C.B.Ü. Fen Bilimleri Dergisi ISSN 1305-1385 C.B.U. Journal of Science
6.2 (2010) 155 – 160 6.2 (2010) 155 –160

THE RULED SURFACES GENERATED BY FRENET VECTORS OF A CURVE IN IR_1^3

Keziban ORBAY[1*], İsmail AYDEMİR[2]

[1] AmasyaUniversity, Faculty of Education, Department of Mathematics, Amasya – TURKEY
[2]Department of Mathematics, Faculty of Art and Sciences, Ondokuz Mayis University, Samsun – TURKEY

Abstract: In this paper, the space-like surface with space-like directional vector and the ruled surfaces generated by Frenet vectors of the base curve of this surface have been investigated in the Minkowski space IR_1^3. As a first step, it is obtained the distribution parameters, the main curvatures and the Gaussian curvatures of these surfaces. Some new results and theorems have been given related to be developable and minimal of these surfaces. Moreover, some new relationships among geodesic curvature, normal curvature and geodesic torsion of the base curve of the space-like ruled surface with space-like directional vector have been found.

Key words: *Space-like ruled surface, distribution parameter, mean curvature, Gaussian curvature.*

IR_1^3 UZAYINDA BİR EĞRİNİN FRENET VEKTÖRLERİ İLE OLUŞTURULAN REGLE YÜZEYLER

Özet: Bu çalışmada**,** IR_1^3 Minkowski uzayında space-like doğrultman vektörlü space-like yüzey ile bu yüzeyin dayanak eğrisinin Frenet vektörleri ile oluşturulan regle yüzeyler incelendi. İlk olarak bu yüzeylerin dağılma parametreleri, ortalama eğrilikleri ve Gauss eğrilikleri hesaplandı. Bu yüzeylerin açılabilir ve minimal olmaları ile ilgili bazı yeni teoremler ve sonuçlar verildi. Ayrıca space-like doğrultman vektörlü space-like regle yüzeyin dayanak eğrisinin geodezik eğriliği, normal eğriliği ve geodezik torsiyonu arasında bazı yeni bağıntılar elde edildi.

Anahtar kelimeler: *Space-like regle yüzey, dağılma parametresi, ortalama eğrilik, Gauss eğriliği.*

MSC 2000: 53 C 50, 53 A 10.

*** Corresponding Author**
keziban.orbay@amasya.edu.tr

1. Introduction

The ruled surfaces have been a powerful subject in the Minkowski space IR_1^3 for line geometry for a long time. In literature, Kobayashi [1] was the first author to address this problem and examined minimal space like ruled surfaces in the Minkowski IR_1^3. Then, recently, Woestijne [2], Kim and Yoon [3] have classified the Lorentz surfaces. In addition to Refs.[1-3], for understanding the space kinematics and mechanism, Refs.[4-6] and references quoted therein are advised for the readers' interest in this field.

In this paper, the ruled surfaces generated by Frenet vectors $(\vec{t},\vec{n},\vec{b})$ of the base curve of space-like ruled surface with space-like directional vector have been studied and some new theorems and results related to the distribution parameters, the mean curvatures and the Gaussian curvatures of these surfaces have been obtained.

2. Preliminaries

Let IR_1^3 be the Minkowski 3-space with the scalar product given by

$$\langle \vec{X},\vec{Y}\rangle = x_1y_1 + x_2y_2 - x_3y_3$$

where $\vec{X}=(x_1,x_2,x_3)$, $\vec{Y}=(y_1,y_2,y_3)$ $\in IR^3$. If $\vec{X}$ and $\vec{Y}$ are perpendicular in Lorentzian space, then $\langle \vec{X},\vec{Y}\rangle = 0$. **The** norm of $\vec{X}\in IR_1^3$ is defined by $\|\vec{X}\| = \sqrt{|\langle \vec{X},\vec{X}\rangle|}$.

