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Research Article

The First Study of Mersenne–Leonardo Sequence

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Abstract
In this study, we introduce a new class of numbers, referred to as Modified Mersenne–Leonardo numbers. The aim of this paper is to define the Modified Mersenne–Leonardo sequence and investigate some of its properties, including the recurrence relation, summation formula, and generating function. Additionally, classical identities such as the Tagiuri–Vajda, Catalan, Cassini, and d'Ocagne identities are derived for the Modified Mersenne–Leonardo numbers.

Keywords: Binet's formula, Generating function, Leonardo numbers, Mersenne numbers, Mersenne–Leonardo numbers
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1. Introduction

The Mersenne sequence is made up of non-negative integers in the form of a power of two minus one, and is best known for some of the prime numbers that make it up, which are called Mersenne primes. These numbers are defined by the recurrence relation

$$M_n = 3M_{n-1} - 2M_{n-2}, \text{ for all } n \geq 2; \tag{1.1}$$

with initial terms $M_0 = 0$ and $M_1 = 1$. This recurrence serves as the foundation for their exploration and application in number theory, forming the sequence:

$$\{M_n\}_{n\geq 0} = \{0, 1, 3, 7, 15, 31, 63, 127, 255, \dots\}.$$

which is referred as sequence A000225 in the OEIS [1]. Considering the initial values $m_0 = 2$ and $m_1 = 3$, with the identical recurrence relation $m_n = 3m_{n-1} - 2m_{n-2}$, for all $n \geq 2$ we have the Mersenne–Lucas numbers. The terms of this sequence are called Mersenne-Lucas numbers and are expressed in the form $m_n = 2^n + 1$, which is identified as sequence A000051 in OEIS [1]. These two classes of numbers are an indispensable concept in number theory, exhibiting significant implications in domains such as cryptography and the identification of large prime numbers.

In mathematical literature, there have been many studies of the sequences of Mersenne and Mersenne-Lucas numbers. For example, [2] offers a thorough and detailed examination of these two types of special numbers; in particular they are intimately tied to classical problems in the theory of prime numbers, as seen in [3], [4] and [5]. It also looks at practical applications, and is particularly relevant in the specific context of cryptography, as seen in [6] and [7]. Some generalizations or extensions of the

Mersenne or Mersenne-Lucas sequences, generation functions, and several identities can be found in [8–10] and [11], among others.

The Leonardo sequence has similarities to the Fibonacci sequence, wherein each term is derived from the sum of the preceding two terms, with the addition of the constant value one. The recurrence for the Leonardo sequence is:

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \text{ for all } n \geq 2;$$

with initial values $Le_0 = Le_1 = 1$. The first few Leonardo numbers are

$$\{Le_n\}_{n\geq 0} = \{1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, 287, \ldots\}.$$

This particular sequence is referenced as $A001595$ in the OEIS [1]. According [12] and [13, 14], this sequence is very similar to the Fibonacci sequence, including a relationship between Leonardo's numbers and Fibonacci's numbers that is

$$Le_n = 2F_{n+1} - 1.$$

The resurgence of interest in these Leonardo sequences can be attributed to the seminal paper by [15]. Primarily utilized in the domain of data structure and algorithmic analysis, it is particularly prevalent in the context of balanced trees, see [13] and [14], among others. Some generalizations or extensions of the Leonardo sequence, generating functions and several identities can be found at [16]- [23], among others.

In a recent publications, [24] proposed the Pell–Leonardo sequence and presented a new sequence with a recurrence third-order, [25] in the section entitled *Generalized Bronze Leonardo sequences* deals with the sequences Bronze Leonardo, Bronze Leonardo–Lucas, and Modified Bronze Leonardo. Another work that is in line with the same logic of these one is [26]. These works motivated the formulation of our work about the Modified Mersenne-Leonardo numbers presented in Section 3 and the subsequent study.

The structure of the present paper is divided into two more sections, as follows. In Section 2, we briefly remember the Mersenne $\{M_n\}_{n\geq 0}$, and summarize the results of this sequence used in this study. In Section 3, we define the Modified Mersenne-Leonardo sequence; in particular, in the Subsection 3.1, we introduce the new sequence, detailing its characteristics and properties in connection with the classical Mersenne sequence. In Subsection 3.2 we present Binet's formula, which gives an explicit expression for the terms of the sequence, and the generating function associated with the sequence is presented. In Subsection 3.3 we present several fundamental identities for Modified Mersenne–Leonardo such as Tagiuri-Vajda, d'Ocagene and their consequences, accompanied by some numerical examples to illustrate their application. Finally, summation formulas involving the Modified Mersenne-Leonardo numbers are presented in Subsection 3.4. We conclude with some final considerations and state some future work on this topic.

2. Mersenne Numbers: A Background

Recall that the characteristic equation associated with equation (1.1) for the Mersenne sequence is given by $r^2 = 3r - 2$. This equation corresponds to the characteristic equation of a Horadam-type sequence, see [27] and [28]. Solving for r yields that the roots of the equation are $r_1 = 2$ and $r_2 = 1$.

The Binet formula provides a direct method to compute the n-th Mersenne number without having to iterate through the sequence. This formula offers an efficient way to calculate Mersenne-type numbers based on the sequence structure. Numerous identities, including Catalan's identity, Cassini's identity, and d'Ocagne's identity, related to Mersenne numbers are presented in classical literature.

