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A Study on Fourth-Order Coupled Boundary Value Problems: Existence, Uniqueness and Approximations

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Abstract

This study examines the existence and approximation of solutions for a coupled system of fourth-order boundary value problems (4th-BVPs), which model the interactions between two distinct but interrelated physical systems. These coupled boundary value problems arise in various applications in engineering and physics, including the analysis of bending behaviors in beams and vibrations in interconnected structural components. By leveraging Green's functions and building upon prior research in fourth-order differential equations, we derive sufficient conditions for the existence and uniqueness of solutions to the system. Additionally, we provide a numerical framework for approximating these solutions, offering practical insights for real-world applications.

1. Introduction

4th-BVPs play a crucial role in many engineering and physics applications, such as analyzing the bending behavior of elastic beams, the stability of mechanical systems, fluid dynamics, biomechanical processes, and vibration models. These problems enable the mathematical modeling and analysis of system behaviors by incorporating higher-order derivatives, which are essential for capturing complex physical phenomena.

In real-world scenarios, physical systems rarely function in isolation; they often involve intricate interactions among multiple structural elements, variables, or external forces. Modeling and analyzing such systems, particularly those with multiple degrees of freedom or coupled dynamics, necessitate the formulation of systems of interdependent differential equations. These systems provide a robust framework for understanding how the behavior of one component affects the entire system.

In this study, we focus on a coupled system of two fourth-order differential equations that represent two distinct yet interrelated physical systems. The coupled system is described as follows:

$$\left.\begin{array}{c}\varphi_1''''(x)+\beta_1^2\varphi_1''(x)=\Gamma_1(x,\varphi_1(x),\varphi_2(x))\\ \varphi_2''''(x)+\beta_2^2\varphi_2''(x)=\Gamma_2(x,\varphi_1(x),\varphi_2(x))\\ \varphi_1(0)=\varphi_2(0)=\varphi_1(L)=\varphi_2(L)=0\\ \varphi_1'(0)=\varphi_2'(0)=\varphi_1'(L)=\varphi_2'(L)=0\end{array}\quad, \ x\in[0,L]\right\} \tag{1.1}$$

Here, φ_1 and φ_2 represent the solutions corresponding to two distinct yet interacting physical systems, such as the bending behaviors of two beams or the vibrations of two structural components. The functions Γ_1 and Γ_2 model the mutual interactions between the two systems.

Such coupled systems are particularly significant in engineering disciplines, where the analysis of interconnected structures is critical. They provide insight into how individual components influence the overall system behavior, enabling more effective designs and analyses of complex structures.

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The motivation for this study stems from the need to model and analyze singular systems frequently encountered in engineering and physics. Such systems are characterized by interdependent components, often described by a complex network of equations due to their inherent interactions. For instance, in structural mechanics, beam systems or load-bearing elements interact with one another in ways that cannot be adequately captured by isolated models. To address these challenges, systems of coupled equations, such as those considered here, are essential for understanding the interplay between different components. The analysis and solutions of such systems are of paramount importance for designing and optimizing physical systems.

4th-BVPs, in particular, pose significant challenges due to their nonlinearity and complex boundary conditions. These difficulties make the investigation of existence, uniqueness, and approximation of solutions critical. The importance of such analyses is underscored by their wide-ranging applications in engineering and physics, where understanding system behaviors requires accurate mathematical modeling and solution methodologies.

Previous studies have substantially advanced the understanding of 4th-BVPs. For example, Agarwal [1] explored the existence and uniqueness of solutions to 4th-BVPs in the context of elastic beam bending. Kaufmann and Kosmatov [2] and Habib [3] extended this work to other applications. More recently, Almuthaybiri and Tisdell [4] established stricter conditions for the existence and uniqueness of solutions, while Chen and Cui [5] investigated the continuity of derivatives for solutions to 4th-BVPs.

Despite this progress, studies addressing coupled systems of dependent differential equations remain relatively rare. Interest in this area has grown in recent years, as seen in the work of Zhai and Anderson [6], who established existence and uniqueness results for doubly dependent differential equation systems. Granas and Guenther [7] contributed analytical techniques for solving more general systems of this type.

The objective of this work is to analyze the coupled system of 4th-BVPs defined by (1.1), focusing on the conditions for the existence and uniqueness of solutions. Additionally, we aim to develop iterative methods for approximating solutions when they exist, providing a comprehensive understanding of the system and its numerical treatment.

In a related study, Rao and Jagan [8] investigated the following boundary value problem (BVP):

$$\left.\begin{array}{c}\varphi''''(x)+\beta^2\varphi''(x)=\Gamma(x,\varphi(x))\\ \varphi(0)=\varphi'(0)=\varphi'(L)=\varphi(L)=0\end{array}\quad, x\in[0,L]\right\} \tag{1.2}$$

Using Green's method, they demonstrated the existence of a solution for this equation, thereby contributing to the growing body of work on 4th-BVPs.

Proposition 1.1. *(see [8]) Let $\Gamma(x,\varphi(x))$ be a continuous function on $[0,L]\times\mathbb{R}$ and Lipschitz with a Lipschitz constant K with respect to the second variable. Assume that $\omega=2-\beta L\sin(\beta L)-2\cos(\beta L)\neq 0$, $\Gamma(x,0)\neq 0$, and*

$$M<\frac{1}{K}$$

where $M=\frac{L^3}{6}k_1+\frac{L^4}{24}(1+k_2)$ with

$$k_1=\left\|\frac{(\sin(\beta t)-\beta t))(1-\cos(\beta L)+(1-\cos(\beta t))(\beta L-\sin(\beta L))}{\beta\omega}\right\|_\infty,$$

and

$$k_2=\left\|\frac{(\cos(\beta t)-1))(1-\cos(\beta L)+(\beta t-\sin(\beta t))\sin(\beta L)}{\omega}\right\|_\infty.$$

