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SOME RESULTS ON PURELY REAL SURFACES AND SLANT SURFACES IN COMPLEX SPACE FORMS

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ABSTRACT

A surface M in a Kaehler surface N is called purely real if it contains no complex points. A slant immersion which was introduced by B.Y. Chen in [1] is an isometric immersion of a Riemannian manifold into an almost Hermitian manifold with constant Wirtinger angle. In this article, we study slant surfaces and purely real surfaces and also give a general optimal inequality for purely real surfaces in complex space forms proved by Chen.

Key words and phrases: Purely real surfaces; slant surfaces, Wirtinger angle; optimal inequality.

ÖZET

N Kaehler yüzeyinin bir M yüzeyi hiçbir kompleks nokta kapsamıyor ise bu yüzeye sırf reeldir denir. Sabit eğilimli (Slant) bir immersiyon [1] de B.Y.Chen tarafından bir Riemann manifoldunun sabit Wirtinger açılı hemen hemen hermityen bir manifoldu içine olan izometrik bir immersiyonu olarak tanımlanmıştır. Bu makalede, sabit eğilimli ve sırf reel yüzeyler çalışılmış ve kompleks uzay formlarındaki sırf reel yüzeyler için Chen tarafından ispatlanan genel optimal bir eşitsizlik verilmiştir.

Anahtar kelimeler: Sırf reel yüzeyler; sabit eğilimli yüzeyler; Wirtinger açısı; optimal eşitsizlik.

1. INTRODUCTION

Let N be a Kaehler surface endowed with a complex structure J and a Riemannian metric $\tilde{g}$ which is J- Hermitian, namely;

$$\begin{aligned} &\tilde{g}(JX,JY)=\tilde{g}(X,Y), \quad \forall X,Y\in T_pN \\ &\tilde{\nabla}J=0 \end{aligned} \tag{1.1}$$

for $p\in N$, where $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}$. Then the curvature tensor $\tilde{R}$ of N satisfies the following equations:

$$\begin{aligned} &\tilde{R}(X,Y;Z,W)=-\tilde{R}(Y,X;Z,W), \\ &\tilde{R}(X,Y;Z,W)=\tilde{R}(Z,W;X,Y), \\ &\tilde{R}(X,Y;JZ,W)=-\tilde{R}(X,Y;Z,JW), \end{aligned} \tag{1.2}$$

where $\tilde{R}(X,Y;Z,W)=\tilde{g}(\tilde{R}(X,Y)Z,W)$.

Let M be a surface in a Kaehler surface N with induced metric g from $\tilde{g}$. Denote by ∇ and R the Levi-Civita connection and the curvature tensor of M, respectively. So the formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y=\nabla_X Y+h(X,Y), \tag{1.3}$$

$$\tilde{\nabla}_X \xi=-A_\xi X+D_X\xi \tag{1.4}$$

for vector fields X, Y tangent to M and ξ normal to M, where h, A and D are the second fundamental form, the shape operator and the normal connection.

The shape operator and the second fundamental form are related by

$$\tilde{g}(h(X,Y),\xi)=g(A_\xi X,Y) \tag{1.5}$$

for X, Y tangent to M and ξ normal to M. The equations of Gauss, Codazzi and Ricci are given respectively by

$$R(X,Y;Z,W)=\tilde{R}(X,Y;Z,W)+\langle h(X,W),h(Y,Z)\rangle-$$

$$\langle h(X,Z), h(Y,W)\rangle,$$

$$(\tilde{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \qquad (1.6)$$

$$\tilde{g}(R^D(X,Y)\xi,\eta) = \tilde{R}(X,Y;\xi,\eta) + \tilde{g}([A_\xi, A_\eta]X,Y), \qquad (1.7)$$

where X,Y,Z,W are vectors tangent to M and $\langle,\rangle$ is the inner product associated with the metric $\tilde{g}$, $\bar{\nabla}h$ and R^D are defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \quad (1.8)$$

$$R^D(X,Y) = [D_X, D_Y] - D_{[X,Y]}. \qquad (1.9)$$

The mean curvature vector H of the surface is defined by

$$H = \frac{1}{2} trace\ h \qquad (1.10)$$

The surface is called *minimal* if H vanishes identically.

