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QUASI-SASAKIAN STRUCTURES ON 5-DIMENSIONAL NILPOTENT LIE ALGEBRAS

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Abstract. In this study, we examine the existence of quasi-Sasakian structures on nilpotent Lie algebras of dimension five. In addition, we give some results about left invariant quasi-Sasakian structures on Lie groups of dimension five, whose Lie algebras are nilpotent. Moreover, subclasses of quasi-Sasakian structures are studied for some certain classes.

1. Introduction

It is known that there is a left invariant almost contact metric structure on any connected odd dimensional Lie group. These structures induce almost contact metric structures on corresponding Lie algebras [1]. Many authors have studied the concept of left invariant almost contact metric structures. In [2], 5-dimensional Lie algebras having Sasakian structures were studied and it was shown that the real Heisenberg group is the unique nilpotent Lie group with a left invariant Sasakian structure. In [3], 5-dimensional K-contact Lie algebras were studied. Also in [4], 5-dimensional cosymplectic, nearly cosymplectic, β-Kenmotsu, semi cosymplectic and almost cosymplectic structures are examined.

In this paper the existence of quasi-Sasakian structures on 5-dimensional nilpotent Lie algebras is investigated. Moreover, we state some theorems on the corresponding Lie groups.

2. Preliminaries

Assume that M^{2n+1} is a smooth manifold of dimension $2n+1$. An almost contact structure (ϕ, ξ, η) on M consists of a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η on M satisfying

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1. \tag{2.1}$$

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An almost contact manifold is a manifold with an almost contact structure. If M is also equipped with a Riemannian metric g holding

$$g(\phi(X), \phi(Y)) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for all vector fields X and Y, then M is called an almost contact metric manifold. We use the abbreviation a.c.m.s. for an almost contact metric structure. The metric g is called a compatible metric. The fundamental 2-form of the almost contact metric manifold (M, ϕ, ξ, η, g) is defined as

$$\Phi(X, Y) = g(X, \phi(Y)) \tag{2.3}$$

for all vector fields X, Y. In [5], a classification of almost contact metric manifolds was given. A space with the same symmetries as the covariant derivative of the fundamental 2-form was obtained and decomposed into twelve $U(n) \times 1$ irreducible components $C_1, \ldots, C_{12}$. Thus, there are 2^{12} invariant subclasses, see also [6].

Assume that (ϕ, ξ, η, g) is an a.c.m.s. on M having the fundamental 2-form Φ. The structure is said to be

- cosymplectic if $\nabla\Phi = 0$,
- normal if $[\phi, \phi] + d\eta \otimes \xi = 0$, where $[\phi, \phi]$ denotes the Nijenhuis torsion of ϕ,
- quasi-Sasakian ($C_6 \oplus C_7$) if the structure is normal and $d\Phi = 0$,
- α-Sasakian (C_6) if $\nabla_X \phi(Y) = \alpha(g(X, Y)\xi - \eta(Y)X)$ for some $\alpha \in \mathbb{R}$,
- C_7 if

$$(\nabla_X \Phi)(Y, Z) = \eta(Z)(\nabla_Y \eta)\phi(X) + \eta(Y)(\nabla_{\phi X}\eta)Z$$

  and $\delta\Phi = 0$ for all vector fields X, Y, Z on M.
- semi-cosymplectic ($C_1 \oplus C_2 \oplus C_3 \oplus C_7 \oplus C_8 \oplus C_9 \oplus C_{10} \oplus C_{11}$) if $\delta\Phi = 0$ and $\delta\eta = 0$, where δ is used for coderivative.

Note that the classes C_6 and $C_6 \oplus C_7 - (C_6 \cup C_7)$ are not contained in the class of semi-cosymplectic structures.

An a.c.m.s. (ϕ, ξ, η, g) on a connected Lie group G is called left invariant if the left multiplication $L_a : G \longrightarrow G$, $L_a(x) = a.x$ satisfies

$$\phi \circ L_a = L_a \circ \phi, \quad L_a(\xi) = \xi$$

for all $a \in G$ and g is left invariant.

For a Lie algebra $\mathfrak{g}$, let η be a 1-form, ϕ be an endomorphism and $\xi \in \mathfrak{g}$ with the property that

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1.$$

Then (ϕ, ξ, η, g) is called an a.c.m.s on the Lie algebra $\mathfrak{g}$, with the positive definite compatible inner product g. An a.c.m.s. (ϕ, ξ, η, g) on a Lie algebra $\mathfrak{g}$ is called nearly cosymplectic if $\nabla_X \Phi(X, Y) = 0$ for any X, Y in $\mathfrak{g}$, etc.

