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Numerical Simulation of Diffusion Equation by Means of He's Variational Iteration Method and Adomian's Decomposition Method

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Özet. Bu çalışmada, sonlu bir aralıkta üç durumda difüzyon denklemi için ADM ve VIM yöntemleri kullanılarak yaklaşık çözümler elde edilmiş ve bulunan bu çözümler karşılaştırılmıştır. Elde edilen sonuçlar ADM'nin daha etkili sonuçlar verdiğini göstermiştir. Sayısal sonuçlar, sadece birkaç terimin tam çözümler elde etmek için yeterli olduğunu göstermiştir.

Anahtar Kelimeler. Difüzyon operatorü, Adomian ayrışım metodu, He'nin varyasyonel iterasyon metodu.

Abstract. In this study, we obtain approximate solutions for diffusion equation on a finite interval by the Adomian decomposition method (ADM) and variational iteration method (VIM) for three cases and then the numerical results are compared. These results show that the ADM leads to more accurate results, and they indicate that only a few terms are sufficient to obtain accurate solutions.

Keywords. Diffusion operator, Adomian's decomposition method, He's variational iteration method.

1. Introduction

The problem of describing the interactions between colliding particles is of fundamental interest in physics. It is interested in collisions of two spinless particles, and it is supposed that the s-wave scattering matrix and the s-wave binding energies are exactly known from collision experiments. With a radial static potential $V(x)$ the s-wave Schrödinger equation is written as

$$y'' + [E - V(E,x)]y = 0,$$

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where $V(E,x)$ is the following form for the energy dependence [1]

$$V(E,x) = 2\sqrt{E}\,p(x) + q(x).$$

Let $E = \lambda^2$. In this case, the Schrödinger equation transforms to the following equation

$$-\frac{d^2y}{dx^2} + [q(x) + 2\lambda p(x)]y = \lambda^2 y, \tag{1.1}$$

which is known as a diffusion equation in the literature, where the function $q(x) \in L^1[0,\pi]$, $p(x) \in L^2[0,\pi]$. Some spectral problems were extensively solved for the diffusion operator in references [2-5].

Consider the problem

$$-y'' + [q(x) + 2\lambda p(x)]y = \lambda^2 y, \tag{1.2}$$

$$y(0) = 1,\ y'(0) = -h, \tag{1.3}$$

where h is a finite number. Let us denote by $\varphi(x,\lambda)$ the solution of (1.2) satisfying the initial conditions (1.3). Let [3]

$$\lambda_n = n + c_0 + \frac{c_1}{n} + \frac{c_{1,n}}{n},$$

be the nth eigenvalue where

$$\begin{aligned} c_0 &= \frac{1}{\pi}\int_0^\pi p(x)dx, \quad \sum_n |c_{1,n}|^2 < \infty, \\ c_1 &= \frac{1}{\pi}\left(h + H + \frac{1}{2}\int_0^\pi [q(x) + p^2(x)]dx\right), \end{aligned}$$

and H is a finite number.

The aim of this study is to approach the diffusion equation differently, but effectively, by using VIM and ADM. The VIM and ADM avoid the complexity involved in other purely numerical methods. We use VIM and ADM to investigate the problem of diffusion equations slowly approaching each other. The paper is organized as follows; in Section 2, we give an application of ADM to a Volterra type integral equation. An application of ADM to the diffusion equation is given in Section 3. In Section 4, we introduce a framework theoretical analysis of the VIM. In Section 5, VIM is carried out to obtain the solution of a diffusion problem. The exact solutions obtained by VIM and ADM are compared in Section 6. The conclusion can be found in Section 7.