Let $N(4\varepsilon)$ denote a complex space form with constant holomorphic sectional curvature 4ε. Then the Riemannian curvature tensor of $N(4\varepsilon)$ satisfies

$$\tilde{R}(X,Y;Z,W) = \varepsilon\{\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle + \langle JX,W\rangle\langle JY,Z\rangle$$

$$-\langle JX,Z\rangle\langle JY,W\rangle + 2\langle X,JY\rangle\langle JZ,W\rangle\}. \qquad (1.11)$$

2. BASIC FORMULAS ON SLANT SURFACES

An immersion of a surface M into a Kaehler surface is called *purely real* if the complex structure J on N carries the tangent bundle of M into a transversal bundle, [6]. Obviously, every purely real surface admits no complex points. A point on a purely real surface M is called *Lagrangian point* if J carries the tangent space T_pM into its normal space $T_p^{\perp}M$. A purely real surface is called *Lagrangian* if every point on M is a Lagrangian point.

Let M be an oriented surface immersed in a Kaehler surface N. For any vector X tangent to M, we put

$$JX = PX + FX, \tag{2.1}$$

where PX and FX are the tangential and the normal componenets of JX, respectively. Thus, P is an endomorphism of the tangent bundle TM and F a normal bundle valued 1-form on TM. The submanifold M is called a *complex surface* if $F = 0$ and is called a *totally real surface* if $P = 0$, and called *proper* if it is neither a complex surface nor a totally real surface,[9]. P and F defined by (2.1) are the endomorphisms on the tangent bundle of M. Since N is almost Hermitian, we have

$$\langle PX, Y\rangle = -\langle X, PY\rangle, \quad X, Y \in T(M).$$

Hence if we define $Q = P^2$ then Q is also a symmetric endomorphism of the tangent bundle of M. Therefore, at each point $x \in M$ the tangent space T_xM admits an orthogonal direct decomposition of eigenspaces of Q $T_xM = D_1(x) \oplus \ldots \oplus D_s(x)$.

Since P is skew-symmetric and $J^2 = -1$, each eigenvalue λ_i of Q lies in $[-1, 0]$. If $\lambda_i \neq 0$, then the eigenspace $D_i(x)$ corresponding to λ_i is of even dimension and invariant under P, that is $P(D_i(x)) = D_i(x)$. Furthermore, for each $\lambda_i \neq -1$, $\dim F(D_i(x)) = \dim D_i(x)$ and the normal subspaces $F(D_i(x)),\ i = 1, \ldots, s$, are mutually perpendicular. Hence $\dim N \geq 2\dim M - \dim D_{-1}(x)$, where $D_{-1}(x)$ denotes the eigenspace of Q corresponding to eigenvalue -1. For $X, Y \in T_xM$ let us define

$$(\nabla_X Q)Y = \nabla_X (QY) - Q(\nabla_X Y). \qquad (2.2)$$

Then the following lemmas can be proved.

Lemma 2.1 [1,8] Let M be a submanifold of an almost Hermitian manifold N. Then the symmetric endomorphism Q is parallel, that is $\nabla Q = 0$, if and only if each eigenvalue λ_i of Q is constant on M.

Each distribution D_i corresponding to the eigenvalue λ_i of Q is completely integrable. M is locally the Riemann product $M_1 \times M_2 \times ... \times M_s$ of the leaves of distributions.

Lemma 2.2 [1,4] Let M be a submanifold of an almost Hermitian manifold N. Then $\nabla P = 0$ if and only if M is locally the Riemannian product $M_1 \times M_2 \times ... \times M_s$, where each M_i is either a complex submanifold, a totally real submanifold or a Kaehlerian slant submanifold of N.

From Lemma 2.1 and Lemma 2.2 we get easily the following

Proposition 2.1 [3] Let M be an irreducible submanifold of an almost Hermitian manifold N. If M is neither complex nor totally real, then M is a Kaehlerian slant submanifold if and only if the endomorphism P is parallel, that is $\nabla P = 0$.

Then we may prove the following theorem for surfaces in an almost Hermitian manifold.

Theorem 2.1, [5] Let M be a surface in an almost Hermitian manifold N, then the following statements are equivalent:

(i) M is neither totally real nor complex in N and $\nabla P = 0$.

(ii) M is a Kaehlerian slant surface.

(iii) M is a proper slant surface.

Proof. Since every proper slant submanifold is of even dimension, Lemma 2.2 implies that if the endomorphism P is parallel then M is a Kaehlerian surface, or a totally real surface, or a kaehlerian slant surface. Thus if M is neither totally real nor complex by definition the statemenets (i) and (ii) are equivalent.