Let G be a connected Lie group with a left invariant almost contact metric structure (ϕ, ξ, η, g) and $\mathfrak{g} \cong T_eG$ be the corresponding Lie algebra of G. Then this structure uniquely induces an a.c.m.s. (ϕ, ξ, η, g) on $\mathfrak{g}$.

The nilpotent Lie algebras of dimension ≤ 5 were classified into nine classes $\mathfrak{g}_i$, $i = 1, 2, \cdots, 9$ with the basis $\{e_1, \ldots, e_5\}$ as follows[7] (refer also to [8, 9]):

$$\begin{aligned}
\mathfrak{g}_1 &: [e_1, e_2] = e_5, [e_3, e_4] = e_5 \\
\mathfrak{g}_2 &: [e_1, e_2] = e_3, [e_1, e_3] = e_5, [e_2, e_4] = e_5 \\
\mathfrak{g}_3 &: [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5 \\
\mathfrak{g}_4 &: [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5 \\
\mathfrak{g}_5 &: [e_1, e_2] = e_4, [e_1, e_3] = e_5 \\
\mathfrak{g}_6 &: [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_2, e_3] = e_5
\end{aligned}$$

The classes $\mathfrak{g}_7, \mathfrak{g}_8, \mathfrak{g}_9$ are abelian. In [4], it was proved that an a.c.m.s. on $\mathfrak{g}_i$, $i = 1, \ldots, 6$, is cosymplectic if and only if the fundamental 2-form of the structure is zero and also that almost contact metric structures on abelian Lie algebras are cosymplectic .

3. Quasi-Sasakian Structures on $\mathfrak{g}_i$

Consider a left invariant a.c.m.s. (ϕ, ξ, η, g) on a connected Lie group G. Same notations are used for the structures on $\mathfrak{g}$. The basis $\{e_1, \ldots, e_5\}$ is chosen such that basis elements are g-orthonormal.

It is known that the characteristic vector field of a quasi-Sasakian structure is Killing [10].

The algebra $\mathfrak{g}_1$: The nonzero covariant derivatives are computed using Kozsul's formula as:

$$\begin{aligned}
\nabla_{e_1} e_2 &= \tfrac{1}{2} e_5, & \nabla_{e_1} e_5 &= -\tfrac{1}{2} e_2, \\
\nabla_{e_2} e_1 &= -\tfrac{1}{2} e_5, & \nabla_{e_2} e_5 &= \tfrac{1}{2} e_1, \\
\nabla_{e_3} e_4 &= \tfrac{1}{2} e_5, & \nabla_{e_3} e_5 &= -\tfrac{1}{2} e_4, \\
\nabla_{e_4} e_3 &= -\tfrac{1}{2} e_5, & \nabla_{e_4} e_5 &= \tfrac{1}{2} e_3, \\
\nabla_{e_5} e_1 &= -\tfrac{1}{2} e_2, & \nabla_{e_5} e_2 &= \tfrac{1}{2} e_1, \\
\nabla_{e_5} e_3 &= -\tfrac{1}{2} e_4, & \nabla_{e_5} e_4 &= \tfrac{1}{2} e_3.
\end{aligned}$$

Let $\Phi = \sum_{i,j} b_{ij} e^{ij}$ be the fundamental 2-form of a quasi-Sasakian structure (ϕ, ξ, η, g) on $\mathfrak{g}_1$. From now on, Φ will denote the fundamental 2-form of a quasi-Sasakian structure on the corresponding Lie algebras. Since the characteristic vector field

ξ is Killing, $\xi = e_5$, see [4]. Since $\Phi(X,\xi) = 0$ for any vector field X, we have $b_{15} = b_{25} = b_{35} = b_{45} = 0$. Also, since $de^1 = de^2 = de^3 = de^4 = 0$ and $de^5 = -e^{12} - e^{34}$, we get $d\Phi = 0$. From the definition of the fundamental 2-form (2.3), the endomorphism ϕ is

$$\begin{aligned} \phi(e_1) &= -b_{12}e_2 - b_{13}e_3 - b_{14}e_4, \\ \phi(e_2) &= b_{12}e_1 - b_{23}e_3 - b_{24}e_4, \\ \phi(e_3) &= b_{13}e_1 + b_{23}e_2 - b_{34}e_4, \\ \phi(e_4) &= b_{14}e_1 + b_{24}e_2 + b_{34}e_3, \\ \phi(e_5) &= 0. \end{aligned} \tag{3.1}$$