2. Application of ADM to a Volterra Type Integral Equation

We consider nonhomogeneous Volterra type integral equation

$$u(x) = f(x) + \gamma \int_0^x K(x,t)u(t)dt, \tag{2.1}$$

where $K(x,t)$ is the kernel of the integral equation, and γ is a parameter. Our concern will be to apply ADM to determine the $u(x)$ of (2.1). In this method, $u(x)$ will be decomposed into components, that will be determined, given by the series form

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{2.2}$$

with u_0 identified as all terms out of the integral sign

$$u_0(x) = f(x). \tag{2.3}$$

Substituting (2.2) into (2.1) yields

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \gamma \int_0^x K(x,t) \left(\sum_{n=0}^{\infty} u_n(t) \right) dt. \tag{2.4}$$

The above-mentioned scheme for the determination of the components $u_0(x)$, $u_1(x)$, $u_2(x), \ldots$ of the solution $u(x)$ for equation (2.1) can be written in a recursive scheme by

$$u_0 = f(x), \tag{2.5}$$

$$u_{n+1}(x) = \gamma \int_0^x K(x,t)u_n(t)dt, \quad n \geq 0. \tag{2.6}$$

With these components determined, the solution $u(x)$ of (2.1) is readily determined in a series form upon using (2.2) [6].

Lemma 2.1 ([7])**.** *The solution of problem* (1.2)-(1.3) *has the following form*

$$\varphi(x,\lambda) = \cos(\lambda x) - \frac{h}{\lambda}\sin(\lambda x) + \int_0^x \frac{\sin\lambda(x-t)}{\lambda}[q(t) + 2\lambda p(t)]\varphi(t,\lambda)dt. \tag{2.7}$$

This is a nonhomogeneous Volterra type integral equation of the second kind. Our concern will be to apply ADM to determine the solution $\varphi(x,\lambda)$ of (2.7). It is clear that

$$f(x,\lambda) = \cos(\lambda x) - \frac{h}{\lambda}\sin(\lambda x), \quad \gamma = 1, \quad K(x,t) = \frac{\sin\lambda(x-t)}{\lambda}[q(t) + 2\lambda p(t)]$$

in (2.7). Using the decomposition series solution (2.2) and the recursive scheme (2.5) and (2.6) to determine the components $\varphi_n(x,\lambda)$, $n \geq 0$ for three cases, we find the following results.

Case 1. In the case $p(x) = x^2$ and $q(x) = 0$, we get

$$\begin{aligned}
\varphi_0(x,\lambda) &= \cos(\lambda x) - \frac{h}{\lambda}\sin(\lambda x),\\
\varphi_1(x,\lambda) &= \frac{1}{6\lambda^4}[x\lambda(3x\lambda^2 + h(-3+2x^2\lambda^2))\cos(\lambda x)\\
&\qquad + (3h - 3x\lambda^2 - 3hx^2\lambda^2 + 2x^3\lambda^4)\sin(\lambda x)],\\
\varphi_2(x,\lambda) &= \frac{1}{3\lambda^7}[(-2 + x^2\lambda^2 + 2\cos(\lambda x))(x\lambda(3x\lambda^2 + h(-3+2x^2\lambda^2))\cos(\lambda x)\\
&\qquad + (3h - 3x\lambda^2 - 3hx^2\lambda^2 + 2x^3\lambda^4)\sin(\lambda x))],\\
&\ \ \vdots
\end{aligned}$$

and so on. Noting that

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \varphi_1(x,\lambda) + \varphi_2(x,\lambda) + \dots,$$

we can easily obtain the solution in a series form given by

$$\begin{aligned}
\varphi(x,\lambda) = \cos(\lambda x) - \frac{h}{\lambda}\sin(\lambda x) &+ \frac{1}{6\lambda^4}[x\lambda(3x\lambda^2 + h(-3+2x^2\lambda^2))\cos(\lambda x)\\
&+ (3h - 3x\lambda^2 - 3hx^2\lambda^2 + 2x^3\lambda^4)\sin(\lambda x)] + \cdots .
\end{aligned}$$

