathcal{J}) = (\text{Ʊ},\mathcal{D}\text{x}\mathcal{J}) = \{((e_3,e_2),\{h_1,h_4\})\}$$

Assume that $(\mho,\mathcal{M})\wedge_{\backslash}(\text{Ʊ},\mathcal{D}\text{x}\mathcal{J}) = (\mathfrak{X},\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J}))$, where

$$(\mho,\mathcal{M})\wedge_{\backslash}(\text{Ʊ},\mathcal{D}\text{x}\mathcal{J}) = (\mathfrak{X},\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J})) = \{((e_1,(e_3,e_2)),\{h_6\}),((e_5,(e_3,e_2)),\{h_2,h_5\})\}$$

Let $(\mho,\mathcal{M})\wedge_{\backslash}(\mathfrak{F},\mathcal{D}) = (\text{ƻ},\mathcal{M}\text{x}\mathcal{D})$, where

$$(\mho,\mathcal{M})\wedge_{\backslash}(\mathfrak{F},\mathcal{D}) = (\text{ƻ},\mathcal{M}\text{x}\mathcal{D}) = \{((e_1,e_3),\{h_6\}),((e_5,e_3),\{h_2,h_5\})\}$$

Suppose that $(\mho,\mathcal{M})\wedge_{\backslash}(\text{ꝟ},\mathcal{J}) = (\text{☼},\mathcal{M}\text{x}\mathcal{J})$, where

$$(\mho,\mathcal{M})\wedge_{\backslash}(\text{ꝟ},\mathcal{J}) = (\text{☼},\mathcal{M}\text{x}\mathcal{J}) = \{((e_1,e_2),\{h_6\}),((e_5,e_2),\{h_2\})\}$$

Let $(\text{ƻ},\mathcal{M}\text{x}\mathcal{D})\vee(\text{☼},\mathcal{M}\text{x}\mathcal{J}) = (\text{ღ},(\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J}))$. Then,

$$\begin{aligned}(\text{ƻ},\mathcal{M}\text{x}\mathcal{D})\vee(\text{☼},\mathcal{M}\text{x}\mathcal{J} &= (\text{ღ},(\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J}))\\ &= \{(((e_1,e_3),(e_1,e_2)),\{h_6\}),(((e_1,e_3),(e_5,e_2)),\{h_2,h_6\}),(((e_5,e_3),(e_1,e_2)),\{h_{[illegible]}\\ &\quad (((e_5,e_3),(e_5,e_2)),\{h_2,h_5\})\}\end{aligned}$$

Thus, $\text{ღ}((e_1,e_3),(e_5,e_2)) \neq \mathfrak{X}(e_1,(e_3,e_2))$, $\text{ღ}((e_1,e_3),(e_5,e_2)) \neq \mathfrak{X}((e_5,(e_3,e_2))$, $\text{ღ}((e_5,e_3),(e_1,e_2)) \neq \mathfrak{X}(e_1,(e_3,e_2))$, and $\text{ღ}((e_5,e_3),(e_1,e_2)) \neq \mathfrak{X}((e_5,(e_3,e_2))$, implying that $(\text{ღ},(\mathcal{M}\text{x}\mathcal{D})\text{x}(\mathcal{M}\text{x}\mathcal{J})) \nsubseteq_{\mathrm{L}} (\mathfrak{X},\mathcal{M}\text{x}(\mathcal{D}\text{x}\mathcal{J}))$.

## 5. *uni-int* Decision-Making Method Applied to Soft Difference-Product

In this section, the *uni-int* operator and *uni-int* decision function defined by Çağman and Enginoğlu [11] are applied for the soft difference-product for the *uni-int* decision-making method. This method reduces a set to its subset according to the criteria given by the decision-makers. Therefore, instead of considering a large number of possibilities, decision-makers concentrate on a small number.

Throughout this section, all the soft difference-products ($\Lambda_\backslash$) of the SSs over $U$ are assumed to be contained in the set $\Lambda_\backslash(U)$, and the approximation function of the soft difference-product of $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$, that is, $(\mho,\mathcal{M})\Lambda_\backslash(\mathfrak{F},\mathcal{D})$ is

$$\mho_{\mathcal{M}}\Lambda_\backslash\mathfrak{F}_{\mathcal{D}}: \mathcal{M}\mathrm{x}\mathcal{D} \to P(U)$$

where $\mho_{\mathcal{M}}\Lambda_\backslash\mathfrak{F}_{\mathcal{D}}(m,d) = \mho(m) \cap \mathfrak{F}'(d)$ for all $(m,d) \in \mathcal{M}\mathrm{x}\mathcal{D}$.