It is clear that (ii) implies (iii) . Now we will prove the converse. Let M be a proper slant surface of N with the slant angle α . Let us choose an orthonormal frame $\{e_1, e_2\}$ such that $Pe_1 = (\cos\alpha)e_2,\ Pe_2 = -(\cos\alpha)e_1.$ On the other hand we may write that

$$\nabla_X e_1 = \omega_1^1(X)e_1 + \omega_1^2(X)e_2, \quad \nabla_X e_2 = \omega_2^1(X)e_1 + \omega_2^2(X)e_2$$

which implies that $(\nabla_X P)e_1 = 0$ and $(\nabla_X P)e_2 = 0$. Therefore, $\nabla P = 0$, that is P is parallel and this implies that M is a Kaehlerian slant surface. For any vector field ξ normal to the submanifold M in N, we put

$$J\xi = t\xi + n\xi, \tag{2.3}$$

where $t\xi$ and $n\xi$ are the tangential and the normal componenets of $J\xi$, respectively.

Now, for each nonzero vector X tangent to M at point p, we will define the angle $\alpha(X)$ between JX and T_pM . For an oriented orthonormal frame $\{e_1, e_2\}$ of T_pM , it follows from (2.1) that

$$Pe_1 = (\cos\alpha)e_2,\ Pe_2 = -(\cos\alpha)e_1 \tag{2.4}$$

for some function α . This function α is called the *Wirtinger angle*. The Wirtinger angle is independent of the choice of e_1, e_2 which preserves the orientation. Thus, it defines a function α on M , called the Wirtinger function of M. For oriented purely real surface in N, the Wirtinger function is differentiable function. A purely real surface is called a *slant surface* if its Wirtinger angle is constant. The Wirtinger angle of a slant surface is called the slant angle, [3]. A slant immersion with slant angle α is said to be α*-slant*. An isometric immersion $f: M \to N$ of M in N is called *holomorphic* if at each point $p \in M$ we have $J(T_pM) = T_pM$ and it is called *totally real* if $J(T_pM) \subset T_p^{\perp}M$ for each $p \in M$, where $T_p^{\perp}M$ is the normal space of M at p. Then a totally real immersion will be called Lagrangian if $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} N$ as defined above. Holomorphic and totally real immersions are slant immersions with slant angle 0 and $\frac{\pi}{2}$, respectively. A slant immersion is called *proper slant* if it is neither holomorphic nor totally real.

For submanifolds for a Kaehlerian manifold we may prove in general the following important lemma, [7].

Lemma 2.3 Let M be a submanifold in a Kaehlerian manifold N. Then

1) For $X, Y \in T(M)$, we have

$$(\nabla_X P)Y = th(X,Y) + A_{FY}X$$

and hence $\nabla P = 0$ if and only if $A_{FX}Y = A_{FY}X, \quad X, Y \in T(M)$.

2) For any $X, Y \in T(M)$, we have

$$(\nabla_X F)Y = nh(X,Y) - h(X, PY)$$

and hence $\nabla F = 0$ if and only if

$$A_{n\xi}X = -A_{\xi}PX, \quad \xi \in N(M), X \in T(M),$$

such that

$$Jh(X,Y) = th(X,Y) + nh(X,Y).$$

Proof. For N is Kaehlerian, $\tilde{\nabla} J = 0$. Then for all $X, Y \in T(M)$

$$0 = \tilde{\nabla}_X JY - J\tilde{\nabla}_X Y = \tilde{\nabla}_X (PY + FY) - J(\nabla_X Y + h(X,Y))$$

$$= \nabla_X PY + h(X, PY) - A_{FY} X + D_X FY - P(\nabla_X Y) - F(\nabla_X Y) - th(X,Y) - nh(X,Y).$$

If we equate the tangential and the normal parts of both sides, then we get

$$(\nabla_X P)Y = th(X,Y) + A_{FY} X,$$
$$(\nabla_X F)Y = nh(X,Y) - h(X, PY).$$

Thus P is parallel if and only if $\langle th(X,Y) + A_{FY} X, Z\rangle = 0$ which is equivalent to

$$\langle A_{FY} X, Z\rangle = -\langle th(X,Y), Z\rangle = \langle A_{FX} Y, Z\rangle.$$

Besides, $\nabla F = 0$ if and only if $\langle nh(X,Y) - h(X, PY), \xi\rangle = 0$

$$\Leftrightarrow \langle h(X, PY), \xi\rangle = \langle nh(X,Y), \xi\rangle = -\langle A_{n\xi} Y, X\rangle \Leftrightarrow$$

$$\langle h(PY, X), \xi\rangle = -\langle A_{n\xi} Y, X\rangle$$

$$\Leftrightarrow \langle A_\xi PY, X\rangle = -\langle A_{n\xi} Y, X\rangle \Leftrightarrow -A_\xi PY = A_{n\xi} Y.$$

Corollary 2.1 Let M be a surface in a Kaehlerian manifold N. Then M is slant if and only if for $X, Y \in T(M)$ $A_{FY} X = A_{FX} Y$.