A vector $\vec{X}$ of IR_1^3 is said to be space-like if $\langle \vec{X},\vec{X}\rangle > 0$ or $\vec{X}=\vec{0}$, time-like if $\langle \vec{X},\vec{X}\rangle < 0$ and light-like(null) if $\langle \vec{X},\vec{X}\rangle = 0$ and $\vec{X}\neq\vec{0}$ [7]. The cross product of $\vec{X}$ and $\vec{Y}$ is defined by

$$\vec{X}\Lambda\vec{Y} = (x_3y_2 - x_2y_3, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1)$$

[8].

Let us consider a regular curve $\vec{\alpha}: I \to IR_1^3$. For all $s\in I\subset IR$, $\dot{\vec{\alpha}}(s)$ is the tangent vector to the curve α at point $\alpha(s)$.

i) If $\langle \dot{\vec{\alpha}}(s),\dot{\vec{\alpha}}(s)\rangle > 0$ then $\vec{\alpha}$ is a space-like curve,

ii) If $\langle \dot{\vec{\alpha}}(s),\dot{\vec{\alpha}}(s)\rangle < 0$ then $\vec{\alpha}$ is a time-like curve,

iii) If $\langle \dot{\vec{\alpha}}(s),\dot{\vec{\alpha}}(s)\rangle = 0$ then $\vec{\alpha}$ is a null curve [7].

A surface in IR_1^3 is called a space-like surface if the induced metric on the surface is a positive definite Riemannian metric [9]. A normal vector on the space-like surface is a time-like vector [10]. Let $\vec{\alpha}(s)$ be a unit speed space-like curve in IR_1^3 with $\kappa(s)$, $\tau(s)$ the natural curvature and torsion of $\vec{\alpha}$ respectively. Let us consider the Frenet frame $\{\vec{t},\vec{n},\vec{b}\}$ of the space-like curve $\vec{\alpha}(s)$ such that $\vec{t}(s)$ is space-like the unit tangent vector, $\vec{n}(s)$ is time-like the unit principal normal vector and $\vec{b}(s)$ is space-like the unit binormal vector. So scalar product and cross product of the vectors $\vec{t}$, $\vec{n}$ and $\vec{b}$ is given by

(2.1) $\langle \vec{t},\vec{t}\rangle = -\langle \vec{n},\vec{n}\rangle = \langle \vec{b},\vec{b}\rangle = 1,$

$\langle \vec{t},\vec{n}\rangle = \langle \vec{t},\vec{b}\rangle = \langle \vec{n},\vec{b}\rangle = 0,$

(2.2) $\vec{t}\Lambda\vec{n} = -\vec{b}$, $\vec{n}\Lambda\vec{b} = -\vec{t}$, $\vec{b}\Lambda\vec{t} = \vec{n}$.

Frenet formulas are given by

(2.3) $\dot{\vec{t}} = \kappa(s)\vec{n}$, $\dot{\vec{n}} = -\kappa(s)\vec{t} + \tau(s)\vec{b}$,
$\dot{\vec{b}} = \tau(s)\vec{n}$, [2].

Let $\vec{\alpha} = \vec{\alpha}(s)$ be a curve in IR_1^3 and $\vec{Z} = \vec{Z}(s)$ be a transversal vector field along α. We have a parametrization for a ruled surface M in IR_1^3

(2.4) $\phi(s,v) = \vec{\alpha}(s) + v\vec{Z}(s)$, $s, v \in I \subset IR$,

where the curve $\vec{\alpha} = \vec{\alpha}(s)$ is called the base curve and $\vec{Z} = \vec{Z}(s)$ is called the rulings. The distribution parameter, the mean curvature and the Gaussian curvature of the ruled surface $\phi(s,v)$ is given by

(2.5) $$\delta = -\frac{\det\left(\dot{\vec{\alpha}}, \vec{Z}, \dot{\vec{Z}}\right)}{\left\|\dot{\vec{Z}}\right\|^2},$$

(2.6) $$H = \frac{1}{2}\left[\frac{GL + EN - 2FM}{EG - F^2}\right]$$

and

(2.7) $$K = \frac{LN - M^2}{EG - F^2},$$

respectively [7]. The foot on the main ruling of the common perpendicular of two constructive rulings in the ruled surface is called a central point. The locus of the central point is called striction curve. The parametrization of the striction curve on the ruled surface is given by

(2.8) $$\vec{\beta}(s) = \vec{\alpha}(s) - \frac{\left\langle \dot{\vec{\alpha}}, \dot{\vec{Z}} \right\rangle}{\left\|\dot{\vec{Z}}\right\|^2}\vec{Z},$$

[7].