Then the equation (1.2) has a unique solution, and

$$\int_0^L|G(x,t)|dt\leq M.$$

is satisfied, where the Green's function associated with (1.2) is defined as follows

$$G(x,\xi)=\begin{cases}G_1(x,\xi), & 0\leq\xi\leq x\leq L,\\ G_2(x,\xi), & 0\leq x\leq\xi\leq L.\end{cases}$$

where

$$
\begin{aligned}
K(x,\xi) &= \frac{1}{\beta^3}\left[\beta(x-\xi)-\sin\beta(x-\xi)\right],\\
K_x(x,\xi) &= \frac{1}{\beta^2}\left[1-\cos\beta(x-\xi)\right],\\
G_1(x,\xi) &= \frac{K_x(L,\xi)\left[(\beta L-\sin\beta L)(1-\cos\beta x)\right]}{\beta(2-2\cos\beta L-\beta L\sin\beta L)}+\frac{K_x(L,\xi)\left[(1-\cos\beta L)(\sin\beta x-\beta x)\right]}{\beta(2-2\cos\beta L-\beta L\sin\beta L)}\\
&+ \frac{K(L,\xi)\left[\sin\beta L(\beta x-\sin\beta x)\right]}{(2-2\cos\beta L-\beta L\sin\beta L)}+\frac{K(L,\xi)\left[(1-\cos\beta L)(\cos\beta x-1)\right]}{(2-2\cos\beta L-\beta L\sin\beta L)}+K(x,\xi),\\
G_2(x,\xi) &= \frac{K_x(L,\xi)\left[(\beta L-\sin\beta L)(1-\cos\beta x)\right]}{\beta(2-2\cos\beta L-\beta L\sin\beta L)}+\frac{K_x(L,\xi)\left[(1-\cos\beta L)(\sin\beta x-\beta x)\right]}{\beta(2-2\cos\beta L-\beta L\sin\beta L)}\\
&+ \frac{K(L,\xi)\left[\sin\beta L(\beta x-\sin\beta x)\right]}{(2-2\cos\beta L-\beta L\sin\beta L)}+\frac{K(L,\xi)\left[(1-\cos\beta L)(\cos\beta x-1)\right]}{(2-2\cos\beta L-\beta L\sin\beta L)}.
\end{aligned}
$$

From now on, let X denote the space of all functions that are four times differentiable, $C^{(4)}[0,L]$ where the norm $||\varphi||_\infty$ on X is the supremum norm. Additionally, the norm $||(\varphi_1,\varphi_2)||$ on X^2 is defined by $||(\varphi_1,\varphi_2)||=||\varphi_1||_\infty+||\varphi_2||_\infty$.

2. Main Results

Building on Proposition 1, we present our first result concerning the existence of solutions and their approximation for the 4th-BVPs system (1.1) in the following theorem.

Theorem 2.1. *If*

$$
\begin{aligned}
\|\Gamma_i(x,\varphi_1(x),\varphi_2(x))-\Gamma_i(x,\widetilde{\varphi}_1(x),\varphi_2(x)\|_\infty &\le K_i||\varphi_1(x)-\widetilde{\varphi}_1(x)||_\infty\\
||\Gamma_i(x,\varphi_1(x),\varphi_2(x))-\Gamma_i(x,\varphi_1(x),\widetilde{\varphi}_2(x))||_\infty &\le L_i||\varphi_2(x)-\widetilde{\varphi}_2(x)||_\infty
\end{aligned}
$$

for $i=1,2$, $\Gamma_1(x,0,\varphi_2(x))\neq 0$, $\Gamma_2(x,\varphi_1(x),0)\neq 0$, *and*

$$\theta=\max\{K_1+K_2,L_1+L_2\}M<1$$

where $M=\max\{M_1,M_2\}$, *and* M_1,M_2 *are given as in Proposition 1.1 for the first and second equations, respectively, then the system (1.1) has a solution which is unique. Furthermore, the iteration* $\{(\varphi_{1,n},\varphi_{2,n})\}_{n\ge 0}$ *defined by*

$$
\begin{aligned}
\varphi_{1,n+1}(x) &= \int_0^L G(x,t)\Gamma_1(t,\varphi_{1,n}(t),\varphi_{2,n}(t))dt \qquad (2.1)\\
\varphi_{2,n+1}(x) &= \int_0^L G(x,t)\Gamma_2(t,\varphi_{1,n}(t),\varphi_{2,n}(t))dt
\end{aligned}
$$

where $(\varphi_{1,0},\varphi_{2,0})\in X^2$, *is convergent to the solution.*

Proof. Let $T(\varphi_1,\varphi_2)=\left(\int_0^L G(x,t)\Gamma_1(t,\varphi_1(t),\varphi_2(t))dt,\int_0^L G(x,t)\Gamma_2(t,\varphi_1(t),\varphi_2(t))dt\right)$. Since we have

$$
\begin{aligned}
\|T(\varphi_1,\varphi_2)-T(\widetilde{\varphi}_1,\widetilde{\varphi}_2)\| &= \left\|\int_0^L G(x,t)\Gamma_1(t,\varphi_1(t),\varphi_2(t))dt-\int_0^L G(x,t)\Gamma_1(t,\widetilde{\varphi}_1(t),\widetilde{\varphi}_2(t))dt\right\|_\infty\\
&+\left\|\int_0^L G(x,t)\Gamma_2(t,\varphi_1(t),\varphi_2(t))dt-\int_0^L G(x,t)\Gamma_2(t,\widetilde{\varphi}_1(t),\widetilde{\varphi}_2(t))dt\right\|_\infty\\
&\le M_1\left\|\Gamma_1(x,\varphi_1(x),\varphi_2(x))-\Gamma_1(x,\widetilde{\varphi}_1(x),\widetilde{\varphi}_2(x))\right\|_\infty+M_2\left\|\Gamma_2(x,\varphi_1(x),\varphi_2(x))-\Gamma_2(x,\widetilde{\varphi}_1(x),\widetilde{\varphi}_2(x))\right\|_\infty\\
&\le K_1M||\varphi_1-\widetilde{\varphi}_1||_\infty+L_1M||\varphi_2-\widetilde{\varphi}_2||_\infty+K_2M||\varphi_1-\widetilde{\varphi}_1||_\infty+L_2M||\varphi_2-\widetilde{\varphi}_2||_\infty\\
&\le M\max\{K_1+K_1,L_1+L_2\}||(\varphi_1,\varphi_2)-(\widetilde{\varphi}_1,\widetilde{\varphi}_2)||\\
&= \theta||(\varphi_1,\varphi_2)-(\widetilde{\varphi}_1,\widetilde{\varphi}_2)||,
\end{aligned}
$$