3. SLANT SURFACES AND A GENERAL INEQUALITY FOR A PURELY REAL SURFACE

For a purely real surface M immersed in N, if we put

$$e_3 = (\csc\alpha)Fe_1, \quad e_4 = (\csc\alpha)Fe_2, \tag{3.1}$$

then we may derive from (2.1),(2.4) and (3.1) that

$$Je_1 = \cos\alpha e_2 + \sin\alpha e_3, \quad Je_2 = -\cos\alpha e_1 + \sin\alpha e_4 \tag{3.2}$$

$$Je_3 = -\sin\alpha e_1 - \cos\alpha e_4, \quad Je_4 = -\sin\alpha e_2 + \cos\alpha e_3 \tag{3.3}$$

$$\langle e_3, e_3\rangle = \langle e_4, e_4\rangle = 1, \quad \langle e_3, e_4\rangle = 0. \tag{3.4}$$

We call such a frame $\{e_1, e_2, e_3, e_4\}$ an *adapted orthonormal frame* for M.

Let ω^1, ω^2 denote the dual 1-forms of e_1, e_2. For an adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, we may put

$$\nabla_X e_1 = \omega(X)e_2, \quad \nabla_X e_2 = -\omega(X)e_1 \tag{3.5}$$

$$\nabla_X e_3 = \Phi(X)e_4, \quad \nabla_X e_4 = -\Phi(X)e_3 \tag{3.6}$$

for some 1-forms ω and Φ known as the connection forms. Then we have

$$d\omega^1 = \omega \wedge \omega^2, \quad d\omega^2 = -\omega \wedge \omega^1. \tag{3.7}$$

Now for any vector η normal to M, we may write $\eta = \langle \eta, e_3\rangle e_3 + \langle \eta, e_4\rangle e_4$. For the second fundamental form h of M, we have $h(e_i, e_j) \in T^{\perp}(M)$, and therefore, we may write

$$h(e_i, e_j) = h_{ij}^3 e_3 + h_{ij}^4 e_4, \quad 1 \le i, j \le 2 \tag{3.8}$$

$$\langle h(e_i, e_j), e_3\rangle = h_{ij}^3 \ , \ \langle h(e_i, e_j), e_4\rangle = h_{ij}^4 \ . \tag{3.9}$$

$$\sum_j h_{ij}^r \omega^j(X) = \sum_j \langle h(e_i, e_j), e_r \rangle \langle X, e_j \rangle$$
$$= \left\langle h\left(e_i, \sum_j \langle X, e_j \rangle e_j\right), e_r \right\rangle = \langle h(X, e_i), e_r \rangle = \langle e_i, A_{e_r} X \rangle, \quad r = 3, 4.$$

where A_{e_r} is Weingarten map and $\tilde{\nabla}_X e_r = -A_{e_r} X + D_X e_r$. Here $\tilde{\nabla}$ is the connection on N and $\langle D_X e_r, e_1 \rangle = \langle D_X e_r, e_2 \rangle = 0$. So we get

$$\sum_{j=1}^{2} h_{ij}^r \omega^j(X) = \langle e_i, A_{e_r} X \rangle - \langle e_i, D_X e_r \rangle = -\langle e_i, \tilde{\nabla}_X e_r \rangle$$
$$= \langle \tilde{\nabla}_X e_i, e_r \rangle - X\langle e_i, e_r \rangle = \langle \tilde{\nabla}_X e_i, e_r \rangle$$
$$= \omega_i^r(X).$$