3. Space-Like Ruled Surfaces With Space-Like Rulings

Let $\vec{\alpha} = \vec{\alpha}(s)$ be a space-like curve, $\{\vec{t}, \vec{n}, \vec{b}\}$ be its Frenet frame defined as in (2.1) and a space-like oriented line $\vec{X}$ in IR_1^3 be firmly connected to this Frenet frame. With scalar functions $x_1(s)$, $x_2(s)$ and $x_3(s)$ of the arc length parameter of the base curve $\vec{\alpha}(s)$, the parametrization of the unit space-like oriented line $\vec{X}$ is given by

(3.1)
$$\vec{X}(s) = x_1(s)\vec{t}(s) + x_2(s)\vec{n}(s) + x_3(s)\vec{b}(s)$$

The ruled surfaces generated by $\vec{t}$, $\vec{n}$, $\vec{b}$ and $\vec{X}$ are

(3.2)
$$M_1 \to \phi_1(s,v) = \vec{\alpha}(s) + v\vec{t}(s),$$
$$M_2 \to \phi_2(s,u) = \vec{\alpha}(s) + u\vec{n}(s),$$
$$M_3 \to \phi_3(s,z) = \vec{\alpha}(s) + z\vec{b}(s),$$
$$M_4 \to \phi_4(s,w) = \vec{\alpha}(s) + w\vec{X}(s),$$

resceptively. We have the following theorem using (2.1)-(2.3) and (2.5).

Theorem 3.1. *The distribution parameters of surfaces* M_1, M_2 *and* M_3 *are*

$$\delta_t = 0, \ \delta_n = \frac{\tau}{\kappa^2 + \tau^2} \ \text{and} \ \delta_b = -\frac{1}{\tau}$$

respectively.

From (2.1), (2.3) and (2.6), the following theorem may be given.

Theorem 3.2. *The mean curvatures of surfaces* M_1, M_2 *and* M_3 *are*

$$H_t = -\frac{\tau}{2v\kappa}, \ H_n = \frac{(\kappa\dot{\tau} - \dot{\kappa}\tau)u^2 - \dot{\tau}u}{2\left[(\kappa u - 1)^2 + \tau^2 u^2\right]^{\frac{3}{2}}},$$

and

$$H_b = \frac{\kappa\tau^2 z^2 + \dot{\tau}z + \kappa}{2\left(1 - \tau^2 z^2\right)\sqrt{\left|1 - \tau^2 z^2\right|}},$$

respectively.

From (2.1), (2.3) and (2.7), the following theorem may be given.

Theorem 3.3. *The Gaussian curvatures of surfaces M_1, M_2 and M_3 are*

$$K_t = 0\,,\quad K_n = \frac{\tau^2}{\left[(\kappa u-1)^2+\tau^2u^2\right]^2}\,,$$

and

$$K_b = \frac{\tau^2}{\left(1-\tau^2 z^2\right)^2}\,,$$

respectively.

We obtain the following results and theorem for the ruled surfaces. $M_1 - M_3$ generated by $\vec{t}$, $\vec{n}$ and $\vec{b}$, using theorem3.1-3.3.

Result 3.1.

i) $K_t = \delta_t = 0$

ii) M_1- *surface is minimal if and only if $\vec{\alpha}$ is a planer curve.*

Result 3.2.