As demonstrated in [8], the Binet formula for Mersenne sequence is given in the next result:

Lemma 2.1. *Let $\{M_n\}_{n\geq 0}$ be the Mersenne sequence. Then:*

$$M_n = 2^n - 1\ . \tag{2.1}$$

Some of the most important properties that Mersenne numbers satisfy are summarized in the following result:

Lemma 2.2. *[8, Proposition 2.5] If M_j is the $j-$th term of the Mersenne sequence then:*

(a) $M_j^2 = 4^j - M_{j+1}$

(b) $\sum_{j=0}^{n} M_j = M_{n+1} - (n+1) = 2M_n - n\ .$

According Soykan [11] the sequence $\{M_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$M_{-n} = \frac{3}{2}M_{-(n-1)} - \frac{1}{2}M_{-(n-2)} \tag{2.2}$$

for $n = 1,2,3,\ldots$. Therefore, the equation of the recurrence (1.1) holds for all integer n.

An examination of the Table 1, and applying the equation (2.2), it is possible to conclude that:

$$M_{-n} = -\frac{M_n}{2^n},$$

for all integers $n \geq 1$.

n	0	1	2	3	4	5	6	7	8	9	10
M_n	0	1	3	7	15	31	63	127	255	511	1023
M_{-n}	0	$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{7}{8}$	$-\frac{15}{16}$	$-\frac{31}{32}$	$-\frac{63}{64}$	$-\frac{127}{128}$	$-\frac{255}{256}$	$-\frac{511}{512}$	$-\frac{1023}{1024}$

Table 2.1. The first few values of the Mersenne numbers with positive and negative subscripts

To facilitate the understanding of the subsequent result, it is first necessary to establish some preliminary auxiliary results.

Lemma 2.3. *Let n,s,k be an non-negative integers and $\{M_k\}_{k\geq 0}$ the Mersenne sequence. Then the following identity hold:*

$$2^{n+s+k} - 2^{n+s} - 2^{n+k} + 2^n = 2^n M_s M_k .$$

Proof. Note that

$$\begin{aligned} 2^{n+s+k} - 2^{n+s} - 2^{n+k} + 2^n &= 2^{n+s}(2^k-1) - 2^n(2^k-1) \\ &= 2^n M_s M_k . \end{aligned}$$

since $M_j = 2^j - 1$ the result follows. □

3. The Modified Mersenne–Leonardo Sequence

In this section, we introduce a new numerical sequence called the Modified Mersenne–Leonardo sequence. We begin with its definition and some of its properties. Next, we present the corresponding Binet formula and generating function. Finally, we conclude this section by stating several identities.

3.1 Introduction to Modified Mersenne–Leonardo sequence

As we have mentioned before, we found the motivation for this our work, mainly, by the recent publications of [24] and [25], which introduced a new sequence of numbers with third order recurrence, respectively, the Pell-Leonardo sequence and Generalized Bronze Leonardo sequences. In these works, various identities are established for these sequences. Following to the literature, now, let us define the Modified Mersenne-Leonardo sequence and explore its implications.

We define the Modified Mersenne–Leonardo numbers by using a recurrence relation, that it is stated in what follows:

Definition 3.1. *For all integer $n \geq 2$, the Modified Mersenne–Leonardo sequence $\{\mathcal{M}_n\}_{n\geq 0}$ satisfies the following recurrence relation:*

$$\mathcal{M}_{n+1} = 3\mathcal{M}_n - 2\mathcal{M}_{n-1} + 1, \tag{3.1}$$

with the initial values $\mathcal{M}_0 = 0$ and $\mathcal{M}_1 = 1$.

The first thirteen Modified Mersenne–Leonardo numbers are:

$$\{\mathcal{M}_n\}_{n\geq 0} = \{0,1,4,11,26,57,120,247,512,1013,2036,4083,8178,\ldots\}.$$

We start with elementary observation that the sequence $\{\mathcal{M}_n\}_{n\geq 0}$ satisfies the second non-homogeneous linear recurrence. An equivalent way to write equation (3.1) is

$$\mathcal{M}_{n+1} = 3\mathcal{M}_n - 2\mathcal{M}_{n-1} + 1, \tag{3.2}$$

by subtracting equations (3.1) and (3.2), we obtain a homogeneous recurrence relation,

$$\mathcal{M}_{n+1} = 4\mathcal{M}_n - 5\mathcal{M}_{n-1} + 2\mathcal{M}_{n-2} .$$

The preceding discussion demonstrates that:

Proposition 3.2. *Let be $\{\mathcal{M}_n\}_{n\geq 0}$ the Modified Mersenne–Leonardo sequence that satisfies the homogeneous recurrence relation*

$$\mathcal{M}_{n+1} = 4\mathcal{M}_n - 5\mathcal{M}_{n-1} + 2\mathcal{M}_{n-2}\,, \tag{3.3}$$

with initial terms $\mathcal{M}_0 = 0$, $\mathcal{M}_1 = 1$, and $\mathcal{M}_2 = 4$.

The relationship between the Modified Mersenne–Leonardo numbers and the Mersenne numbers is expressed in the following proposition.