Replacing X and Y by the vectors given below in the normality condition, we have

$$X = e_1, Y = e_2 \Rightarrow b_{12}^2 + b_{13}b_{24} - b_{14}b_{23} = 1, \tag{3.2}$$

$$X = e_3, Y = e_4 \Rightarrow b_{34}^2 + b_{13}b_{24} - b_{14}b_{23} = 1, \tag{3.3}$$

$$X = e_1, Y = e_3 \Rightarrow b_{13}(b_{12} + b_{34}) = 0, \tag{3.4}$$

$$X = e_1, Y = e_4 \Rightarrow b_{14}(b_{12} + b_{34}) = 0, \tag{3.5}$$

$$X = e_2, Y = e_4 \Rightarrow b_{24}(b_{12} + b_{34}) = 0, \tag{3.6}$$

$$X = e_2, Y = e_3 \Rightarrow b_{23}(b_{12} + b_{34}) = 0. \tag{3.7}$$

From the equations (3.2)-(3.7) and the relation (2.2), we get $b_{12}^2 = b_{34}^2$, $b_{13}^2 = b_{24}^2$ and $b_{14}^2 = b_{23}^2$. In addition, the coderivative $\delta\Phi$ is

$$\delta\Phi(X) = -\sum(\nabla_{e_i}\Phi)(e_i, X) = x_5(b_{12} + b_{34}) \tag{3.8}$$

for any vector $X = \sum x_i e_i$.

There are three cases:

First case: If $b_{12} = b_{34} = 0$, then $\delta\Phi = 0$. This means that the structure is in C_7, otherwise the quasi-Sasakian structure would be semi-cosymplectic, which is not the case.

Second case: If $b_{12} = b_{34} \neq 0$, then $b_{13} = b_{14} = b_{23} = b_{24} = 0$. Then from (3.1) and (2.2), we obtain that $b_{12} = b_{34} = \pm 1$. Thus the fundamental 2-form is given by $\Phi = \pm(e^{12} + e^{34})$. Obviously, this structure is α-Sasakian for $\alpha = \mp\frac{1}{2}$.

Third case: If $b_{12} = -b_{34} \neq 0$, then $\delta\Phi = 0$. This implies that the structure is in C_7 by similar arguments to the first case.

Therefore a quasi-Sasakian structure ($C_6 \oplus \mathcal{C}_7$) on $\mathfrak{g}_1$ is either in C_6 (α-Sasakian), or in C_7. That is, $C_6 \oplus C_7 = C_6 \cup C_7$.

The algebra $\mathfrak{g}_2$: Since a quasi-Sasakian structure has a Killing vector field, $\xi = e_5$, refer to [4]. Thus a quasi-Sasakian structure on $\mathfrak{g}_2$ has the characteristic vector field e_5. For $\Phi = \sum b_{ij}e^{ij}$, the relation $\Phi(X,\xi) = 0$ for any vector field X implies that $b_{15} = b_{25} = b_{35} = b_{45} = 0$. Besides, since $de^3 = -e^{12}$ and $de^5 = -e^{13} - e^{24}$, we get

$$d\Phi = 0 \text{ if and only if } \quad b_{34} = 0.$$

Also, from the definition of Φ, we get

$$\begin{aligned}\phi(e_1) &= -b_{12}e_2 - b_{13}e_3 - b_{14}e_4,\\ \phi(e_2) &= b_{12}e_1 - b_{23}e_3 - b_{24}e_4,\\ \phi(e_3) &= b_{13}e_1 + b_{23}e_2,\\ \phi(e_4) &= b_{14}e_1 + b_{24}e_2,\\ \phi(e_5) &= 0.\end{aligned}$$

Now we check the normality condition setting $X = e_1$, $Y = e_2$. In this case we have

$$\begin{aligned}&[\phi,\phi](e_1,e_2) + d\eta(e_1,e_2)e_5\\ &= (b_{12}^2 - 1)e_3 + (b_{12}(b_{13} + b_{24})e_5 = 0,\end{aligned}$$

which implies $b_{12}^2 = 1$ and $b_{13} = -b_{24}$. Since

$$g(\phi(e_1),\phi(e_1)) = b_{12}^2 + b_{13}^2 + b_{14}^2 = g(e_1,e_1) = 1,$$

we obtain $b_{13} = b_{14} = b_{24} = 0$. This yields that $\phi(e_4) = 0$, which is not the case since $g(\phi(e_4),\phi(e_4)) = g(e_4,e_4) = 1$. Thus there does not exist any quasi-Sasakian structure on $\mathfrak{g}_2$.