i) M_2- *surface is developable if and only if $\vec{\alpha}$ is a planer curve,*

ii) *If M_2-surface is developable then M_2-surface is minimal,*

iii) *If M_2-surface is minimal then $u=0$ or*

$$u = \frac{\dot{\tau}}{\kappa\dot{\tau}-\dot{\kappa}\tau}\,.$$

iv) *There is a relationship between K_n and H_n as the following*

$$\frac{H_n}{K_n} = \frac{\left[(\kappa\dot{\tau}-\dot{\kappa}\tau)u^2-\dot{\tau}u\right]\sqrt{(\kappa u-1)^2+\tau^2u^2}}{2\tau^2}$$

v) *There is a relationship between K_n and δ_n as the following*

$$\frac{K_n}{\delta_n} = \frac{\tau(\kappa^2+\tau^2)}{\left[(1-Ku)^2+\tau^2u^2\right]^2}\,.$$

Result 3.3.

i) *Since $\delta_b \neq 0$, M_3- surface is not developable,*

ii) *There is a relationship between K_b and H_b as the following*

$$\frac{H_b}{K_b} = -\frac{\left(\kappa\tau^2 z^2+\dot{\tau}z+\kappa\right)\sqrt{\tau^2 z^2-1}}{2\tau^2}\,,$$

ii) *There is a relationship between K_b and δ_b as the following*

$$\frac{K_b}{\delta_b} = -\frac{\tau^3}{\left(1-\tau^2 z^2\right)^2}\,.$$

We reach the following theorem using theorem3.1.

Theorem 3.4. *$\vec{\alpha}$ is a helix if and only if $\dfrac{\delta_b}{\delta_n}$ is constant.*

The unit normal vector $\vec{N}$ on the ruled surface of M_4 is given by

(3.3)

$$\vec{N} = \frac{\phi_{4s}\Lambda\phi_{4w}}{\|\phi_{4s}\Lambda\phi_{4w}\|} = \frac{\dot{\vec{\alpha}}(s)\Lambda\vec{X}(s)+w\dot{\vec{X}}(s)\Lambda\vec{X}(s)}{\|\phi_{4s}\Lambda\phi_{4w}\|}$$

.

Thus, from (2.2) and (3.3), the unit normal vector to surface M_4 at the point $(s,0)$ is

$$\text{(3.4)}\quad \vec{N}(s,0) = -\frac{x_2\vec{b}+x_3\vec{n}}{\sqrt{x_2^2-x_3^2}}\,.$$

Therefore, from (2.1)-(2.3), (2.5)-(2.7) and (3.4), the following theorem may be given.

Theorem 3. 5. *The distribution parameter, the mean curvature and the Gaussian curvature of surface M_4 are*

$$\delta_X = \frac{x_2\dot{x}_3+x_3\dot{x}_2+\left(x_2^2+x_3^2\right)\tau+x_1x_3\kappa}{\left(\dot{x}_1-x_2\kappa\right)^2-\left(\dot{x}_2+x_1\kappa+x_3\tau\right)^2+\left(\dot{x}_3+x_2\tau\right)^2}$$

,

$$H_X = \frac{2x_1x_2\dot{x}_3-2x_1\dot{x}_2x_3-2x_1\left(1-x_1^2\right)\tau+(x_3-2x_1^2x_3)\kappa}{2\left(1-x_1^2\right)\sqrt{1-x_1^2}}$$

and

$$K_X = \frac{\left[x_3\dot{x}_2 - x_2\dot{x}_3 + \left(1 - x_1^2\right)\tau + x_1x_3\kappa\right]^2}{\left(x_1^2 - 1\right)^2},$$

respectively.

So, we give the following results using theorem 3.5.

Result 3.4.

i) *If* M_4*- surface is minimal and* $\vec{X}$ *is oriented space-like line in the nb-plane then* $\kappa = 0$ *or* $x_3 = 0$.

ii) *If* M_4*- surface is minimal and* $\vec{X}$ *is oriented space-like line in the tb-plane then* $\frac{\tau}{\kappa} = \frac{x_3 - 2x_1^2x_3}{2x_1(1 - x_1^2)}$.

iii) *If* M_4*- surface is minimal and* $\vec{X}$ *is oriented space-like line in the tn-plane then* $\tau = 0$ *or* $x_1 = 0$.