Proposition 3.3. *Let be $\{\mathcal{M}_n\}_{n\geq 0}$ the Modified Mersenne–Leonardo sequence and $\{M_n\}_{n\geq 0}$ the Mersenne sequence, then*

$$\mathcal{M}_{n+1} - \mathcal{M}_n = M_{n+1} \tag{3.4}$$

for all non-negative integer n.

Proof. We will prove this by induction on n. From the definition of Modified Mersenne–Leonardo numbers, we know that $\mathcal{M}_0 = M_0 = 0$ and $\mathcal{M}_1 = M_1 = 1$. Now, assume that equation (3.4) is true for all $1 < n \leq k$, and we will show that equation (3.4) also holds for $n = k+1$. Indeed, by applying the induction hypothesis and the homogeneous recurrence relation give by equation (3.3), we can express:

$$\begin{aligned}
\mathcal{M}_{k+1} &= 4\mathcal{M}_k - 5\mathcal{M}_{k-1} + 2\mathcal{M}_{k-2}\\
&= 3\mathcal{M}_k - 3\mathcal{M}_{k-1} - 2\mathcal{M}_{k-1} + 2\mathcal{M}_{k-2} + \mathcal{M}_k\\
&= 3(\mathcal{M}_k - \mathcal{M}_{k-1}) - 2(\mathcal{M}_{k-1} - \mathcal{M}_{k-2}) + \mathcal{M}_k\\
&\overset{\text{hip. ind.}}{=} 3M_k - 2M_{k-1} + \mathcal{M}_k\,.
\end{aligned}$$

As $M_{k+1} = 3M_k - 2M_{k-1}$, by equation (1.1), we obtain the result required. □

3.2 The Binet formula and generating functions

In this subsection, we introduce the Binet formula as well as the generating and exponential functions associated with the Modified Mersenne–Leonardo sequence. We also found the limit of the ratio $\mathcal{M}_{k+1}/\mathcal{M}_k$, for all $k \in \mathbb{N}$.

The characteristic equation associated with equation (3.3) for the Modified Mersenne–Leonardo sequence is given by $r^3 = 4r^2 - 5r + 2$. The roots are $r_1 = 1$ (double root of the equation) and $r_2 = 2$. With these roots, the Binet formula provides a direct method to compute the n-th Modified Mersenne–Leonardo number without having to iterate through the sequence.

Now we will determine the Binet formula for Modified Mersenne–Leonardo sequence, and we obtain:

Proposition 3.4 (Binet's formula)**.** *Let $\{\mathcal{M}_n\}_{n\geq 0}$ be the Modified Mersenne–Leonardo sequence. Then:*

$$\mathcal{M}_n = 2^{n+1} - (n+2)\,. \tag{3.5}$$

Proof. Using equations (2.1) and (3.4), a straightforward calculation gives us:

$$\begin{aligned}
\mathcal{M}_1 - \mathcal{M}_0 &= 2 - 1\\
\mathcal{M}_2 - \mathcal{M}_1 &= 2^2 - 1\\
\vdots\quad & \quad\vdots\\
\mathcal{M}_n - \mathcal{M}_{n-1} &= 2^n - 1\,.
\end{aligned}$$

So,

$$\begin{aligned}
\mathcal{M}_n - \mathcal{M}_0 &= (2-1) + (2^2-1) + \ldots + (2^n - 1)\\
&= (1 + 2 + 2^2 + \ldots + 2^n) - (n+1)\\
&= 2^{n+1} - (n+2)\,.
\end{aligned}$$

Since $\mathcal{M}_0 = 0$, we arrive at the result. □

It follows directly from Proposition 3.4 , and by making use of Lemma 2.2(b), that:

Corollary 3.5. *Let $\{\mathcal{M}_n\}_{n\geq 0}$ be the Modified Mersenne–Leonardo sequence. Then:*

$$\mathcal{M}_n = M_{n+1} - (n+1) = 2M_n - n\,.$$

Which implies that

Corollary 3.6. *Let $\{\mathcal{M}_n\}_{n\geq 0}$ be the Modified Mersenne–Leonardo sequence. Then:*

$$\mathcal{M}_n = \sum_{j=0}^{n} M_j \ .$$

In the literature, it is important to note that the function $GF_{a_n}(x)$ is referred to as the ordinary generating function for the sequence $\{a_n\}_{n\geq 0}$, with

$$GF_{a_n}(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots \tag{3.6}$$

To make notation easier, let us denote $GF_{a_n}(x)$ by $L(x)$.

Additionally, making use the homogeneous recurrence relation (Proposition 3.2), the following result states the ordinary generating function for the Modified Mersenne–Leonardo sequence.

Proposition 3.7. *The ordinary generating function for the Modified Mersenne–Leonardo sequence $\{\mathcal{M}_n\}_{n\geq 0}$, denoted by $L(x)$, is given by*

$$L(x) = \frac{4x}{2x^3 - 5x^2 + 4x + 1} \ .$$

Proof. According to equation (3.6), the ordinary generating function for the Modified Mersenne–Leonardo sequence is $L(x) = \sum_{n=0}^{\infty} \mathcal{M}_n x^n$; then using the equations $4x \cdot L(x)$, $-5x^2 \cdot L(x)$ and $2x^3 \cdot L(x)$, we obtain