for any $\varphi_1,\varphi_2,\widetilde{\varphi}_1,\widetilde{\varphi}_2 \in X$, T is contraction and by Banach contraction principle, T has a unique fixed point which is also the solution of (1.1) . Let $(\varphi_{1,p},\varphi_{2,p})$ be the fixed point of T. Then, we have

$$\begin{aligned}
\left\|(\varphi_{1,n+1},\varphi_{2,n+1})-(\varphi_{1,p},\varphi_{2,p})\right\| &= \left\|T(\varphi_{1,n},\varphi_{2,n})-T(\varphi_{1,p},\varphi_{2,p})\right\| \\
&\le \theta||(\varphi_{1,n},\varphi_{2,n})-(\varphi_{1,p},\varphi_{2,p})|| \\
&\le \theta^2||(\varphi_{1,n-1},\varphi_{2,n-1})-(\varphi_{1,p},\varphi_{2,p})|| \\
&\quad \ldots \\
&\le \theta^{n+1}||(\varphi_{1,0},\varphi_{2,0})-(\varphi_{1,p},\varphi_{2,p})||.
\end{aligned}$$

Since $\theta<1$, we conclude that $\lim_{n\to\infty}||(\varphi_{1,n+1},\varphi_{2,n+1})-(\varphi_{1,p},\varphi_{2,p})||=0$. □

Example 2.2. *Let $X=C^{(4)}[0,1]$ and consider the following system of BVPs*

$$\left.\begin{array}{c}
\varphi_1''''(x)+2^2\varphi_1''(x)=2\varphi_1(x)-\frac{2}{3}\varphi_2(x)+1 \\
\varphi_2''''(x)+3^2\varphi_2''(x)=\frac{6}{5}\varphi_1(x)-4\varphi_2(x)+1 \\
\varphi_1(0)=\varphi_2(0)=\varphi_1(1)=\varphi_2(1)=0 \\
\varphi_1'(0)=\varphi_2'(0)=\varphi_1'(1)=\varphi_2'(1)=0
\end{array} \;,\; x\in[0,1]\right\} \tag{2.2}$$

Then $M_1=1.104e-01$ and $M_2=6.985e-02$. Since $\Gamma_1(x,\varphi_1(x),\varphi_2(x))=2\varphi_1(x)-\frac{2}{3}\varphi_2(x)+1$ and $\Gamma_2(x,\varphi_1(x),\varphi_2(x))=\frac{6}{5}\varphi_1(x)-4\varphi_2(x)+1$,it is also satisfied that

$$\begin{aligned}
||\Gamma_1(x,\varphi_1(x),\varphi_2(x))-\Gamma_1(x,\widetilde{\varphi}_1(x),\varphi_2(x))||_\infty &= \left\|2\varphi_1(x)-\frac{2}{3}\varphi_2(x)+1-(2\widetilde{\varphi}_1(x)-\frac{2}{3}\varphi_2(x)+1)\right\|_\infty \\
&\le 2\left\|\varphi_1(x)-\widetilde{\varphi}_1(x)\right\|_\infty,\ K_1=2, \\
||\Gamma_2(x,\varphi_1(x),\varphi_2(x))-\Gamma_2(x,\widetilde{\varphi}_1(x),\varphi_2(x))||_\infty &= \left\|\frac{6}{5}\varphi_1(x)-4\varphi_2(x)+1-\left(\frac{6}{5}\widetilde{\varphi}_1(x)-4\varphi_2(x)+1\right)\right\|_\infty \\
&\le \frac{6}{5}||\varphi_1(x)-\widetilde{\varphi}_1(x)||_\infty,\ K_2=\frac{6}{5}, \\
||\Gamma_1(x,\varphi_1(x),\varphi_2(x))-\Gamma_1(x,\varphi_1(x),\widetilde{\varphi}_2(x))||_\infty &= \left\|2\varphi_1(x)-\frac{2}{3}\varphi_2(x)+1-(2\varphi_1(x)-\frac{2}{3}\widetilde{\varphi}_2(x)+1)\right\|_\infty \\
&\le \frac{2}{3}||\varphi_2(x)-\widetilde{\varphi}_2(x)||_\infty,\ L_1=\frac{2}{3}, \\
||\Gamma_2(x,\varphi_1(x),\varphi_2(x))-\Gamma_2(x,\varphi_1(x),\widetilde{\varphi}_2(x))||_\infty &= \left\|\frac{6}{5}\varphi_1(x)-4\varphi_2(x)+1-\left(\frac{6}{5}\varphi_1(x)-4\widetilde{\varphi}_2(x)+1\right)\right\|_\infty \\
&\le 4||\varphi_2(x)-\widetilde{\varphi}_2(x)||_\infty,\ L_2=4,
\end{aligned}$$

for all $\varphi_1,\varphi_2,\widetilde{\varphi}_1,\widetilde{\varphi}_2\in X$. Obviously, since $K_1M_1<1$ and $K_2M_2<1$, by Proposition 1.1,

$$\varphi_1(x)=\int_0^1 G(x,t)\Gamma_1(t,\varphi_1(t),\varphi_2(t))dt$$

has a solution for fixed $\varphi_2\in X$ and

$$\varphi_2(x)=\int_0^1 G(x,t)\Gamma_2(t,\varphi_1(t),\varphi_2(t))dt$$

has a solution for fixed $\varphi_1\in X$. Let

$$\begin{aligned}
T(\varphi_1,\varphi_2) &= \left(\int_0^1 G(x,t)\Gamma_1(t,\varphi_1(t),\varphi_2(t))dt,\int_0^1 G(x,t)\Gamma_2(t,\varphi_1(t),\varphi_2(t))dt\right) \\
&= \left(\int_0^1 G(x,t)\left(\frac{4}{7}\varphi_1(t)+\frac{1}{4}\varphi_2(t)+1\right)dt,\int_0^1 G(x,t)\left(\frac{2}{3}\varphi_1(t)-\frac{1}{2}\varphi_2(t)+1\right)dt\right).
\end{aligned}$$