The algebra $\mathfrak{g}_3$: For a quasi-Sasakian structure on $\mathfrak{g}_3$, ξ should be e_5, otherwise ξ is not Killing [4]. For $\Phi = \sum b_{ij}e^{ij}$, we have $b_{15} = b_{25} = b_{35} = b_{45} = 0$ since $\Phi(X,\xi) = 0$. Since $de^3 = -e^{12}$, $de^4 = -e^{13}$ and $de^5 = -e^{14} - e^{23}$,

$$d\Phi = 0 \text{ if and only if } \quad b_{24} = b_{34} = 0.$$

From the equation (2.3), we get

$$\begin{aligned}\phi(e_1) &= -b_{12}e_2 - b_{13}e_3 - b_{14}e_4,\\ \phi(e_2) &= b_{12}e_1 - b_{23}e_3,\\ \phi(e_3) &= b_{13}e_1 + b_{23}e_2,\\ \phi(e_4) &= b_{14}e_1,\\ \phi(e_5) &= 0.\end{aligned}$$

The normality condition for $X = e_1$, $Y = e_2$ is

$$\begin{aligned}0 &= [\phi,\phi](e_1,e_2) + d\eta(e_1,e_2)e_5\\ &= b_{14}b_{23}e_1 + \left(b_{12}^2 - 1\right)e_3 + b_{12}b_{13}e_4 + b_{12}\left(b_{14} + b_{23}\right)e_5.\end{aligned}$$

Then $b_{12}^2 = 1$ and $b_{13} = b_{14} = b_{23} = 0$. This means that $\phi(e_4) = 0$, which is a contradiction since $g(\phi(e_4),\phi(e_4)) = g(e_4,e_4) = 1$. As a result, there does not exist any quasi-Sasakian structure on $\mathfrak{g}_3$.

The algebra $\mathfrak{g}_4$: The space of Killing vector fields on $\mathfrak{g}_4$ is $\langle e_5\rangle$ [4]. Thus e_5 is the characteristic vector field of a quasi-Sasakian structure. Let $\Phi = \sum b_{ij}e^{ij}$. Then

$b_{15} = b_{25} = b_{35} = b_{45} = 0$ since $\Phi(X, \xi) = 0$. Since $de^3 = -e^{12}$, $de^4 = -e^{13}$ and $de^5 = -e^{14}$,

$$d\Phi = 0 \text{ if and only if } \quad b_{24} = b_{34} = 0.$$

From the defining relation (2.3), we have

$$\begin{aligned} \phi(e_1) &= -b_{12}e_2 - b_{13}e_3 - b_{14}e_4, \\ \phi(e_2) &= b_{12}e_1 - b_{23}e_3, \\ \phi(e_3) &= b_{13}e_1 + b_{23}e_2, \\ \phi(e_4) &= b_{14}e_1, \end{aligned}$$

Set $X = e_1$, $Y = e_2$, then we have

$$\begin{aligned} &[\phi, \phi](e_1, e_2) + d\eta(e_1, e_2)e_5 \\ &\qquad = b_{14}b_{23}e_1 + \left(b_{12}^2 - 1\right) e_3 + b_{12}b_{13}e_4 + b_{12}b_{14}e_5 = 0. \end{aligned}$$

Then $b_{12}^2 = 1$ and $b_{13} = b_{14} = 0$. Thus $\phi(e_4) = 0$, which contradicts with the condition (2.2). Thus there does not exist any quasi-Sasakian structure on $\mathfrak{g}_4$.

The algebra $\mathfrak{g}_5$: On this Lie algebra, the space of Killing vector fields is spanned by e_4, e_5 [4]. Thus the characteristic vector field is $\xi = a_4e_4 + a_5e_5$ and $\eta = b_4e^4 + b_5e^5$. If $\Phi = \sum b_{ij}e^{ij}$, then since $de^1 = de^2 = de^3 = 0$, $de^4 = -e^{12}$, $de^5 = -e^{13}$,

$$d\Phi = 0 \text{ if and only if } \quad b_{45} = 0 \text{ and } b_{25} = b_{34}.$$