Thus we proved that

$$\omega_1^3 = h_{11}^3 \omega^1 + h_{12}^3 \omega^2, \quad \omega_2^3 = h_{21}^3 \omega^1 + h_{22}^3 \omega^2 \tag{3.10}$$

and

$$\omega_1^4 = h_{11}^4 \omega^1 + h_{12}^4 \omega^2, \quad \omega_2^4 = h_{21}^4 \omega^1 + h_{22}^4 \omega^2 \tag{3.11}$$

where h is symmetric that is $h_{ij}^r = h_{ji}^r$. Also , since the Weingarten map is symmetric we have

$$\begin{aligned} \omega_i^r(X) &= \langle \tilde{\nabla}_X e_i, e_r \rangle = X\langle e_i, e_r \rangle - \langle e_i, \tilde{\nabla}_X e_r \rangle \\ &= -\langle e_i, -A_{e_r} X + D_X e_r \rangle = \langle e_i, A_{e_r} X \rangle = \langle A_{e_r} e_i, X \rangle. \end{aligned} \tag{3.12}$$

On the other hand, since $A_{e_r} e_j$ is tangential, we get

$$A_{e_r} e_j = \sum_i \langle A_{e_r} e_j, e_i \rangle e_i = \sum_i \langle h(e_i, e_j), e_r \rangle e_i = \sum_i h_{ij}^r e_i .$$

Then we have

$$A_{e_3}e_j = h^3_{1j}e_1 + h^3_{2j}e_2, \quad A_{e_4}e_j = h^4_{1j}e_1 + h^4_{2j}e_2. \tag{3.13}$$

It is known that the Gauss and normal curvatures of the surface M in a Kaehler surface N are given by

$$\begin{aligned} G &= h^3_{11}h^3_{22} - (h^3_{12})^2 + h^4_{11}h^4_{22} - (h^4_{12})^2 \\ G^D &= h^3_{11}h^4_{12} + h^3_{12}h^4_{22} - h^3_{12}h^4_{11} - h^3_{22}h^4_{12} \end{aligned} \tag{*}$$

Theorem 3.1 Let M be a slant surface in a Kaehler surface N. Gauss and normal curvatures of M are identically equal, namely $G \equiv G^D$.

Proof. Let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal adapted frame as in above. By using Corollary 2.1 we have

$$\begin{aligned} h^3_{12} &= \langle h(e_1,e_2), e_3\rangle = \langle h(e_1,e_2), (\csc\alpha)Fe_1\rangle \\ &= (\csc\alpha)\langle h(e_1,e_2), Fe_1\rangle = (\csc\alpha)\langle A_{Fe_1}e_1, e_2\rangle \\ &= (\csc\alpha)\langle A_{Fe_1}e_2, e_1\rangle = (\csc\alpha)\langle A_{Fe_2}e_1, e_1\rangle \\ &= (\csc\alpha)\langle h(e_1,e_1), Fe_2\rangle = \langle h(e_1,e_1), e_4\rangle = h^4_{11}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} h^3_{22} &= \langle h(e_2,e_2), e_3\rangle = \langle h(e_2,e_2), (\csc\alpha)Fe_1\rangle \\ &= (\csc\alpha)\langle h(e_2,e_2), Fe_1\rangle = (\csc\alpha)\langle A_{Fe_1}e_2, e_2\rangle \\ &= (\csc\alpha)\langle A_{Fe_1}e_2, e_2\rangle = (\csc\alpha)\langle A_{Fe_2}e_1, e_2\rangle \\ &= (\csc\alpha)\langle h(e_1,e_2), Fe_2\rangle = \langle h(e_1,e_2), e_4\rangle = h^4_{12}. \end{aligned}$$

Therefore, using (*) we get $R \equiv R^D$. We need the following lemma, [1,6].

Lemma 3.1 Let M be a purely real surface in a Kaehler surface N. Then, with respect to an adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, we have

$$\begin{aligned} &e_1\alpha = h_{11}^4 - h_{12}^3, \quad e_2\alpha = h_{12}^4 - h_{22}^3, \\ &\Phi_1 = \omega_1 - (h_{11}^3 + h_{12}^4)\cot\alpha, \ \Phi_2 = \omega_2 - (h_{12}^3 + h_{22}^4)\cot\alpha \end{aligned} \tag{3.14}$$

where $\omega_j = \omega(e_j)$ and $\Phi_j = \Phi(e_j)$ for $j = 1, 2$.

Now we will give a general optimal inequality for purely real surfaces proved by B. Y. Chen in [6].

Theorem 3.2, Let M be a purely real surface in a complex space form $N(4\varepsilon)$. Then we have

$$H^2 \geq 2\left\{K - \|\nabla\alpha\|^2 - (1 + 3\cos^2\alpha)\varepsilon\right\} + 4\langle\nabla\alpha, Jh(e_1, e_2)\rangle\csc\alpha$$

(3.15)

with respect to an orthonormal frame $\{e_1, e_2\}$ satisfying $\langle\nabla\alpha, e_2\rangle = 0$, where H^2 and K are the squared mean curvature and the Gauss curvature of M, respectively.