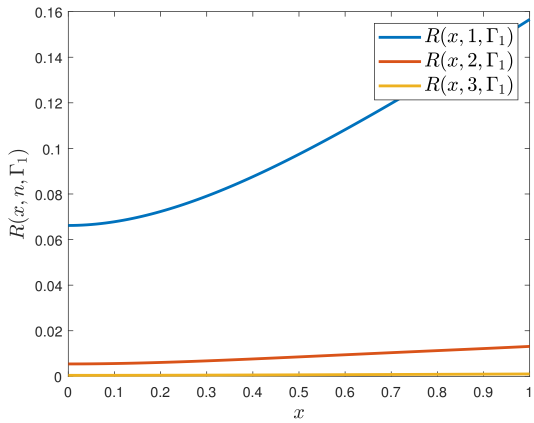
Since

$$
\begin{aligned}
\theta &= \max\{K_1+K_2, L_1+L_2\}M \\
&= \max\left\{2+\frac{6}{5}, \frac{2}{3}+4\right\} 1.104e-01 \\
&= 5.155e-01 < 1,
\end{aligned}
$$

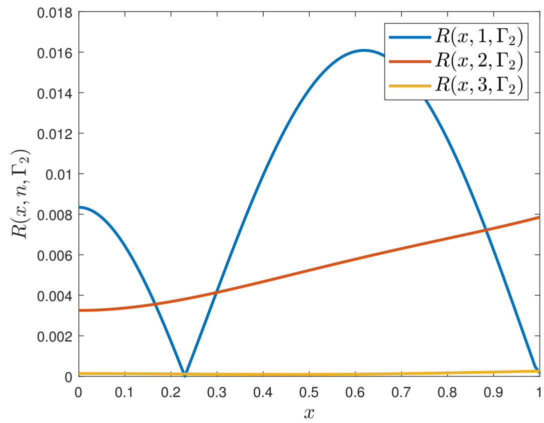
the system (2.2) has the solution by Theorem 2.1. In addition, the iteration $\{(\varphi_{1,n}(x), \varphi_{2,n}(x))\}_{n\geq 0}$ *defined by*

$$
\begin{aligned}
\varphi_{1,n}(x) &= \int_0^1 G(x,t)\left(\frac{4}{7}\varphi_{1,n}(t)+\frac{1}{4}\varphi_{2,n}(t)+1\right)dt \\
\varphi_{2,n}(x) &= \int_0^1 G(x,t)\left(\frac{2}{3}\varphi_{1,n}(t)-\frac{1}{2}\varphi_{2,n}(t)+1\right)dt
\end{aligned}
\tag{2.3}
$$

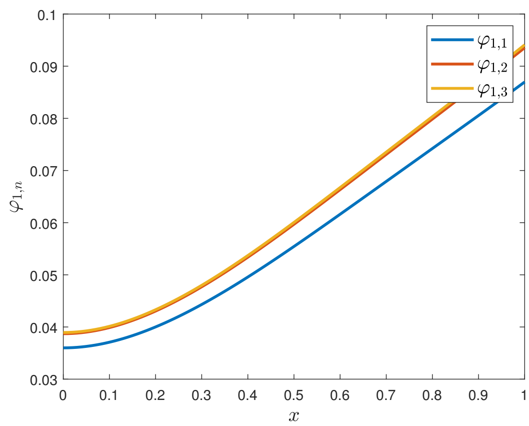
is convergent to the solution staring with $(\varphi_{1,0}, \varphi_{2,0}) = (x,x)$*. Let* $R(x,n,\Gamma_i) = |\varphi''''_{i,n}(x)+\beta^2\varphi''_{i,n}(x)-\Gamma_i(x,\varphi_{1,n}(x),\varphi_{2,n}(x))|$ *be the residual error for* $i=1,2$ *and* $n>0$*. The Residual errors for* $n=1,2,3$ *are shown in Figure 1 and Table 1.*



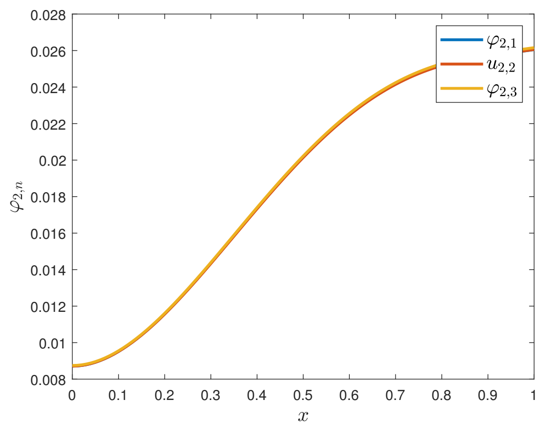
(a) $R(x,n,\Gamma_1)$ for $n=1,2,$ and 3



(b) $R(x,n,\Gamma_2)$ for $n=1,2,$ and 3



(c) $\varphi_{1,n}$ for $n=1,2,$ and 3



(d) $\varphi_{2,n}$ for $n=1,2,$ and 3

Figure 1

	n=1		n=2		n=3	
	$R(x,1,\Gamma_1)$	$R(x,1,\Gamma_2)$	$R(x,2,\Gamma_1)$	$R(x,2,\Gamma_2)$	$R(x,3,\Gamma_1)$	$R(x,3,\Gamma_2)$
0	6,619E-02	8,347E-03	5,435E-03	3,253E-03	4,366E-04	1,394E-04
0.1	6,782E-02	6,436E-03	5,613E-03	3,372E-03	4,492E-04	1,343E-04
0.2	7,231E-02	1,722E-03	6,091E-03	3,689E-03	4,834E-04	1,222E-04
0.3	7,906E-02	4,229E-03	6,787E-03	4,143E-03	5,339E-04	1,088E-04
0.4	8,759E-02	9,935E-03	7,622E-03	4,674E-03	5,957E-04	9,953E-05
0.5	9,743E-02	1,416E-02	8,530E-03	5,233E-03	6,646E-04	9,895E-05
0.6	1,082E-01	1,604E-02	9,461E-03	5,783E-03	7,373E-04	1,102E-04
0.7	1,197E-01	1,518E-02	1,038E-02	6,305E-03	8,115E-04	1,342E-04
0.8	1,316E-01	1,169E-02	1,128E-02	6,805E-03	8,862E-04	1,697E-04
0.9	1,439E-01	6,216E-03	1,219E-02	7,306E-03	9,623E-04	2,131E-04
1.0	1,566E-01	1,674E-04	1,313E-02	7,856E-03	1,042E-03	2,587E-04