$$\begin{array}{rcccccc}
-L(x) = & -\mathcal{M}_0 - & \mathcal{M}_1 x - & \mathcal{M}_2 x^2 - & \mathcal{M}_3 x^3 & \ldots \\
 & & & & -\mathcal{M}_n x^n - & \ldots \\
4x \cdot L(x) = & & 4\mathcal{M}_0 x + & 4\mathcal{M}_1 x^2 + & 4\mathcal{M}_2 x^3 + & \ldots \\
 & & & & +4\mathcal{M}_{n-1} x^n + & \ldots \\
-5x^2 \cdot L(x) = & & & -5\mathcal{M}_0 x^2 - & 5\mathcal{M}_1 x^3 & \ldots \\
 & & & & -5\mathcal{M}_{n-2} x^n - & \ldots \\
2x^3 \cdot L(x) = & & & & 2\mathcal{M}_0 x^3 + & \ldots \\
 & & & & +2\mathcal{M}_{n-3} x^n + & \ldots
\end{array}$$

When we add both sides of these equations, we have:

$$\begin{aligned}
(2x^3 - 5x^2 + 4x - 1)L(x) &= -\mathcal{M}_0 + (4\mathcal{M}_1 - \mathcal{M}_0)x + (-5\mathcal{M}_0 + 4\mathcal{M}_1 - \mathcal{M}_2)x^2 + \\
&\quad (2\mathcal{M}_0 - 5\mathcal{M}_1 + 4\mathcal{M}_2 - \mathcal{M}_3)x^3 + \ldots \\
&\quad +(2\mathcal{M}_{n-3} - 5\mathcal{M}_{n-2} + 4\mathcal{M}_{n-1} - \mathcal{M}_n)x^n + \ldots \\
&= -\mathcal{M}_0 + (4\mathcal{M}_1 - \mathcal{M}_0)x + (-5\mathcal{M}_0 + 4\mathcal{M}_1 - \mathcal{M}_2)x^2 + 0 + 0 \ldots
\end{aligned}$$

Since $\mathcal{M}_0 = 0$, $\mathcal{M}_1 = 1$, and $\mathcal{M}_2 = 4$, the result follows easily. □

The exponential generating function $E_{a_n}(x)$ of a sequence $\{a_n\}_{n\geq 0}$ is defined as a power series of the form:

$$E_{a_n}(x) = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \cdots + \frac{a_n x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \ .$$

In the next result, we consider $a_n = \mathcal{M}_n$ and making use of equation (3.5), the Binet formula for Modified Mersenne–Leonardo sequence and then we obtain the exponential generating function for this sequence.

Proposition 3.8. *For all $n \geq 0$ the exponential generating function for the Modified Mersenne–Leonardo sequence $\{\mathcal{M}_n\}_{n\geq 0}$ is*

$$E_{\mathcal{M}_n}(x) = 2e^{2x} - (x+2)e^x \ .$$

Proof. Note that

$$\begin{aligned} E_{\mathcal{M}_n}(x) &= \sum_{n=0}^{\infty} \frac{\mathcal{M}_n x^n}{n!} = \sum_{n=0}^{\infty} \frac{2^{n+1}-(n+2)}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{2^{n+1}}{n!} x^n - \sum_{n=0}^{\infty} \frac{n+2}{n!} x^n = 2e^{2x} - (x+2)e^x. \end{aligned}$$

□

The Poisson generating function $P_{a_n}(x)$ for a sequence $\{a_n\}_{n\geq 0}$ is given by:

$$P_{a_n}(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} e^{-x} \ .$$

This function generates the sequence $\{a_n\}_{n\geq 0}$ in terms of the parameter x. A relationship can be observed between exponential generation $E_{a_n}(x)$ and Poisson generation $P_{a_n}(x)$, which may be expressed by the following equation:

$$P_{a_n}(x) = e^{-x} E_{a_n}(x) \ .$$

As a consequence, the corresponding Poisson generating function is obtained.

Corollary 3.9. *For all $n \geq 0$ the Poisson generating function for the Modified Mersenne–Leonardo sequence $\{\mathcal{M}_n\}_{n\geq 0}$ is*

$$P_{\mathcal{M}_n}(x) = 2e^x - (x+2) \ .$$

To conclude this section, we determine the limit of the ratio $\frac{\mathcal{M}_{n+1}}{\mathcal{M}_n}$, where $\mathcal{M}_n$ be the n-th term of Modified Mersenne–Leonardo sequence. Again, using Binet's formula, the equation (3.5), we get another property of Modified Mersenne-Leonardo sequences $\{\mathcal{M}_n\}_{n\in\mathbb{Z}}$, which is stated by the following proposition.

Proposition 3.10. *For all non-negative integer n, let $\mathcal{M}_n$ be the n-th term of Modified Mersenne–Leonardo sequence, then*

$$\lim_{n\to\infty} \frac{\mathcal{M}_{n+1}}{\mathcal{M}_n} = 2 \ . \tag{3.7}$$

Proof. We have that

$$\begin{aligned} \lim_{n\to\infty} \frac{\mathcal{M}_{n+1}}{\mathcal{M}_n} &= \lim_{n\to\infty} \frac{2^{n+2}-(n+3)}{2^{n+1}-(n+2)} \\ &= \lim_{n\to\infty} \frac{2}{1-\dfrac{n+2}{2^{n+1}}} - \lim_{n\to\infty} \frac{\frac{n+3}{2^{n+1}}}{1-\dfrac{n+2}{2^{n+1}}} = 2, \end{aligned}$$

since $\lim_{n\to\infty} \dfrac{n+k}{2^{n+1}} = 0$, for some integer k fixed. □

Furthermore, the following result can be demonstrated by using the basic techniques of the calculation of the limits and equation (3.7).