**Definition 5.1.** Let $(\mho,\mathcal{M})$ and $(\mathfrak{F},\mathcal{D})$ be SS over $U$. Then, *uni-int* operators for soft difference-product, denoted by $uni_x int_y$ and $uni_y int_x$ are defined respectively as

$uni_x int_y$: $\Lambda_\backslash \to P(U)$, $\quad uni_x int_y(\mho_{\mathcal{M}}\Lambda_\backslash\mathfrak{F}_{\mathcal{D}}) = \cup_{m\in\mathcal{M}}(\cap_{d\in\mathcal{D}}(\mho_{\mathcal{M}}\Lambda_\backslash\mathfrak{F}_{\mathcal{D}}(m,d)))$

$uni_y int_x$: $\Lambda_\backslash \to P(U)$, $\quad uni_y int_x(\mho_{\mathcal{M}}\Lambda_\backslash\mathfrak{F}_{\mathcal{D}}) = \cup_{d\in\mathcal{D}}(\cap_{m\in\mathcal{M}}(\mho_{\mathcal{M}}\Lambda_\backslash\mathfrak{F}_{\mathcal{D}}(m,d)))$

**Definition 5.2.** [11] Let $(\mho,\mathcal{M})\Lambda_\backslash(\mathfrak{F},\mathcal{D}) \in \Lambda_\backslash(U)$. Then, *uni-int* decision function for soft difference-product, denoted by *uni-int* are defined by

*uni-int*: $\Lambda_\backslash \to P(U)$, $\quad$ *uni-int*$(\mho_{\mathcal{M}}\Lambda_\backslash\mathfrak{F}_{\mathcal{D}}) = uni_x int_y(\mho_{\mathcal{M}}\Lambda_\backslash\mathfrak{F}_{\mathcal{D}}) \cup uni_y int_x(\mho_{\mathcal{M}}\Lambda_\backslash\mathfrak{F}_{\mathcal{D}})$

that reduces the size of the universe $U$. Thus, the values *uni-int*$(\mho_{\mathcal{M}}\Lambda_\backslash\mathfrak{F}_{\mathcal{D}})$ is a subset of $U$ called *uni-int* decision set of $\mho_{\mathcal{M}}\Lambda_\backslash\mathfrak{F}_{\mathcal{D}}$.

Presume that a set of choices and a set of parameters are given. Then, taking into account the problem, a set of ideal alternatives is selected utilizing the *uni-int* decision-making technique, which is organized as follows:

Step 1: Select the feasible subsets from the parameter collection.

Step 2: Construct the SSs for every parameter set.

Step 3: Determine the SSs' soft difference-product.

Step 4: Determine the product's *uni-int* decision set.

The application of soft set theory to the *uni-int* decision-making for the soft difference-products is demonstrated as follows:

**Example 5.3.** A private club has announced a recruitment notice for contracted coaches, and candidates will be selected based on the success in oral and practical exams from among those invited, which will be up to three (3) times the number of available positions. The selected candidates will then undergo a comprehensive training program, and those who successfully complete it will qualify to join the private club professional caoch team. Applications will first undergo a preliminary elimination to check compliance with the required qualifications for the applied position, and applications found to not meet any of the requirements will be canceled. Among the candidates whose applications are considered valid, Mrs Çam, the club's Human Resources manager, will carry out a selection process based on the results of the oral and practical exams administered to the candidates. Taking into account the characteristics that should be present in a coach candidate, Mrs Çam will first determine the parameters he wants in the selected coach candidates, and then the parameters he definitely does not want. Based on this, he will make his decision using the *uni-int* decision-making method of the soft difference-product. Assume that the coach candidate whose application is considered valid is as follows: $U = \{z_1, z_2, \cdots, z_{25}\}$. Let the set of parameters to be used for determining the selected coaches be $\{c_1, c_2, \dots, c_8\}$. Here, $c_i$ are parameters, where $i \in \{1,2,\cdots,8\}$ correspond to the following descriptions, respectively:

$c_1$ = "having the professional knowledge required by the position,"

$c_2$ = "having a low level of general culture and general ability",

$c_3$ = "having practical experience in the professional field required by the position",

$c_4$ = “having strong communication and reasoning abilities”,

$c_5$ = “having low physical endurance and conditioning”,

$c_6$ = “being closed to innovations, scientific and technological developments”,

$c_7$ = “being motivated and determined”

$c_8$ = “having low self-confidence, persuasion skills, and credibility”.