Table 1: Residual errors for $n = 1,2,$ and 3

Theorem 2.3. *Let Γ_i for $i = 1,2$ and θ be as in Theorem 2.1 and assume that there exist $\widetilde{\Gamma}_i(x,\varphi_1(x),\varphi_2(x))$ functions on $[0,L]\times X^2$ such that*

$$||\Gamma_i(x,\varphi_1(x),\varphi_2(x)) - \widetilde{\Gamma}_i(x,\varphi_1(x),\varphi_2(x))||_\infty \le \xi_i$$

for $i = 1,2$, and the following system

$$\left.\begin{array}{r}\varphi_1''''(x)+\beta^2\varphi_1''(x) = \widetilde{\Gamma}_1(x,\varphi_1(x),\varphi_2(x)) \\ \varphi_2''''(x)+\beta_1^2\varphi_2''(x) = \widetilde{\Gamma}_2(x,\varphi_1(x),\varphi_2(x)) \\ \varphi_1(0) = \varphi_2(0) = \varphi_1(L) = \varphi_2(L) = 0 \\ \varphi_1'(0) = \varphi_2'(0) = \varphi_1'(L) = \varphi_2'(L) = 0 \end{array} \quad , \ x\in[0,L] \right\} \tag{2.4}$$

has a solution. Then

$$||(\varphi_{1,p},\varphi_{2,p}) - (\widetilde{\varphi}_{1,p},\widetilde{\varphi}_{2,p})|| \le M\frac{\xi_1+\xi_2}{1-\theta}$$

holds for $(\varphi_{1,p},\varphi_{2,p})$ and $(\widetilde{\varphi}_{1,p},\widetilde{\varphi}_{2,p})$, where $(\varphi_{1,p},\varphi_{2,p})$ and $(\widetilde{\varphi}_{1,p},\widetilde{\varphi}_{2,p})$ are the the solutions of systems (1.1) and (2.4),respectively, and $M = \max\{M_1,M_2\}$, and M_1,M_2 are given as in Propositon 1.1 for first and second equation, respectively.

Proof. Let

$$T(\varphi_1,\varphi_2) = \left(\int_0^L G(x,t)\Gamma_1(t,\varphi_1(t),\varphi_2(t))dt, \int_0^L G(x,t)\Gamma_2(t,\varphi_1(t),\varphi_2(t))dt\right)$$

and

$$S(\varphi_1,\varphi_2) = \left(\int_0^L G(x,t)\widetilde{\Gamma}_1(t,\varphi_1(t),\varphi_2(t))dt, \int_0^L G(x,t)\widetilde{\Gamma}_2(t,\varphi_1(t),\varphi_2(t))dt\right).$$

Then, by Theorem 2.1, T has a fixed point $(\varphi_{1,p},\varphi_{2,p})$ which is the unique solution of system (1.1). Let $(\varphi_{1,0},\varphi_{2,0}) = (\widetilde{\varphi}_{1,p},\widetilde{\varphi}_{2,p})$ be a fixed point of S, which is also a solution of system (2.4), and define $(\varphi_{1,n+1},\varphi_{2,n+1}) = T(\varphi_{1,n},\varphi_{2,n})$. Then, $\{(\varphi_{1,n+1},\varphi_{2,n+1})\}_{n\ge 0}$ converges to $(\varphi_{1,p},\varphi_{2,p})$ by Theorem 2.1. Since

$$\begin{aligned} ||(\varphi_{1,n+1},\varphi_{2,n+1}) - (\varphi_{1,n},\varphi_{2,n})|| &= ||T(\varphi_{1,n},\varphi_{2,n}) - T(\varphi_{1,n-1},\varphi_{2,n-1})|| \\ &\le \theta||(\varphi_{1,n},\varphi_{2,n}) - (\varphi_{1,n-1},\varphi_{2,n-1})|| \\ &\quad \dots \\ &\le \theta^n||(\varphi_{1,1},\varphi_{2,1}) - (\varphi_{1,0},\varphi_{2,0})||, \end{aligned}$$

then

$$
\begin{aligned}
||(\varphi_{1,n},\varphi_{2,n})-(\varphi_{1,0},\varphi_{2,0})|| &\leq \sum_{k=1}^{n}||(\varphi_{1,k},\varphi_{2,k})-(\varphi_{1,k-1},\varphi_{2,k-1})|| \\
&\leq \sum_{k=1}^{n}\theta^{k-1}||(\varphi_{1,1},\varphi_{2,1})-(\varphi_{1,0},\varphi_{2,0})|| \\
&\leq \frac{1}{1-\theta}||(\varphi_{1,1},\varphi_{2,1})-(\varphi_{1,0},\varphi_{2,0})|| \\
&= \frac{1}{1-\theta}||T(\varphi_{1,0},\varphi_{2,0})-S(\varphi_{1,0},\varphi_{2,0})|| \\
&\leq \frac{1}{1-\theta}\left(\begin{array}{c}\left\| \int_0^L G(x,t)\Gamma_1(t,\varphi_{1,0}(t),\varphi_{2,0}(t))dt-\int_0^L G(x,t)\widetilde{\Gamma}_1(t,\varphi_{1,0}(t),\varphi_{2,0}(t))dt \right\|_\infty \\ +\left\| \int_0^L G(x,t)\Gamma_2(t,\varphi_{1,0}(t),\varphi_{2,0}(t))dt-\int_0^L G(x,t)\widetilde{\Gamma}_2(t,\varphi_{1,0}(t),\varphi_{2,0}(t))dt \right\|_\infty\end{array}\right) \\
&\leq \frac{1}{1-\theta}\left(\begin{array}{c}\left\|\int_0^L G(x,t)dt\right\|_\infty \left\| \Gamma_1(x,\varphi_{1,0}(x),\varphi_{2,0}(x))-\widetilde{\Gamma}_1(x,\varphi_{1,0}(x),\varphi_{2,0}(x)) \right\|_\infty \\ +\left\|\int_0^L G(x,t)dt\right\|_\infty \left\| \Gamma_2(x,\varphi_{1,0}(x),\varphi_{2,0}(x))-\widetilde{\Gamma}_2(x,\varphi_{1,0}(x),\varphi_{2,0}(x)) \right\|_\infty\end{array}\right) \\
&\leq M\frac{\xi_1+\xi_2}{1-\theta}
\end{aligned}
$$

which implies that

$$||(\varphi_{1,p},\varphi_{2,p})-(\widetilde{\varphi}_{1,p},\widetilde{\varphi}_{2,p})|| \leq M\frac{\xi_1+\xi_2}{1-\theta}.$$