Corollary 3.11. *For all non-negative integer n, let $\mathcal{M}_n$ be the n-th term of Modified Mersenne–Leonardo sequence, then*

$$\lim_{n\to\infty} \frac{\mathcal{M}_{n-1}}{\mathcal{M}_n} = \frac{1}{2} \ .$$

3.3 Some identities

In this section, we derive and examine several identities associated with the Modified Mersenne–Leonardo sequence $\{\mathcal{M}_n\}_{n\geq 0}$. Through an exploration of these fundamental identities, our objective is to deepen the understanding of the structural properties and behavior of the Modified Mersenne–Leonardo sequence, clarifying its mathematical significance.

A direct calculation and employing the Binet formula, equation (3.5), yields the following result:

Proposition 3.12. *Let n and k be non-negative integers with $n \geq k$, and $\{\mathcal{M}_k\}_{k\geq 0}$ the Modified Mersenne–Leonardo sequence. Then the following identity holds:*

$$\mathcal{M}_{n+k}\mathcal{M}_{n-k} = 4^{n+1} + \Big((k(4^k-1)-(n+2)(4^k+1))2^{n-k+1}\Big) + \big(n^2+4n+4-k^2\big) \ . \tag{3.8}$$

Proof. Note that

$$\begin{aligned}
\mathcal{M}_{n+k}\mathcal{M}_{n-k} &= (2^{n+k+1}-(n+k+2))(2^{n-k+1}-(n-k+2)) \\
&= 2^{2n+2}-(n-k+2)2^{n+k+1}-(n+k+2)2^{n-k+1}+(n+k+2)(n-k+2) \\
&= 4^{n+1} + \Big((k(4^k-1)-(n+2)(4^k+1))2^{n-k+1}\Big) + ((n\cdot n)+(4\cdot n)+4-(k\cdot k)) \ ,
\end{aligned}$$

as required. □

As an example, consider $n=4$ and $k=3$, so we have

$$\mathcal{M}_7\mathcal{M}_1 = 247\cdot 1 = 247,$$

and we get,

$$4^5+(3(4^3-1)-6(4^3+1))2^2+(4^2+4^2+4-9) = 1024+(189-390)4+27 = 247 \ .$$

From Proposition 3.12 follows the next result.

Corollary 3.13. *Let n be non-negative integers, and $\{\mathcal{M}_n\}_{n\geq 0}$ the Modified Mersenne–Leonardo sequence. Then the following identities hold:*

(a) $\mathcal{M}_{n+2}\mathcal{M}_{n-2} = n^2 + \big(2^{n+3}-17n-4\big)2^{n-1}+4n$;

(b) $\mathcal{M}_{n+1}\mathcal{M}_{n-1} = n^2 + \big(2^{n+2}-5n-7\big)2^{n}+4n+3$.

Proof. Setting $k=2$ and $k=1$, respectively, in equation (3.8). Since $M_2=3$ and $M_4=15$, we have the result. □

Other interesting result:

Proposition 3.14. *Let be non-negative the integers n and $\{\mathcal{M}_k\}_{k\geq 0}$ the Modified Mersenne–Leonardo sequence. Then the following identity hold:*

$$\mathcal{M}_{m+3}\mathcal{M}_{m+4} - \mathcal{M}_{m+1}\mathcal{M}_{m+6} = (84n+160)2^n+6 \ .$$

Proof. Note that

$$\begin{aligned}
\mathcal{M}_{m+3}\mathcal{M}_{m+4} - \mathcal{M}_{m+1}\mathcal{M}_{m+6} &= [2^{n+4}-(n+5)][2^{n+5}-(n+6)]-[2^{n+2}-(n+3)][2^{n+7}-(n+9)] \\
&= 2^{n+2}[(n+8)-4(n+6)]+2^{n+5}[4(n+3)-(n+5)]+6 \\
&= (84n+160)2^n+6,
\end{aligned}$$

as required. □

Take $n=5$ and we present the following example, and we have

$$\mathcal{M}_8\mathcal{M}_9 - \mathcal{M}_6\mathcal{M}_{11} = 502\cdot 1013 - 120\cdot 4083 = 18566 \ ;$$

on the other hand

$$(84\cdot 5+160)2^5+6 = 580\cdot 32+6 = 18566.$$

The following result demonstrates the efficacy of the Binet formula and helps to illustrate the convolution identity for the Modified Mersenne–Leonardo sequence.

Proposition 3.15 (Convolution's Identity)**.** *Let $\{\mathcal{M}_n\}_{n\geq 0}$ be the Modified Mersenne–Leonardo sequence, we have the following identities:*

$$\mathcal{M}_{m-1}\mathcal{M}_n + \mathcal{M}_m\mathcal{M}_{n+1} = 10\cdot 2^{m+n} - (6m+10)2^n - (3n+8)2^m + (2mn+5m+3n+8)$$

for all m and n non-negative integers.