Case 2. In the case $p(x) = 0$ and $q(x) = x^2$, we get

$$\begin{aligned}
\varphi_0(x,\lambda) &= \cos(\lambda x) - \frac{h}{\lambda}\sin(\lambda x),\\
\varphi_1(x,\lambda) &= \frac{1}{12\lambda^5}[x\lambda(3x\lambda^2 + h(-3+2x^2\lambda^2))\cos(\lambda x)\\
&\qquad + (3h - 3x\lambda^2 - 3hx^2\lambda^2 + 2x^3\lambda^4)\sin(\lambda x)],\\
\varphi_2(x,\lambda) &= \frac{1}{12\lambda^9}[(-2 + x^2\lambda^2 + 2\cos(\lambda x))(x\lambda(3x\lambda^2 + h(-3+2x^2\lambda^2))\cos(\lambda x)\\
&\qquad + (3h - 3x\lambda^2 - 3hx^2\lambda^2 + 2x^3\lambda^4)\sin(\lambda x))],\\
&\ \ \vdots
\end{aligned}$$

and

$$\varphi(x,\lambda) = \cos(\lambda x) - \frac{h}{\lambda}\sin(\lambda x) + \frac{1}{12\lambda^5}[x\lambda(3x\lambda^2 + h(-3+2x^2\lambda^2))\cos(\lambda x) + (3h - 3x\lambda^2 - 3hx^2\lambda^2 + 2x^3\lambda^4)\sin(\lambda x)] + \cdots .$$

Case 3. In the case $p(x) = x^2$ and $q(x) = x^2$, we get

$$\begin{aligned}
\varphi_0(x,\lambda) &= \cos(\lambda x) - \frac{h}{\lambda}\sin(\lambda x),\\
\varphi_1(x,\lambda) &= \frac{1}{12\lambda^5}(1+2\lambda)[x\lambda(3x\lambda^2 + h(-3+2x^2\lambda^2))\cos(\lambda x)\\
&\qquad + (3h - 3x\lambda^2 - 3hx^2\lambda^2 + 2x^3\lambda^4)\sin(\lambda x)],\\
\varphi_2(x,\lambda) &= \frac{1}{12\lambda^9}(1+2\lambda)^2[(-2 + x^2\lambda^2 + 2\cos(\lambda x))(x\lambda(3x\lambda^2 + h(-3+2x^2\lambda^2))\\
&\qquad \cos(\lambda x) + (3h - 3x\lambda^2 - 3hx^2\lambda^2 + 2x^3\lambda^4)\sin(\lambda x))],\\
&\vdots
\end{aligned}$$

and

$$\varphi(x,\lambda) = \cos(\lambda x) - \frac{h}{\lambda}\sin(\lambda x) + \frac{1}{12\lambda^5}(1+2\lambda)[x\lambda(3x\lambda^2 + h(-3+2x^2\lambda^2))\cos(\lambda x) + (3h - 3x\lambda^2 - 3hx^2\lambda^2 + 2x^3\lambda^4)\sin(\lambda x)) + \cdots .$$

3. Application of ADM to the Diffusion Equation

The decomposition method was introduced by Adomian [8], [9] in the 1980s in order to solve linear and nonlinear functional equations (algebraic, differential, partial differential, integral, integro-differential equations, etc.). Approximate and exact solutions of wide varieties of physically significant problems modeled by nonlinear partial differential equations are easily calculated by the decomposition method [10–18]. The nonlinear partial differential equations and systems are directly solvable preserving the actual physics and involving much less calculations. No linearization, perturbation or discretized methods which result in intensive computation are necessary.

In this section, we describe the algorithm of the ADM as it applies to the diffusion equation. We consider the equation (1.2) in operator form

$$L_x y = [q(x) + 2\lambda p(x) - \lambda]y, \tag{3.1}$$

where $L_x = \dfrac{\partial^2}{\partial x^2}$. Assuming the inverse of the operator L_x^{-1} exists and it can conveniently be taken as the two fold definite integral with respect to x from 0 to x, i.e.,

$$L_x^{-1}(\cdot) = \int_0^x \int_0^x (\cdot)\, dx\, dx,$$

and applying the inverse operator L_x^{-1}, (3.1) yields

$$\begin{aligned} y(x,\lambda) &= y(0,\lambda) + x y_x(0,\lambda) + L_x^{-1}\{[q(x) + 2\lambda p(x) - \lambda] y\} \\ &= 1 - xh + L_x^{-1}\{[q(x) + 2\lambda p(x) - \lambda] y\}. \end{aligned} \tag{3.2}$$