From the equation (2.3), we get

$$\begin{aligned} \phi(e_1) &= -b_{12}e_2 - b_{13}e_3 - b_{14}e_4 - b_{15}e_5, \\ \phi(e_2) &= b_{12}e_1 - b_{23}e_3 - b_{24}e_4 - b_{25}e_5, \\ \phi(e_3) &= b_{13}e_1 + b_{23}e_2 - b_{25}e_4 - b_{35}e_5, \\ \phi(e_4) &= b_{14}e_1 + b_{24}e_2 + b_{25}e_3, \\ \phi(e_5) &= b_{15}e_1 + b_{25}e_2 + b_{35}e_3. \end{aligned}$$

Now we check the normality condition for $X = e_1$, $Y = e_4$. We have

$$\begin{aligned} &[\phi, \phi](e_1, e_4) + d\eta(e_1, e_4)\xi \\ &\qquad = -\left(b_{14}b_{24} + b_{15}b_{25}\right) e_1 - \left(b_{24}^2 + b_{25}^2\right) e_2 \\ &\qquad\qquad - b_{25}\left(b_{24} + b_{35}\right) e_3 + b_{12}b_{14}e_4 + b_{13}b_{14}e_5 = 0, \end{aligned}$$

then $b_{24} = b_{25} = 0$. For $X = e_2$, $Y = e_4$,

$$[\phi, \phi](e_2, e_4) + d\eta(e_2, e_4)\xi = b_{14}^2e_1 + b_{14}b_{23}e_5 = 0.$$

Then $b_{14} = 0$. This implies that $\phi(e_4) = 0$, which is a contradiction. Thus there does not exist any quasi-Sasakian structure on $\mathfrak{g}_5$.

The algebra $\mathfrak{g}_6$: A vector field ξ on $\mathfrak{g}_6$ is Killing if and only if ξ is in the space $\langle e_4, e_5 \rangle$ [4]. Thus the characteristic vector field should be $\xi = a_4e_4 + a_5e_5$ and

$\eta = b_4 e^4 + b_5 e^5$. Let $\Phi = \sum b_{ij} e^{ij}$ be the fundamental 2-form of a quasi-Sasakian structure on $\mathfrak{g}_6$. Since $de^1 = de^2 = 0$, $de^3 = -e^{12}$, $de^4 = -e^{13}$, $de^5 = -e^{23}$,

$$d\Phi = 0 \text{ if and only if } \quad b_{34} = b_{35} = b_{45} = 0 \text{ and } b_{15} = b_{24}.$$

From (2.3), we obtain

$$\phi(e_1) = -b_{12}e_2 - b_{13}e_3 - b_{14}e_4 - b_{15}e_5,$$

$$\phi(e_2) = b_{12}e_1 - b_{23}e_3 - b_{15}e_4 - b_{25}e_5,$$

$$\phi(e_3) = b_{13}e_1 + b_{23}e_2,$$

$$\phi(e_4) = b_{14}e_1 + b_{15}e_2,$$

$$\phi(e_5) = b_{15}e_1 + b_{25}e_2.$$

From the normality, if $X = e_1$, $Y = e_2$, then we have

$$\begin{aligned}[\phi,\phi](e_1,e_2) &+ d\eta(e_1,e_2)\xi \\ &= (b_{14}b_{23} - b_{13}b_{15})\, e_1 + (b_{15}b_{23} - b_{13}b_{25})\, e_2 \\ &\qquad + \left(b_{12}^2 - 1\right) e_3 + b_{12}b_{13}e_4 + b_{12}b_{23}e_5 = 0.\end{aligned}$$

Then $b_{12}^2 = 1$ and $b_{13} = b_{23} = 0$. This yields $\phi(e_3) = 0$, which contradicts with the fact that $g(\phi(e_3), \phi(e_3)) = g(e_3, e_3) = 1$. Thus there does not exist any quasi-Sasakian structure on $\mathfrak{g}_6$.

We combine our results in the followings.

Theorem 1. *A quasi-Sasakian structure on $\mathfrak{g}_1$ is either α-Sasakian or in C_7. That is,*

$$C_6 \oplus C_7 = C_6 \cup C_7.$$

Theorem 2. *An almost contact metric structure on a five dimensional nilpotent Lie algebra $\mathfrak{g}$ is quasi-Sasakian if and only if $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_1$.*

This theorem yields

Corollary 1. *There is no left invariant quasi-Sasakian structure on a five dimensional connected Lie group whose corresponding Lie algebra is not isomorphic to $\mathfrak{g}_1$.*

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