Proof. Applying the Binet formula for Modified Mersenne–Leonardo sequence, equation (1.1), we have

$$\begin{aligned}\mathcal{M}_{m-1}\mathcal{M}_n + \mathcal{M}_m\mathcal{M}_{n+1} &= [2^m-(m+1)][2^{n+1}-(n+2)] + [2^{m+1}-(m+2)][2^{n+2}-(n+3)]\\ &= [2^{m+n+1}-(n+2)2^m-(m+1)2^{n+1}+(m+1)(n+2)] + [2^{m+n+3}-(n+3)2^{m+1}-(m+2)2^{n+2}+(m+2)(n+3)]\\ &= 10\cdot 2^{m+n}-(6m+10)2^n-(3n+8)2^m+(2mn+5m+3n+8)\end{aligned}$$

as required. □

Now, the Tagiuri-Vajda's identity for the Modified Mersenne–Leonardo sequence $\{\mathcal{M}_n\}_{n\geq 0}$ is stated in as follows.

Theorem 3.16. *Let n,s,k be non-negative integers, and $\{\mathcal{M}_n\}_{n\geq 0}$ the Modified Mersenne–Leonardo sequence. We have*

$$\mathcal{M}_{n+s}\mathcal{M}_{n+k} - \mathcal{M}_n\mathcal{M}_{n+s+k} = 2^{n+1}[(n+2)M_sM_k - kM_s - sM_k] + ks\ , \qquad (3.9)$$

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence.

Proof. Using equation (3.5) again we obtain that

$$\begin{aligned}\mathcal{M}_{n+s}\mathcal{M}_{n+k} - \mathcal{M}_n\mathcal{M}_{n+s+k} &= [2^{n+s+1}-(n+s+2)][2^{n+k+1}-(n+k+2)] - [2^{n+1}-(n+2)][2^{n+s+k+1}-(n+s+k+2)]\\ &= (n+s+k+2)2^{n+1}-(n+k+2)2^{n+s+1}-(n+s+2)2^{n+k+1}+(n+2)2^{n+s+k+1}\\ &= [2n(2^{n+k+s}-2^{n+s}-2^{n+k}+2^n)]+[4(2^{n+k+s}-2^{n+s}-2^{n+k}+2^n)]-[2s(2^{n+k}-2^n]-[2k(2^{n+2}-2^n].\end{aligned}$$

By Lemma 2.3 we have

$$\begin{aligned}\mathcal{M}_{n+s}\mathcal{M}_{n+k} - \mathcal{M}_n\mathcal{M}_{n+s+k} &= 2n\cdot 2^nM_sM_k + 4\cdot 2^nM_sM_k - 2k\cdot 2^nM_s - 2s\cdot 2^nM_k + ks\\ &= 2^n[(2n+4)M_sM_k - 2kM_s - 2sM_k] + ks\end{aligned}$$

and we have the validity of the result. □

In this example, take $n=4$, $s=1$ and $k=3$, and we have

$$\mathcal{M}_5\mathcal{M}_8 - \mathcal{M}_4\mathcal{M}_8 = 57\cdot 247 - 26\cdot 502 = 1027\ ,$$

on the other hand

$$2^5[6M_1M_3 - 3M_1 - 1M_3] + 3 = 32[6\cdot 7 - 3 - 7] + 3 = 1027\ .$$

As a consequence of Tagiuri-Vajda's identity, the subsequent results of this section establish the respective identities of d'Ocagne, Catalan, and Cassini for the Mersenne-Leonard numbers.

First the d'Ocagne identity:

Proposition 3.17. *Let r,n be non-negative integers with $r\geq n$, and $\{\mathcal{M}_n\}_{n\geq 0}$ the Modified Mersenne–Leonardo sequence, then*

$$\mathcal{M}_{n+1}\mathcal{M}_r - \mathcal{M}_n\mathcal{M}_{r+1} = 2^{n+1}[(n+1)M_{r-n} - (r-n)] + (r-n),$$

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence.

Proof. Consider $k=r-n$ and $s=1$ in equation (3.9), then

$$\mathcal{M}_{n+1}\mathcal{M}_r - \mathcal{M}_n\mathcal{M}_{r+1} = 2^{n+1}[(n+2)M_1M_{r-n} - (r-n)M_1 - M_{r-n}] + (r-n)$$

as $M_1 = 1$, we have

$$\mathcal{M}_{n+1}\mathcal{M}_r - \mathcal{M}_n\mathcal{M}_{r+1} = 2^{n+1}[(n+1)M_{r-n} - (r-n)] + (r-n),$$

which proves the result. □

Now, the Catalan identity:

Proposition 3.18. *Let n,k be non-negative integers with $n \geq k$, and $\{\mathcal{M}_n\}_{n\geq 0}$ the Modified Mersenne–Leonardo sequence, then*

$$\mathcal{M}_{n+k}\mathcal{M}_{n-k} - (\mathcal{M}_n)^2 = 2^{n-k+1}[(1+2^k)kM_k - (n+2)M_k^2] - k^2 , \tag{3.10}$$

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence.

Proof. Let us assume that $s = -k$ in equation (3.9), and then

$$\mathcal{M}_{n+k}\mathcal{M}_{n-k} - \mathcal{M}_n^2 = 2^{n+1}[(n+2)M_{-k}M_k - kM_{-k} - (-k)M_k] + k(-k).$$

Since $M_{-k} = \frac{-M_k}{2^k}$, the result follows. □

As a consequence of Catalan's identity is

Corollary 3.19. *For all non-negative integer n, we have*

$$\mathcal{M}_n^2 - \mathcal{M}_{n+2}\mathcal{M}_{n-2} = (9n-12)2^{n-1} + 4,$$

where $\{\mathcal{M}_n\}_{n\geq 0}$ is the Modified Mersenne–Leonardo sequence.