Following ADM [8], [9], we expect the decomposition of the solution into a sum of components to be defined by the decomposition series form

$$y(x,\lambda) = \sum_{n=0}^{\infty} y_n(x,\lambda). \tag{3.3}$$

Substituting the initial conditions into (3.2) identifying the zeroth component $y_0(x,\lambda) = 1 - xh$ by terms arising from initial conditions, we obtain the subsequent components by the following recursive relationship

$$y_0(x,\lambda) = 1 - xh, \tag{3.4}$$

$$y_{n+1}(x,\lambda) = L_x^{-1}\{[q(x) + 2\lambda p(x) - \lambda] y_n(x,\lambda)\}, \tag{3.5}$$

where $n \geq 0$. The remaining components y_1, y_2, $y_3, \ldots$, etc. were computed by a recursive scheme either directly by hand or programmed on Mathematica by using (3.5) in four cases. Some of the symbolically computed components are as follows:

Case 1. In the case $p(x) = 0$ and $q(x) = 0$, we get

$$\begin{aligned} y_0(x,\lambda) &= 1 - xh, \\ y_1(x,\lambda) &= -\left(\frac{x^2}{2} - \frac{hx^3}{6}\right)\lambda, \\ y_2(x,\lambda) &= \left(\frac{x^4}{24} - \frac{hx^5}{120}\right)\lambda^2, \\ &\vdots \end{aligned}$$

In this manner, three components of the decomposition series were obtained of which $y(x,\lambda)$ was evaluated to have the following expansion

$$y(x,\lambda) = 1 - xh - \left(\frac{x^2}{2} - \frac{hx^3}{6}\right)\lambda + \left(\frac{x^4}{24} - \frac{hx^5}{120}\right)\lambda^2 + \cdots .$$

Case 2. In the case $p(x) = 0$ and $q(x) = x^2$, we get

$$\begin{aligned}
y_0(x,\lambda) &= 1 - xh,\\
y_1(x,\lambda) &= \frac{x^4}{12} - \frac{hx^5}{20} - \frac{x^2}{2}\lambda + \frac{1}{6}hx^3\lambda,\\
y_2(x,\lambda) &= \frac{x^8}{672} - \frac{hx^9}{1440} - \frac{7x^6\lambda}{360} + \frac{13hx^7\lambda}{2520} + \frac{x^4\lambda^2}{24} - \frac{1}{120}hx^5\lambda^2,\\
&\vdots
\end{aligned}$$

In this manner, three components of the decomposition series were obtained of which $y(x,\lambda)$ was evaluated to have the following expansion

$$\begin{aligned}
y(x,\lambda) = 1 - xh + \frac{x^4}{12} - \frac{hx^5}{20} - \frac{x^2}{2}\lambda + \frac{1}{6}hx^3\lambda + \frac{x^8}{672} - \frac{hx^9}{1440}\\
- \frac{7x^6\lambda}{360} + \frac{13hx^7\lambda}{2520} + \frac{x^4\lambda^2}{24} - \frac{1}{120}hx^5\lambda^2 + \cdots .
\end{aligned}$$

Case 3. In the case $p(x) = x^2$ and $q(x) = 0$, we get

$$\begin{aligned}
y_0(x,\lambda) &= 1 - xh,\\
y_1(x,\lambda) &= \left(-\frac{x^2}{2} + \frac{hx^3}{6} + \frac{x^4}{6} - \frac{1}{10}hx^5\right)\lambda,\\
y_2(x,\lambda) &= \left(\frac{x^4}{24} - \frac{hx^5}{120} - \frac{7x^6}{180} + \frac{13hx^7}{1260} + \frac{x^8}{168} - \frac{1}{360}hx^9\right)\lambda^2,\\
&\vdots
\end{aligned}$$