Result 3.5.

ii) *If* M_4*- surface is developable and* $\vec{X}$ *is oriented space-like line in the nb-plane then* $\tau = \frac{x_2\dot{x}_3 + x_3\dot{x}_2}{x_2^2 + x_3^2}$.

ii) *If* M_4*- surface is developable and* $\vec{X}$ *is oriented space-like line in the tb-plane then* $\frac{\tau}{\kappa} = -\frac{x_1}{x_3}$.

iii) *If* M_4*- surface is developable and* $\vec{X}$ *is oriented space-like line in the tn-plane then* $\tau = 0$ or $x_2 = 0$.

From (2.8), the parametrization of the striction curve on the ruled surface generated by $\vec{X}$ oriented space-like line in IR_1^3 is given by

$$\vec{\beta}(s) = \vec{\alpha}(s) - \frac{\dot{x}_1 - x_2\kappa}{\left\|\dot{\vec{X}}\right\|^2}\vec{X}.$$

So, the following result may be given.

Result 3.6. *If the striction curve is the base curve of the ruled surface generated by* $\vec{X}$ *oriented space-like line in* IR_1^3 *then*

$$x_1 = \int x_2\kappa ds + c$$

is satisfied.

From (2.1)-(2.3) and (3.4), the geodesic curvature, normal curvature of the base curve and the geodesic torsion are

(3.6) $k_g = \left\langle \vec{N}\Lambda\vec{t}, \dot{\vec{t}}\right\rangle = -\frac{x_2\kappa}{\sqrt{\left|x_1^2 - 1\right|}}$,

(3.7) $k_n = \left\langle \ddot{\vec{\alpha}}, \vec{N}\right\rangle = \frac{x_3\kappa}{\sqrt{\left|x_1^2 - 1\right|}}$,

(3.8) $\tau_g = \left\langle \vec{N}\Lambda\dot{\vec{N}}, \dot{\vec{t}}\right\rangle = -\frac{x_2x_3\kappa^2}{\left|x_1^2 - 1\right|}$.

Theorem 3. 6. *There are the following relationships*

$k_g^2 - k_n^2 = \kappa^2$ *and* $\tau_g = k_gk_n$

between the geodesic torsion and curvatures of a space-like base curve.

Theorem 3. 7. *If the base curve of surface is a geodesic curve, then* $k_n = \mp\kappa$ *and* $\tau_g = 0$.

References

[1] Kobayashi, O., "Maximal surfaces in the 3-dimensional Minkowski space L^3", Tokyo Journal Mathematics, 6, 297-309, (1983).

[2] Woestijne, I.V., "Minimal surfaces in the 3-dimensional Minkowski space", Geometry and Topology of Submanifolds: II, Word Scientific, Singapore, 344-369, (1990).

[3] Kim, Y.H. and Yoon, D.W., "Classificiation of ruled surfaces in Minkowski 3-spaces", Journal of Geometry and Physics, 49, 89-100, (2004).

[4] Küçük, A., "On the devolopable time-like trajectory ruled surfaces in a Lorentz 3-space IR_1^3", Applied Mathematics and Computation, 157, 483, (2004).

[5] Aslaner, R., “Hyperruled surfaces in Minkowski IR_1^3”, Iranian Journal of science & Technology Transaction : A-Science, 29, A3, 341-347, (2005).
[6] Tosun, M. and Güngör, M.A., “A study on time-like complemantary ruled surfaces in the Minkowski n-space”, Iranian Journal of science & Technology Transaction : A-Science, 29 A2, 325-323, (2005).
[7] O’Neill, B., “Semi-Riemannian Geometry”, Academic Press, New York, (1983).
[8] Akutagawa, K. and Nshkawa, S., “The Gauss map and space-like surfaces with prescribed mean curvature in Minkowski 3-space”, Tohouku Mathematic Journal, 42, 67-82, (1990).
[9] Beem, J.K. and Ehrlich, P.E., “Global Lorentzian Geometry”, Marcel Dekker, New York, (1981).
[10] Turgut A. and Hacısalihoğlu, H.H., “Space-like ruled surfaces in the Minkowski 3-space”, Commun. Fac. Sci. Univ. Ank. Series A1-46, 83-91, (1997).

Geliş Tarihi:03/02/2009 *Kabul Tarihi:01/11/2010*