Proof. By doing $k = 2$ in equation (3.10), we have

$$\mathcal{M}_{n+2}\mathcal{M}_{n-2} - \mathcal{M}_n^2 = 2^{n-2+1}[(1+2^2)2M_2 - (n+2)M_2^2] - 2^2$$

$$= 2^{n-1}[10M_2 - (n+2)M_2^2] - 4,$$

since $M_2 = 3$, we get

$$\mathcal{M}_{n+2}\mathcal{M}_{n-2} - \mathcal{M}_n^2 = 2^{n-1}[30 - 9(n+2)] - 4,$$

and we have the result required. □

Other consequence of Catalan's identity, by doing $k = 1$ in equation (3.10) and since $M_1 = 1$, we have the following result.

Corollary 3.20. *[Cassini's identity] For all $n \in \mathbb{Z}$ then*

$$\mathcal{M}_n^2 - \mathcal{M}_{n+1}\mathcal{M}_{n-1} = (n-1)2^n + 1,$$

where $\{\mathcal{M}_n\}_{n\geq 0}$ is the Modified Mersenne–Leonardo sequence.

As a example, consider $n = 10$, so we have

$$\mathcal{M}_{10}^2 - \mathcal{M}_{11}\mathcal{M}_9 = (2036)^2 - 4083 \cdot 1013 = 9217 ,$$

and we get,

$$9 \cdot 2^{10} + 1 = 9217 .$$

To finish this subsection, making the substitution of $n = 2m$ in Corollary 3.20, we obtain:

Corollary 3.21. *For all integer $m \geq 1$, we have*

$$\mathcal{M}_{2m}^2 - \mathcal{M}_{2m+1}\mathcal{M}_{2m-1} = (2m-1)4^m + 1,$$

where $\{\mathcal{M}_n\}_{n\geq 0}$ is the Modified Mersenne–Leonardo sequence.

This corollary is other Cassini's type identity where in this case, the first term on the left side of the equation is always considered with an even subscript.

3.4 Sum of terms involving the Modified Mersenne–Leonardo numbers

In this section, we present the results of our investigation into the partial sums of the Modified Mersenne–Leonardo numbers, considering a variable number of terms. Specifically, we analyze the sequence of partial sums, defined as the sum of the terms of the Modified Mersenne–Leonardo sequence for a given non-negative value of n,

$$\sum_{k=0}^{n} \boldsymbol{M\!L}_k = \boldsymbol{M\!L}_0 + \boldsymbol{M\!L}_1 + \boldsymbol{M\!L}_2 + \cdots + \boldsymbol{M\!L}_n \ ,$$

for $n \geq 0$, and where $\{\boldsymbol{M\!L}_n\}_{n\geq 0}$ is the Modified Mersenne–Leonardo sequence.

Proposition 3.22. *Let $\{\boldsymbol{M\!L}_n\}_{n\geq 0}$ be the Modified Mersenne–Leonardo sequence, the following identities hold:*

$$\begin{aligned}
&(a) \quad \textstyle\sum_{k=0}^{n} \boldsymbol{M\!L}_k &&= M_{n+2} - \frac{(n+2)(n+3)}{2} \ ,\\
&(b) \quad \textstyle\sum_{k=0}^{n} \boldsymbol{M\!L}_{2k} &&= \frac{2}{3} M_{2n+1} - (n+1)(n+2) \ ,\\
&(c) \quad \textstyle\sum_{k=0}^{n} \boldsymbol{M\!L}_{2k+1} &&= \frac{8}{3} M_{2n+1} - (n+1)(n+3) \ .
\end{aligned}$$

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence.

Proof. (a) Follows from the definition of partial sum of terms of the Modified Mersenne–Leonardo numbers, and making use of the Binet formula for Modified Mersenne–Leonardo sequence, the equation (3.5), we get

$$\begin{aligned}
\sum_{k=0}^{n} \boldsymbol{M\!L}_k &= \boldsymbol{M\!L}_0 + \boldsymbol{M\!L}_1 + \cdots + \boldsymbol{M\!L}_n\\
&= (2-2) + (2^2-3) + (2^3-4) + \cdots + \big(2^{n+1} - (n+2)\big)\\
&= (1+2+2^2+\cdots+2^{n+1}) - \big(1+2+3+\cdots+(n+2)\big)\\
&= 2^{n+2} - 1 - \frac{(n+2)(n+3)}{2} \ ,
\end{aligned}$$

and we have the result required.

(b) See that

$$\begin{aligned}
\sum_{k=0}^{n} \boldsymbol{M\!L}_{2k} &= \boldsymbol{M\!L}_0 + \boldsymbol{M\!L}_2 + +\boldsymbol{M\!L}_4 + \cdots + \boldsymbol{M\!L}_{2n}\\
&= (2-2) + (2^3-4) + (2^5-6) + \cdots + \big(2^{2n+1} - (2n+2)\big)\\
&= 2(1+2^2+2^4 \cdots + 2^{2n}) - 2\big(1+2+3+\cdots+(n+1)\big)\\
&= 2\frac{(2^2)^{n+1}-1}{2^2-1} - (n+1)(n+2) \ ,
\end{aligned}$$

as required.