In this manner, three components of the decomposition series were obtained of which $y(x,\lambda)$ was evaluated to have the following expansion

$$\begin{aligned}
y(x,\lambda) = 1 - xh + \left(-\frac{x^2}{2} + \frac{hx^3}{6} + \frac{x^4}{6} - \frac{1}{10}hx^5\right)\lambda\\
+ \left(\frac{x^4}{24} - \frac{hx^5}{120} - \frac{7x^6}{180} + \frac{13hx^7}{1260} + \frac{x^8}{168} - \frac{1}{360}hx^9\right)\lambda^2 + \cdots .
\end{aligned}$$

Case 4. In the case $p(x) = x^2$ and $q(x) = x^2$, we get

$$
\begin{aligned}
y_0(x,\lambda) &= 1 - xh,\\
y_1(x,\lambda) &= \frac{1}{20}hx^5(-1-2\lambda) - \frac{x^2\lambda}{2} + \frac{1}{6}hx^3\lambda + \frac{1}{12}x^4(1+2\lambda),\\
y_2(x,\lambda) &= \frac{x^4\lambda^2}{24} - \frac{1}{120}hx^5\lambda^2 - \frac{7}{360}x^6\lambda(1+2\lambda)\\
&\qquad + \frac{13hx^7\lambda(1+2\lambda)}{2520} + \frac{1}{672}x^8(1+2\lambda)^2 - \frac{hx^9(1+2\lambda)^2}{1440},\\
&\ \vdots
\end{aligned}
$$

In this manner, three components of the decomposition series were obtained of which $y(x,\lambda)$ was evaluated to have the following expansion

$$
\begin{aligned}
y(x,\lambda) = 1 - xh + \frac{1}{20}hx^5(-1-2\lambda) - \frac{x^2\lambda}{2} + \frac{1}{6}hx^3\lambda + \frac{1}{12}x^4(1+2\lambda) + \frac{x^4\lambda^2}{24} - \frac{1}{120}hx^5\lambda^2\\
- \frac{7}{360}x^6\lambda(1+2\lambda) + \frac{13hx^7\lambda(1+2\lambda)}{2520} + \frac{1}{672}x^8(1+2\lambda)^2 - \frac{hx^9(1+2\lambda)^2}{1440} + \cdots .
\end{aligned}
$$

4. The Basic Idea of He's Variational Iteration Method

The VIM was first proposed by He [19-22]. It has been shown to solve effectively, easily and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions. He applied his method to autonomous ordinary differential systems [23] and nonlinear equations with convolution product nonlinearity [24]. In several papers, VIM has been successfully applied to a wide range of mathematical, physical and engineering problems by many authors [25-34].

The idea of VIM is constructing a correctional functional by a general Lagrange multiplier. The multiplier in the functional should be chosen such that its correction solution is superior to its initial approximation (trial function) and is the best within the flexibility of the trial function; accordingly we can identify the multiplier by variational theory. The initial approximation can be freely chosen with possible unknowns which can be determined by imposing the boundary or initial conditions.

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(t), \tag{4.1}$$

where L and N are linear and nonlinear operators respectively, and $g(t)$ is the source inhomogeneous term. According to VIM, we can write down a correction functional as follows

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda_1 \left\{ Lu_n(\xi) + N\widetilde{u_n}(\xi) - g(\xi) \right\} d\xi, \quad n \geq 0, \tag{4.2}$$

where λ_1 is a general Lagrangian multiplier which can be identified optimally via the variational theory. The subscript n indicates the nth approximation and $\widetilde{u_n}$ is considered as a restricted variation [19-24], i.e. $\delta\widetilde{u_n} = 0$.