The equality case of (3.15) holds at p if and only if, with respect a suitable adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, the shape operator at p take the forms

$$A_{e_3} = \begin{pmatrix} 3\varphi & \delta \\ \delta & \varphi \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} \delta + e_1\alpha & \varphi \\ \varphi & 3\delta + 3e_1\alpha \end{pmatrix}. \tag{3.16}$$

Proof. Assume that M is a purely real surface in M. Without loss of generality, we may choose an adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$ such that the gradient of α is parallel to e_1 at p. So, we have $\nabla\alpha = (e_1\alpha)e_1$. Let us put

$$h(e_1,e_1) = \beta e_3 + \gamma e_4,\quad h(e_1,e_2) = \delta e_3 + \varphi e_4,\quad h(e_2,e_2) = \lambda e_3 + \mu e_4. \tag{3.17}$$

Then by Lemma 3.1 we have

$$A_{e_3} = \begin{pmatrix} \beta & \delta \\ \delta & \varphi \end{pmatrix},\quad A_{e_4} = \begin{pmatrix} \delta + e_1\alpha & \varphi \\ \varphi & \mu \end{pmatrix}. \tag{3.18}$$

From this we see that the squared mean curvature H^2 and the Gauss curvature K of M satisfy

$$4H^2 = (\beta+\varphi)^2 + (\delta+\mu+e_1\alpha)^2, \tag{3.19}$$

$$K = \beta\varphi + \delta\mu + \mu e_1\alpha - \delta^2 - \varphi^2 + (1+3\cos^2\alpha)\varepsilon \tag{3.20}$$

Hence, we obtain

$$H^2 - 2K + 2\|\nabla\alpha\|^2 = \frac{1}{4}\{(\beta-3\varphi)^2 + (\mu - 3(\delta+e_1\alpha)^2\} -$$

$$4\delta e_1\alpha - 2(1+3\cos^2\alpha)\varepsilon \geq -4\delta e_1\alpha - 2(1+3\cos^2\alpha)\varepsilon. \tag{3.21}$$

On the other hand, from $\nabla\alpha = (e_1\alpha)e_1$ and (3.1) we have $F(\nabla\alpha) = (e_1\alpha)\sin\alpha e_3$. Hence, we obtain from (3.17) that

$$\delta e_1\alpha = \langle J(\nabla\alpha), h(e_1,e_2)\rangle \csc\alpha. \tag{3.22}$$

Combining this with (3.21) gives inequality (3.15).

If the equality case of (3.15) holds at a point p, then it follows from (3.21) that $\beta = 3\varphi$ and $\mu = 3\delta + 3e_1\alpha$ hold at p. Hence we obtain (3.16) from (3.18). Conversely, if (3.16) holds at a point p, then it follows from (3.16) and Lemma 3.1 that we have $e_2\alpha = 0$ at p. Thus, we get $\langle \nabla\alpha, Jh(e_1, e_2)\rangle = -\delta e_1\alpha \sin\alpha$ at p. So, it is a straight-forward to show that (3.16) holds at p.

Now, we may express the following two results of the Theorem 3.2 about a slant surface in a complex space form $N(4\varepsilon)$ and a purely real surface in $\mathbb{C}^2$, [6].

Corollary 3.1 If M is a slant surface in a complex space form $N(4\varepsilon)$ with slant angle θ, then we have

$$H^2 \geq 2\left\{K - (1 + 3\cos^2\theta)\varepsilon\right\}. \tag{3.23}$$

Corollary 3.2 Let M be a purely real surface in $\mathbb{C}^2$. Then we have

$$H^2 \geq 2\left\{K - \|\nabla\alpha\|^2 + 2\langle \nabla\alpha, Jh(e_1, e_2)\rangle \csc\alpha\right\} \tag{3.24}$$

with respect to an orthonormal frame $\{e_1, e_2\}$ satisfying $\langle \nabla\alpha, e_2\rangle = 0$.

The equality case of (3.24) holds if and only if, with respect to a suitable adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$, the shape operators of M take the forms

$$A_{e_3} = \begin{pmatrix} 3\varphi & \delta \\ \delta & \varphi \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} \delta + e_1\alpha & \varphi \\ \varphi & 3\delta + 3e_1\alpha \end{pmatrix}.$$

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