(c) Similarly, we have

$$\begin{aligned}
\sum_{k=0}^{n} \boldsymbol{M\!L}_{2k+1} &= \boldsymbol{M\!L}_1 + \boldsymbol{M\!L}_3 + \cdots + \boldsymbol{M\!L}_{2n+1}\\
&= (2^2-3) + (2^4-5) + (2^6-7) + \cdots + \big(2^{2n+2} - (2n+3)\big) \ ,
\end{aligned}$$

making using of (b), which verifies the result. □

Remark 3.23. *Firstly, see that $\frac{(n+2)(n+3)}{2}$ is always integer, if n is even then $n+2$ is even; otherwise, $n+3$ is even. Now, it can be demonstrated that $3 = 2^2 - 1$ divides $M_{2n+1} = (2^2)^{n+1} - 1$ since that the condition $a-b$ divides $a^k - b^k$ for all integers a, b and k non-negative is satisfied.*

A direct consequence of the Proposition 3.22 is the next result.

Proposition 3.24. *Let be $\{\mathcal{M}_n\}_{n\geq 0}$ is the Modified Mersenne–Leonardo sequence and $\{M_n\}_{n\geq 0}$ is the Mersenne sequence. For $n \geq 0$, the following identities hold:*

$$\sum_{j=0}^{n}(-1)^k\mathcal{M}_k = (n+1) - \frac{6}{8}M_{2n+1};$$

if last term is negative, and

$$\sum_{j=0}^{n}(-1)^k\mathcal{M}_k = \frac{2}{3}(M_{2n+3} - 3M_{2n+1}) + (n+1);$$

if last term is positive.

Proof. (a) First, note that the last term is negative, which results in the following considerations:

$$\begin{aligned}\sum_{k=0}^{2n+1}(-1)^k\mathcal{M}_k &= \mathcal{M}_0 - \mathcal{M}_1 + \mathcal{M}_2 - \mathcal{M}_3 + \cdots + \mathcal{M}_{2n} - \mathcal{M}_{2n+1}\\ &= (\mathcal{M}_0 + \mathcal{M}_2 + \cdots + \mathcal{M}_{2n}) - (\mathcal{M}_1 + \mathcal{M}_3 + \cdots + \mathcal{M}_{2n+1})\\ &= \sum_{k=0}^{n}\mathcal{M}_{2k} - \sum_{k=0}^{n}\mathcal{M}_{2k+1}\ .\end{aligned}$$

In accordance with Proposition 3.22, items (b) and (c), the result can be deduced.

(b) In which case that last term is positive, so

$$\begin{aligned}\sum_{k=0}^{2(n+1)}(-1)^k\mathcal{M}_k &= \mathcal{M}_0 - \mathcal{M}_1 + \mathcal{M}_2 - \mathcal{M}_3 + \cdots + \mathcal{M}_{2n} - \mathcal{M}_{2n+1} + \mathcal{M}_{2n+2}\\ &= \sum_{k=0}^{n+1}\mathcal{M}_{2k} - \sum_{k=0}^{n}\mathcal{M}_{(2k+1)}.\end{aligned}$$

Similarly, as in item (a), the result can be obtained by applying Proposition 3.22. □

Finally, in the context of sequences, the difference operator, denoted by Δ, is defined as $\Delta a_n = a_n - a_{n-1}$, where $\{a_n\}_{n\geq 0}$ is a sequence.

Making $S_n = \sum_{k=0}^{n}\mathcal{M}_k$ for all integer $n \geq 0$, consider the sequence $\{S_n\}_{n\geq 0}$, where $\{\mathcal{M}_n\}_{n\geq 0}$ is the Modified Mersenne–Leonardo sequence.

Proposition 3.25. *Let be $\{\mathcal{M}_n\}_{n\geq 0}$ be the Modified Mersenne–Leonardo sequence and $S_n = \sum_{k=0}^{n}\mathcal{M}_k$. For all integer $n \geq 1$, the following identities hold:*

(a) $\Delta S_n = \mathcal{M}_n$,
(b) $\Delta^2 S_n = M_{n-1}$;
(c) $\Delta^3 S_n = 2^{n-1}$.

where $\{M_n\}_{n\geq 0}$ is the Mersenne sequence.

Proof. (a) Using the Proposition 3.22, item (a), we have

$$\begin{aligned}\Delta S_n &= S_n - S_{n-1}\\ &= \left(M_{n+2} - \frac{(n+2)(n+3)}{2}\right) - \left(M_{n+1} - \frac{(n+1)(n+2)}{2}\right)\\ &= 2^{n+1} - (n+2)\ ,\end{aligned}$$

which verifies the result.

(b) By combining Proposition 3.3 and item (a).

(c) A straight calculation. □

4. Final Considerations

In this paper, we introduced the Modified Mersenne-Leonardo numbers and studied their properties. The aim of this work was to define the Modified Mersenne-Leonardo sequence as an extension of the Mersenne sequence and to examine some of its properties, particularly the recurrence relation, summation formula, and generating function.

We hope that this study will serve as motivation for further research, enabling a deeper exploration of the properties and applications of these sequences. We believe that they can be extended to the sets of complex numbers, quaternions, and hybrid numbers.

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