5. VIM Solutions

Now, we apply the variational iteration method to the diffusion problem to obtain an explicit, uniformly valid, and totally analytic solution. In order to solve the equation (1.2) using VIM, we first construct a correction functional, as follows

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda_1(s) \left\{ y_n''(s) + [\lambda - q(s) - 2\lambda p(s)]\widetilde{y_n} \right\} ds, \quad n \geq 0, \tag{5.1}$$

where λ_1 is Lagrange multiplier whose optimal value is found by variational theory. Also, $\widetilde{y_n}$ is chosen suitably to satisfy the restricted variation condition, i.e. $\delta\widetilde{y_n} = 0$. To determine the optimal value of $\lambda_1(s)$, we continue as follows

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda_1(s) \left\{ y_n''(s) + [\lambda - q(s) - 2\lambda p(s)]\widetilde{y_n} \right\} ds, \quad n \geq 0, \tag{5.2}$$

which gives

$$\delta y_{n+1}(x) = [1 - \lambda_1'(x)]\delta y_n(x) + \lambda_1 \delta y_n'(x) + \int_0^x \lambda_1''(s)\delta y_n(s) ds = 0. \tag{5.3}$$

Hence, the stationary conditions can be obtained from equation (5.3) read as

$$1 - \lambda_1'(x) = 0, \quad \lambda_1(x) = 0, \quad \lambda_1''(s)\big|_{x=s} = 0, \tag{5.4}$$

and the Lagrange multiplier is obtained as

$$\lambda_1(s) = s - x. \tag{5.5}$$

Finally, the iteration formula can be given as

$$y_{n+1}(x) = y_n(x) + \int_0^x (s - x) \left\{ y_n''(s) + [\lambda - q(s) - 2\lambda p(s)]y_n \right\} ds, \quad n \geq 0. \tag{5.6}$$

We start with the initial approximation $y_0(x) = 1$. The next iterates y_1, y_2, $y_3, \cdots$ are given below respectively for three cases.

Case 1. Let $p(x) = 0$ and $q(x) = x^2$. In this case, we get the next iterates y_1, y_2, $y_3, ...$ as follows

$$
\begin{aligned}
y_1 &= 1 + \frac{x^4}{12} - \frac{x^2\lambda}{2}, \\
y_2 &= 1 + \frac{x^4}{12} + \frac{x^8}{672} - \frac{x^2\lambda}{2} - \frac{7x^6\lambda}{360} + \frac{x^4\lambda^2}{24}, \\
y_3 &= 1 + \frac{x^4}{12} + \frac{x^8}{672} - \frac{x^{12}}{88704} - \frac{x^2\lambda}{2} - \frac{7x^6\lambda}{360} - \frac{211x^{10}\lambda}{907200} + \frac{x^4\lambda^2}{24} + \frac{11x^8\lambda^2}{10080} - \frac{x^6\lambda^3}{720}, \\
&\vdots
\end{aligned}
$$

Case 2. Let $p(x) = x^2$ and $q(x) = 0$. In this case, we get the next iterates y_1, y_2, $y_3, \cdots$ as follows

$$
\begin{aligned}
y_1 &= 1 - \frac{x^2\lambda}{2} + \frac{x^4\lambda}{6}, \\
y_2 &= 1 - \frac{x^2\lambda}{2} + \frac{x^4\lambda}{6} + \frac{x^4\lambda^2}{24} - \frac{7x^6\lambda^2}{180} + \frac{x^8\lambda^2}{168}, \\
y_3 &= 1 - \frac{x^2\lambda}{2} + \frac{x^4\lambda}{6} + \frac{x^4\lambda^2}{24} - \frac{7x^6\lambda^2}{180} + \frac{x^8\lambda^2}{168} \\
&\quad - \frac{x^6\lambda^3}{720} + \frac{11x^8\lambda^3}{5040} - \frac{211x^{10}\lambda^3}{226800} - \frac{x^{12}\lambda^3}{11088}, \\
&\vdots
\end{aligned}
$$

Case 3. Let $p(x) = x^2$ and $q(x) = x^2$. In this case, we get the next y_1, $y_2, \cdots$ as follows

$$
\begin{aligned}
y_1 &= 1 + \frac{x^4}{12} - \frac{x^2\lambda}{2} + \frac{x^4\lambda}{6}, \\
y_2 &= 1 + \frac{x^4}{12} - \frac{x^2\lambda}{2} + \frac{x^4\lambda}{6} + \frac{x^4\lambda^2}{24} + \frac{7}{72}x^6\lambda(1 + 2\lambda) \\
&\quad + \frac{1}{672}x^8(1 + 2\lambda)^2 - \frac{7}{60}x^6(\lambda + 2\lambda^2), \\
&\vdots
\end{aligned}
$$