To solve this coach recruitment process problem, we can apply the *uni-int* method to soft difference-product as follows:

**Step 1:** Determining the sets of parameters

The decision maker, Mrs Çam from the existing set of parameters, first selects the parameters he would like to have in the selected coach candidates ($i$), and then selects the parameters he DOES NOT want ($ii$): That is,

($i$) Parameters that must absolutely be present in selected candidates: These represent traits or skills that are essential and desired in a coach, and their absence disqualifies a candidate.

($ii$) Parameters that are preferred NOT to be present in candidates to be selected: These represent undesirable traits or deficiencies that make a candidate unsuitable for selection.

By categorizing these parameters into two sets, the selection process ensures clarity and alignment with the decision-maker’s priorities. Let these parameter sets be $\mathcal{M} = \{c_1, c_3, c_4, c_7\}$ and $\mathcal{D} = \{c_2, c_5, c_6, c_8\}$, respectively.

**Step 2:** Construct the soft sets by using the parameter sets determined in Step 1.

Using these parameter sets, the decision-maker constructs the soft sets $(\mho, \mathcal{M})$ and $(\mho, \mathcal{D})$, respectively:

$$(\mho, \mathcal{M}) = \{(c_1, \{z_1, z_3, z_4, z_7, z_9, z_{13}, z_{15}, z_{17}, z_{22}, z_{24}\}), (c_3, \{z_2, z_3, z_5, z_8, z_{11}, z_{17}, z_{21}, z_{23}, z_{25}\}),$$
$$(c_4, \{z_6, z_9, z_{13}, z_{16}, z_{18}, z_{19}, z_{21}, z_{22}, z_{24}\}), (c_7, \{z_1, z_3, z_6, z_{10}, z_{11}, z_{13}, z_{17}, z_{22}, z_{23}, z_{25}\})\}$$

and

$$(\mho, \mathcal{D}) = \{(c_2, \{z_8, z_9, z_{12}, z_{14}, z_{16}, z_{17}, z_{22}, z_{25}\}), (c_5, \{z_1, z_3, z_5, z_7, z_{11}, z_{15}, z_{19}, z_{20}, z_{21}\}),$$
$$(c_6, \{z_3, z_4, z_6, z_9, z_{13}, z_{18}, z_{19}, z_{25}\}), (c_8, \{z_7, z_{10}, z_{11}, z_{14}, z_{19}, z_{20}, z_{21}, z_{22}, z_{23}, z_{25}\})\}$$

While $(\mho, \mathcal{M})$ is a soft set representing candidates close to the ideal by possessing the desired parameters in $\mathcal{M}$, $(\mho, \mathcal{D})$ is a soft set representing candidates to be eliminated due to undesirable parameters in $\mathcal{D}$.

**Step 3:** Determine the $\wedge_\backslash$-product of soft sets.

$$\mho_{\mathcal{M}} \wedge_\backslash \mho_{\mathcal{D}} = \{\big((c_1, c_2), \{z_1, z_3, z_4, z_7, z_{13}, z_{15}, z_{24}\}\big), \big((c_1, c_5), \{z_4, z_9, z_{13}, z_{17}, z_{22}, z_{24}\}\big),$$
$$\big((c_1, c_6), \{z_1, z_7, z_{15}, z_{17}, z_{22}, z_{24}\}\big), \big((c_1, c_8), \{z_1, z_3, z_4, z_9, z_{13}, z_{15}, z_{17}, z_{24}\}\big),$$
$$\big((c_3, c_2), \{z_2, z_3, z_5, z_{11}, z_{21}, z_{23}\}\big), \big((c_3, c_5), \{z_2, z_8, z_{17}, z_{23}, z_{25}\}\big),$$
$$\big((c_3, c_6), \{z_2, z_5, z_8, z_{11}, z_{17}, z_{21}, z_{23}\}\big), \big((c_3, c_8), \{z_2, z_3, z_5, z_8, z_{17}\}\big),$$
$$\big((c_4, c_2), \{z_6, z_{13}, z_{18}, z_{19}, z_{21}, z_{24}\}\big), \big((c_4, c_5), \{z_6, z_9, z_{13}, z_{16}, z_{18}, z_{22}, z_{24}\}\big),$$
$$\big((c_4, c_6), \{z_{16}, z_{21}, z_{22}, z_{24}\}\big), \big((c_4, c_8), \{z_6, z_9, z_{13}, z_{16}, z_{18}, z_{24}\}\big),$$