□

Example 2.4. *Consider the following system of BVP*

$$
\left.\begin{array}{c}
\varphi_1''''(x)+2^2\varphi_1''(x)=2\varphi_1^{\frac{9}{10}}(x)-\frac{2}{3}\varphi_2^{\frac{4}{3}}(x)+\frac{x+9}{x+10} \\
\varphi_2''''(x)+3^2\varphi_2''(x)=\frac{6}{5}\varphi_1^{\cos(\varphi_1(x))}(x)-4\varphi_2^{\frac{\sin(\varphi_2(x))}{\varphi_2(x)}}(x)+1 \\
\varphi_1(0)=\varphi_2(0)=\varphi_1(1)=\varphi_2(1)=0 \\
\varphi_1'(0)=\varphi_2'(0)=\varphi_1'(1)=\varphi_2'(1)=0
\end{array}\right\} ,\ x\in[0,1] \tag{2.5}
$$

Solving the this system of BVP directly is highly challenging or even infeasible due to the nonlinear functions involved in. However, thanks to Theorem 2.3, approximate solutions close to the exact one can be obtained without directly solving the equation.

Let X, Γ_1, Γ_2, β, and β_2 be as defined in Example 2.2. Additionally, let $\overline{X}=\{\varphi\in X: 0\leq\varphi(x)\leq 1\}$. It can be observed from Figure 1 that the solution of system (2.2) belongs to $\overline{X}\times\overline{X}$. Then the functions $\widetilde{\Gamma}_1(x,\varphi_1(x),\varphi_2(x))=2\varphi_1^{\frac{9}{10}}(x)-\frac{2}{3}\varphi_2^{\frac{4}{3}}(x)+\frac{x+9}{x+10}$ and $\widetilde{\Gamma}_2(x,\varphi_1(x),\varphi_2(x))=\frac{6}{5}\varphi_1^{\cos(\varphi_1(x))}-4\varphi_2^{\frac{\sin(\varphi_2(x))}{\varphi_2(x)}}+1$ satisfy the following

$$
\begin{aligned}
||\Gamma_1(x,\varphi_1(x),\varphi_2(x))-\widetilde{\Gamma}_1(x,\varphi_1(x),\varphi_2(x))||_\infty &= \left\| 2\varphi_1(x)-\tfrac{2}{3}\varphi_2(x)+1-\left(2\varphi_1^{\frac{9}{10}}(x)-\tfrac{2}{3}\varphi_2^{\frac{4}{3}}(x)+\frac{x+9}{x+10}\right)\right\|_\infty \\
&\leq \left\|2\varphi_1(x)-2\varphi_1(x)^{\frac{9}{10}}\right\|_\infty+\left\|\frac{2}{3}\varphi_2(x)-\frac{2}{3}\varphi_1^{\frac{4}{3}}(x)\right\|_\infty+\left\|1-\frac{x+9}{x+10}\right\|_\infty \\
&\leq 2.43e-01=\xi_1, \\
||\Gamma_2(x,\varphi_1(x),\varphi_2(x))-\widetilde{\Gamma}_2(x,\varphi_1(x),\varphi_2(x))||_\infty &= \left\| \tfrac{6}{5}\varphi_1(x)-4\varphi_2(x)+1-\left(\tfrac{6}{5}\varphi_1^{\cos(\varphi_1(x))}(x)-4\varphi_2^{\frac{\sin(\varphi_2(x))}{\varphi_2(x)}}(x)+1\right)\right\|_\infty \\
&\leq \left\|\frac{6}{5}\varphi_1(x)-\frac{6}{5}\varphi_1^{\cos(\varphi_1(x))}(x)\right\|_\infty+\left\|4\varphi_2(x)-4\varphi_2^{\frac{\sin(\varphi_2(x))}{\varphi_2(x)}}(x)\right\|_\infty \\
&\leq 1.52e-01=\xi_2,
\end{aligned}
$$

for all $\varphi_1, \varphi_2 \in X$.Since $M_1 = 1.104e-01$ and $M_2 = 6.985e-02$, we have $M = \max\{M_1, M_2\} = 1.104e-01$. Then, by Theorem 2.3, we have:

$$\begin{aligned} ||(\varphi_{1,p}, \varphi_{2,p}) - (\widetilde{\varphi}_{1,p}, \widetilde{\varphi}_{2,p})|| &\leq M\frac{\xi_1 + \xi_2}{1-\theta} \\ &= 9.02e-02 \end{aligned}$$

where $(\varphi_{1,p}, \varphi_{2,p})$ is the solution of the system (2.2) and $(\widetilde{\varphi}_{1,p}, \widetilde{\varphi}_{2,p})$ is the solution of the system (2.5). As a result, without solving the system (2.5) which is more challenging to solve, it is possible to approximate the solution by solving the simpler system (2.2), which closely resembles the original system (2.5).

Theorem 2.5. *Let M be as in Proposition 1.1. If*

$$||\Gamma(x, \varphi_1(x)) - \Gamma(x, \varphi_2(x))||_\infty \leq K||\varphi_1(x) - \varphi_2(x)||_\infty$$

and $\theta = KM < 1$, then the iteration defined by

$$\varphi_{n+1}(x) = \int_0^L G(x,t)\Gamma(t, \varphi_n(t))dt \tag{2.6}$$

is convergent to the solution of the following BVP problem

$$\left.\begin{array}{c} \varphi''''(x) + \beta^2\varphi''(x) = \Gamma(x, \varphi(x)) \\ \varphi(0) = \varphi'(0) = \varphi(L) = \varphi'(L) = 0 \end{array} , x \in [0, L]\right\}. \tag{2.7}$$