6. Comparison Analysis

In this section, we will compare the ADM and VIM analytic solutions in order to confirm the efficiency of ADM. By mathematical experiments, we will give more lively descriptions. The detailed results are shown in Table 1. In Table 1, fix $h = 1.0$,

$\lambda = 5$, vary the value x, for the numerical solutions obtained by the iteration method and decomposition method for 2 and 3 steps. From Table 1, we see that analytic approximations (VIM and ADM) of $y(x, \lambda)$ show good agreement with the numerical ones and ADM leads to more accurate results than VIM.

x	VIM	VIM	ADM	ADM
	(2 order)	(3 order)	(2 order)	(3 order)
0.01	0.00101	0.00101	9.99664×10^{-6}	9.99664×10^{-6}
0.02	0.002039	0.0020399	0.00003997	0.000039972
0.03	0.003089	0.0030898	0.000089907	0.000089908
0.04	0.004159	0.0041597	0.00015978	0.00015978
0.05	0.005249	0.0052494	0.000249567	0.000249568
0.06	0.006359	0.0063590	0.000359246	0.000359248
0.07	0.007488	0.0074885	0.000488794	0.000488797
0.08	0.008637	0.0086377	0.000638187	0.000638191
0.09	0.009806	0.0098068	0.000807399	0.000807406
0.1	0.010995	0.0109956	0.000996407	0.000996417

TABLE 1. Error between VIM, ADM using 2-3 terms and exact solutions for $y(x, \lambda)$ when $\lambda = 5$, $h = 1$, $p(x) = 0$ and $q(x) = x^2$.

As Figure 1 shows, a comparison is made between 3 iterates of VIM solutions, ADM solutions and analytic solutions for the case $\lambda = 5$, $h = 1$, $p(x) = 0$ and $q(x) = x^2$. The results presented in Figure 1 clearly show the good accuracy of the VIM and ADM. If we solve the equation for $p(x) = x^2$, $q(x) = 0$ and $p(x) = x^2$, $q(x) = x^2$, we get analogous results.

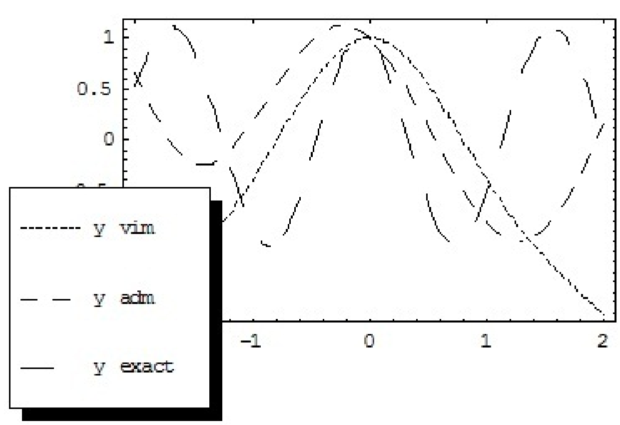

FIGURE 1. A comparison between VIM solutions, ADM solutions and analytic solutions for the case $\lambda = 5$, $h = 1$, $p(x) = 0$ and $q(x) = x^2$.

7. Conclusion

In this paper, we obtain an explicit series solution of the diffusion equation by means of the ADM and VIM for different cases of $p(x)$ and $q(x)$. We made a comparison between ADM and VIM solutions. The results of numerical examples are presented and only a few terms are required to obtain accurate solutions with ADM. The present study shows that ADM is more effective than VIM (See Table 1). Consequently, the present success of the VIM and ADM for the diffusion equation verifies that these methods are useful tools for these kind of problems in science and engineering.

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