$$\big((c_7,c_2),\{z_1,z_3,z_6,z_{10},z_{11},z_{13},z_{23}\}\big),\big((c_7,c_5),\{z_6,z_{10},z_{13},z_{17},z_{22},z_{23},z_{25}\}\big),$$

$$\big((c_7,c_6),\{z_1,z_{10},z_{11},z_{17},z_{22},z_{23}\}\big),\big((c_7,c_8),\{z_1,z_3,z_6,z_{13},z_{17}\}\big)\}$$

**Step 4:** Determine the set of *uni-int*($\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}}$):

$$uni_m-int_d(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})=\cup_{m\in\mathcal{M}}(\cap_{d\in\mathcal{D}}(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(m,d)))$$

We determine first $\cap_{d\in\mathcal{D}}\Big((\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(m,d)\Big)$:

$$(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_1,c_2)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_1,c_5)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_1,c_6)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_1,c_8)$$

$$=\{z_1,z_3,z_4,z_7,z_{13},z_{15},z_{24}\}\cap\{z_4,z_9,z_{13},z_{17},z_{22},z_{24}\}\cap\{z_1,z_7,z_{15},z_{17},z_{22},z_{24}\}$$

$$\cap\{z_1,z_3,z_4,z_9,z_{13},z_{15},z_{17},z_{24}\}=\{z_{24}\}$$

$$(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_3,c_2)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_3,c_5)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_3,c_6)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_3,c_8)$$

$$=\{z_2,z_3,z_5,z_{11},z_{21},z_{23}\}\cap\{z_2,z_8,z_{17},z_{23},z_{25}\}\cap\{z_2,z_5,z_8,z_{11},z_{17},z_{21},z_{23}\}$$

$$\cap\{z_2,z_3,z_5,z_8,z_{17}\}=\{z_2\}$$

$$(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_4,c_2)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_4,c_5)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho)(c_4,c_6)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_4,c_8)$$

$$=\{z_6,z_{13},z_{18},z_{19},z_{21},z_{24}\}\cap\{z_6,z_9,z_{13},z_{16},z_{18},z_{22},z_{24}\}\cap\{z_{16},z_{21},z_{22},z_{24}\}$$

$$\cap\{z_6,z_9,z_{13},z_{16},z_{18},z_{24}\}=\{z_{24}\}$$

$$(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_7,c_2)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_7,c_5)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_7,c_6)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_7,c_8)$$

$$=\{z_1,z_3,z_6,z_{10},z_{11},z_{13},z_{23}\}\cap\{z_6,z_{10},z_{13},z_{17},z_{22},z_{23},z_{25}\}\cap\{z_1,z_{10},z_{11},z_{17},z_{22},z_{23}\}$$

$$\cap\{z_1,z_3,z_6,z_{13},z_{17}\}=\emptyset$$

Thus,

$$uni_m-int_d(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})=\cup_{m\in\mathcal{M}}\Big(\cap_{d\in\mathcal{D}}\Big(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}}(m,d)\Big)\Big)=\{z_{24}\}\cup\{z_2\}\cup\{z_{24}\}\cup\emptyset=\{z_2,z_{24}\}$$

$$uni_d-int_m(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})=\cup_{d\in\mathcal{D}}(\cap_{m\in\mathcal{M}}(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(m,d)))$$

We determine first $\cap_{m\in\mathcal{M}}\Big((\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(m,d)\Big)$ :

$$(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_1,c_2)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_3,c_2)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_4,c_2)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_7,c_2)$$

$$=\{z_1,z_3,z_4,z_7,z_{13},z_{15},z_{24}\}\cap\{z_2,z_3,z_5,z_{11},z_{21},z_{23}\}\cap\{z_6,z_{13},z_{18},z_{19},z_{21},z_{24}\}$$

$$\cap\{z_1,z_3,z_6,z_{10},z_{11},z_{13},z_{23}\}=\emptyset$$

$$(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_1,c_5)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_3,c_5)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_4,c_5)\cap(\mho_{\mathcal{M}}\Lambda_{\backslash}\mho_{\mathcal{D}})(c_7,c_5)$$

$$=\{z_4,z_9,z_{13},z_{17},z_{22},z_{24}\}\cap\{z_2,z_8,z_{17},z_{23},z_{25}\}\cap\{z_6,z_9,z_{13},z_{16},z_{18},z_{22},z_{24}\}$$