Proof. Let $T(\varphi) = \int_0^L G(x,t)\Gamma(t, \varphi(t))dt$. Then T is a contraction, indeed,

$$\begin{aligned} \|T(\varphi_1) - T(\varphi_2)\|_\infty &= \left\| \int_0^L G(x,t)\Gamma(t, \varphi_1(t))dt - \int_0^L G(x,t)\Gamma(t, \varphi_2(t))dt \right\|_\infty \\ &\leq M\|\Gamma(x, \varphi_1(x)) - \Gamma(x, \varphi_2(x))\|_\infty \\ &\leq \theta||\varphi_1 - \varphi_2||_\infty, \end{aligned}$$

for any $\varphi_1, \varphi_2 \in X$, and thus, T has a unique solution by Proposition 1.1. Let $\varphi_p = T(\varphi_p) = \int_0^L G(x,t)\Gamma(t, \varphi_p(t))dt$ be the unique fixed point of T. Then, we have

$$\begin{aligned} \left\|\varphi_{n+1} - \varphi_p\right\|_\infty &= \left\|T(\varphi_n) - T(\varphi_p)\right\|_\infty \\ &\leq \theta||\varphi_n - \varphi_p||_\infty \\ &\leq \theta^2||\varphi_{n-1} - \varphi_p||_\infty \\ &\quad \dots \\ &\leq \theta^{n+1}||\varphi_0 - \varphi_p||_\infty \end{aligned}$$

which gives $\lim_{n\to\infty}||\varphi_{n+1} - \varphi_p|| = 0$, since $\theta < 1$. □

Example 2.6. *Let $X = C^{(4)}[0,1]$ and consider the following BVP*

$$\left.\begin{array}{c} \varphi''''(x) + 2^2\varphi''(x) = 2\varphi(x) + x^2 + 1 \\ \varphi(0) = \varphi(0) = \varphi(1) = \varphi(1) = 0 \end{array} , x \in [0, 1]\right\} \tag{2.8}$$

Then $M = 2.209e-01$. Since $\Gamma(x, \varphi(x)) = 2\varphi(x) + x^2 + 1$,

$$\begin{aligned} ||\Gamma(x, \varphi_1(x)) - \Gamma(x, \varphi_2(x))||_\infty &= ||2\varphi_1(x) + x^2 + 1 - (2\varphi_2(x) + x^2 + 1)||_\infty \\ &\leq 2||\varphi_1(x) - \varphi_2(x)||_\infty, K = 2, \end{aligned}$$

for all $\varphi_1, \varphi_2 \in X$. Obviously, since $KM < 1$, by Proposition 1.1, we have

$$\varphi_p(x) = \int_0^1 G(x,t)\Gamma(t, \varphi_p(t))dt$$

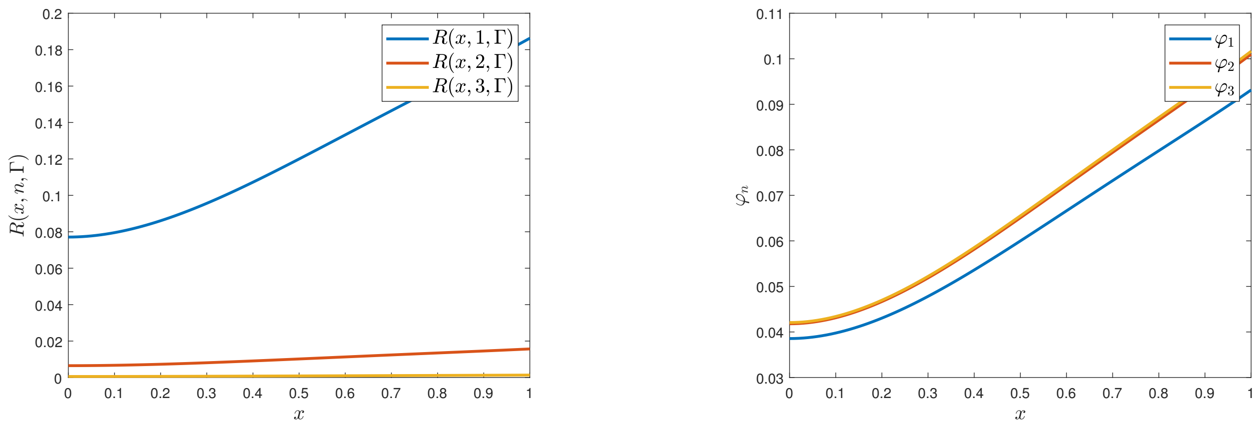
	n=1	**n=2**	**n=3**
	$R(x,1,f)$	$R(x,2,f)$	$R(x,3,f)$
0	7,710E-02	6,489E-03	5,481E-04
0.1	7,953E-02	6,701E-03	5,659E-04
0.2	8,607E-02	7,269E-03	6,139E-04
0.3	9,564E-02	8,095E-03	6,838E-04
0.4	1,072E-01	9,089E-03	7,678E-04
0.5	1,200E-01	1,017E-02	8,593E-04
0.6	1,332E-01	1,128E-02	9,531E-04
0.7	1,465E-01	1,239E-02	1,046E-03
0.8	1,596E-01	1,347E-02	1,138E-03
0.9	1,727E-01	1,455E-02	1,229E-03
1.0	1,863E-01	1,568E-02	1,324E-03

Table 2: Residual errors for $n = 1,2,$and 3

which is the solution of equation (2.8). Also,

$$\varphi_{n+1}(x) = \int_0^1 G(x,t)(2\varphi_n(t) + t^2 + 1)dt \tag{2.9}$$

is convergent to the solution φ_p. *Let* $R(x,n,\Gamma) = |\varphi_n''''(x) + \beta^2\varphi_n''(x) - \Gamma(x,\varphi_n(x))|$ *be the residual error for* $n > 0$. *Residual errors for* $n = 1,2,$*and* 3 *are shown in Figure 2 and Table 2.*



(a) $R(x,n,f)$ for $n = 1,2,$ and 3

(b) φ_n for $n = 1,2,$ and 3

Figure 2

Theorem 2.7. *Let* Γ *and* θ *be as in the Theorem 2.5 and* M *be as in Proposition 1.1. Assume that there exist* $\widetilde{\Gamma}(x,\varphi(x))$ *functions on* $[0,L]\times X$ *such that*

$$||\Gamma(x,\varphi(x)) - \widetilde{\Gamma}(x,\varphi(x))||_\infty \leq \xi.$$

If

$$\left.\begin{aligned} \varphi''''(x) + \beta^2\varphi''(x) = \widetilde{\Gamma}(x,\varphi(x)) \\ \varphi(0) = \varphi'(0) = \varphi(L) = \varphi'(L) = 0 \end{aligned} \quad , \ x \in [0,L] \right\} \tag{2.10}$$

has a solution, then

$$||\varphi_p - \widetilde{\varphi}_p||_\infty \leq M\frac{\xi_1}{1-\theta}$$

holds for $\varphi_p, \widetilde{\varphi}_p$ *which are the the solutions of BVPs (2.7) and (2.10),respectively.*

Proof. Let $T(\varphi)=\int_0^L G(x,t)\Gamma(t,\varphi(t))dt$ and $S(\varphi)=\int_0^L G(x,t)\widetilde{\Gamma}(t,\varphi(t))dt$. Then by Theorem 2.1, T has a fixed point φ_p which is the unique solution of system (1.2). Let $\varphi_0=\widetilde{\varphi}_p$ be the fixed point of S which is also the solution of system (2.10). Define $\varphi_{n+1}=T(\varphi_n)$. Then $\{\varphi_n\}_{n\geq 0}$ converges to φ_p by Theorem 2.5. Since

$$\begin{aligned}
||\varphi_{n+1}-\varphi_n||_\infty &= ||T(\varphi_n)-T(\varphi_{n-1})||_\infty \\
&\leq \theta||\varphi_n-\varphi_{n-1}||_\infty \\
&\quad \dots \\
&\leq \theta^n||\varphi_1-\varphi_0||_\infty,
\end{aligned}$$

$$\begin{aligned}
||\varphi_n-\varphi_0||_\infty &\leq \sum_{k=1}^{n}||\varphi_k-\varphi_{k-1}||_\infty \\
&\leq \sum_{k=1}^{n}\theta^{k-1}||\varphi_k-\varphi_{k-1}||_\infty \\
&\leq \frac{1}{1-\theta}||\varphi_k-\varphi_{k-1}||_\infty \\
&= \frac{1}{1-\theta}||T(\varphi_0)-S(\varphi_0)||_\infty \\
&= \frac{1}{1-\theta}\left(\left\|\int_0^L G(x,t)\Gamma(t,\varphi_0(t))dt-\int_0^L G(x,t)\widetilde{\Gamma}(t,\varphi_0(t))dt\right\|_\infty\right) \\
&\leq \frac{1}{1-\theta}\left(\left\|\int_0^L G(x,t)dt\right\|_\infty\left\|\Gamma(x,\varphi_0(x))-\widetilde{\Gamma}(x,\varphi_0(x))\right\|_\infty\right) \\
&= M\frac{\xi}{1-\theta}
\end{aligned}$$

which implies that

$$||\varphi_p-\widetilde{\varphi}_p||_\infty \leq M\frac{\xi}{1-\theta}$$

□

Example 2.8. *Consider the following BVP:*

$$\left.\begin{aligned}
&\varphi''''(x)+2^2\varphi''(x)=2\varphi^{\varphi^2(x)+\cos(\varphi(x))}(x)+x^2+1 \\
&\varphi(0)=\varphi'(0)=\varphi(1)=\varphi'(1)=0
\end{aligned}\quad ,\ x\in[0,1]\right\} \tag{2.11}$$

Solving this BVP directly is highly challenging or even infeasible due to the nonlinear functions involved in. However, thanks to Theorem 2.3, approximate solutions close to the exact one can be obtained without directly solving the equation.

Let Γ and β be as defined in Example 2.6 and $\overline{X}=\{\varphi\in X:0\leq\varphi(x)\leq 1\}$. It can be observed from figure 2 that the solution of equation (2.5) belongs to $\overline{X}\times\overline{X}$. Then $\widetilde{\Gamma}(x,\varphi(x))=2\varphi^{\varphi^2(x)+\cos(\varphi(x))}(x)+x^2+1$ satisfy the following

$$\begin{aligned}
||\Gamma(x,\varphi(x))-\widetilde{\Gamma}(x,\varphi(x))||_\infty &= \left\|2\varphi(x)+t^2+1-(2\varphi^{\varphi^2(x)+\cos(\varphi(x))}(x)+x^2+1)\right\|_\infty \\
&\leq \left\|2\varphi(x)-2\varphi^{\varphi^2(x)+\cos(\varphi(x))}(x)\right\|_\infty \\
&\leq 0.61e-01=\xi
\end{aligned}$$

for all $\varphi\in X$ and since $M=2.209e-01$ and $\theta=KM=4.419e-01$.Then, by Theorem 2.3 we have the following estimate for the solution of the system (2.5)

$$\begin{aligned}
||\varphi_p-\widetilde{\varphi}_p||_\infty &\leq M\frac{\xi_1}{1-\theta} \\
&= 2.42e-02
\end{aligned}$$

in which φ_p is the solution of the equation (2.2) and $\widetilde{\varphi}_p$ is the solution of the equation (2.11). As a result, without solving the equation (2.11) which is more challenging to solve, it is possible to approximate the solution of the equation (2.11) by solving the simpler equation (2.2) , which closely resembles the original equation.

3. Conclusion

In this study, we have analyzed a system of interdependent fourth-order differential equations that model coupled physical phenomena, such as the bending of elastic beams and the vibrations of structural elements. By establishing conditions for the existence and uniqueness of solutions, we have provided a rigorous mathematical framework for addressing higher-order boundary value problems. Furthermore, our application of iterative methods not only demonstrates the solvability of such systems but also offers practical tools for engineers and scientists working on related applications.

Our findings contribute significantly to the literature by extending classical results on fourth-order boundary value problems and complementing prior works. Beyond the theoretical advancements, our results open several promising directions for future research. One key extension involves exploring more generalized nonlinear coupled systems and their numerical solutions. Additionally, investigating the stability and convergence properties of iterative methods in different boundary conditions could enhance their applicability.

In conclusion, this study underscores the importance of coupled fourth-order differential equation systems in mathematical modeling and highlights the need for advanced analytical and numerical techniques for their solution. The broader impact of this work lies in its potential to bridge theoretical insights with practical applications across multiple scientific and engineering domains.

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