$$\cap\{z_6,z_{10},z_{13},z_{17},z_{22},z_{23},z_{25}\}=\emptyset$$

$$(\mho_{\mathcal{M}}\Lambda_{\setminus}\mho_{\mathcal{D}})(c_1,c_6)\cap(\mho_{\mathcal{M}}\Lambda_{\setminus}\mho_{\mathcal{D}})(c_3,c_6)\cap(\mho_{\mathcal{M}}\Lambda_{\setminus}\mho_{\mathcal{D}})(c_4,c_6)\cap(\mho_{\mathcal{M}}\Lambda_{\setminus}\mho_{\mathcal{D}})(c_7,c_6)$$
$$=\{z_1,z_7,z_{15},z_{17},z_{22},z_{24}\}\cap\{z_2,z_5,z_8,z_{11},z_{17},z_{21},z_{23}\}\cap\{z_{16},z_{21},z_{22},z_{24}\}$$
$$\cap\{z_1,z_{10},z_{11},z_{17},z_{22},z_{23}\}=\emptyset$$

$$(\mho_{\mathcal{M}}\Lambda_{\setminus}\mho_{\mathcal{D}})(c_1,c_8)\cap(\mho_{\mathcal{M}}\Lambda_{\setminus}\mho_{\mathcal{D}})(c_3,c_8)\cap(\mho_{\mathcal{M}}\Lambda_{\setminus}\mho_{\mathcal{D}})(c_4,c_8)\cap(\mho_{\mathcal{M}}\Lambda_{\setminus}\mho_{\mathcal{D}})(c_7,c_8)$$
$$=\{z_1,z_3,z_4,z_9,z_{13},z_{15},z_{17},z_{24}\}\cap\{z_2,z_3,z_5,z_8,z_{17}\}\cap\{z_6,z_9,z_{13},z_{16},z_{18},z_{24}\}$$
$$\cap\{z_1,z_3,z_6,z_{13},z_{17}\}=\emptyset$$

Thus,

$$uni_d-int_m(\mho_{\mathcal{M}}\Lambda_{\setminus}\mho_{\mathcal{D}})=\cup_{d\in\mathcal{D}}\left(\cap_{m\in\mathcal{M}}\left((\mho_{\mathcal{M}}\Lambda_{\setminus}\mho_{\mathcal{D}})(m,d)\right)\right)=\emptyset\cup\emptyset\cup\emptyset\cup\emptyset=\emptyset$$

Hence,

$$\textit{uni-int}\left(\mho_{\mathcal{M}}\Lambda_{\setminus}\mathfrak{F}_{\mathcal{D}}\right)=[uni_m-int_d(\mho_{\mathcal{M}}\Lambda_{\setminus}\mho_{\mathcal{D}})]\cup[uni_d-int_m(\mho_{\mathcal{M}}\Lambda_{\setminus}\mho_{\mathcal{D}})]=\{z_2,z_{24}\}\cup\emptyset=\{z_2,z_{24}\}$$

From this, it can be concluded that out of the 25 applicants whose applications are accepted for the private club's coach recruitment process, the candidates $z_2$ and $z_{24}$ earn the right to undergo a comprehensive training program of this club to join the private club professional coach team.

## 6. Conclusion

In this study, we first presented a new type of soft set product that we term the "soft difference-product" using Molodtsov's soft set. We provided its example and closely analyzed its algebraic characteristics in terms of different types of soft subset and soft equality, including M-subset/equality, F-subset/equality, L-subset/equality, and J-subset/equality. Furthermore, the distributions of soft difference-product over various kinds of soft set operations were obtained. Finally, we applied the *uni-int* soft decision-making method that selects the optimal elements among potential options without the need of rough or fuzzy soft sets. Additionally, we included an example that shows how the method may be used successfully for a real-world scenario problem. This work may enable several applications, such as new approaches to decision-making and novel soft set-based cryptography techniques. Future research may propose some more new types of soft product operations and further examine fundamental characteristics associated with different kinds of soft equal relations to contribute to the soft set literature from a theoretical and practical standpoint.

## Author Contributions

All the authors equally contributed to this work. This paper is derived from the second author's master's thesis supervised by the first author. They all read and approved the final version of the paper.

## Conflict of Interest

All the authors declare no conflict of interest.

## Ethical Review and Approval

No approval from the Board of Ethics is required.

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