



Two Dimensional Čebyšev Type Inequalities for Functions Whose Second Derivatives are Co-ordinated (h_1, h_2) -Preinvex

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Abstract

In this paper, we extend the identity established in [2] for preinvex functions. Using this novel identity we establish some new Čebyšev type inequalities involving functions of two independent variable whose mixed derivatives are co-ordinated (h_1, h_2) -preinvex.

Keywords: Čebyšev type inequalities, co-ordinated (h_1, h_2) -preinvex, integral inequality

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1. Introduction

It is well known that the convexity plays an important and central role in many areas, such as economic, finance, optimization, and game theory. Due to its diverse applications this concept has been extended and generalized in several directions. One of the significant generalization is that introduced by Hanson [6], where he introduced the concept of invexity. In [3], the authors gave the notion of preinvex functions which is special case of invexity. Many authors have study the basic properties of invex set and preinvex functions, and their role in optimization, variational inequalities and equilibrium problems, see [13, 14, 21, 24, 25]. About some papers involving the generalized convexity one can see [7, 15, 16, 17, 18, 23] and references therein.

In 1882, Čebyšev [4] gave the following inequality

$$|T(f, g)| \leq \frac{1}{12} (b-a)^2 \|f'\|_{\infty} \|g'\|_{\infty}, \quad (1.1)$$

where $f, g : [a, b] \rightarrow \mathbb{R}$ are absolutely continuous functions, whose first derivatives f' and g' are bounded, with

$$T(f, g) = \frac{1}{b-a} \int_a^b f(x)g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \quad (1.2)$$

and $\|\cdot\|_{\infty}$ denotes the norm in $L_{\infty}[a, b]$ defined as $\|f\|_{\infty} = \text{ess sup}_{t \in [a, b]} |f(t)|$.

During the past few years, many researchers have given considerable attention to the inequality (1.1). Various generalizations, extensions and variants of the above inequality have appeared in the literature, we can mention the works [1, 19, 20], and references therein.

Recently, Barnett and Dragomir [2] gave an analogue of the functional (1.2) for functions of two variables. Guezane-Lakoud and Aissaoui [5] have used this identity and established some new Čebyšev type inequalities for functions whose mixed derivatives are bounded. Sarikaya et al. [22] treated the case where the mixed derivatives are convex on the co-ordinates. Meftah et al. [9, 10, 11, 12] have discussed the cases where the mixed derivatives are on co-ordinated (s_1, m_1) - (s_2, m_2) -convex, (s, r) -convex in the first sense, (h_1, h_2) -convex, and log-convex. Motivated by the above results, in this paper we extend the identity given in [2] for preinvex functions. Using this novel identity we derive some new Čebyšev type inequalities for functions of two independent variable whose modulus of their mixed derivatives are (h_1, h_2) -preinvex on the co-ordinates.

2. Preliminaries

In this section we recall some definitions

Let K_1, K_2 be a nonempty closed subset of \mathbb{R} , and let $f : K \rightarrow \mathbb{R}$ and $\eta_1 : K_1 \times K_1 \rightarrow \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \rightarrow \mathbb{R}$ be continuous bifunctions.

Definition 2.1. [8] Let $(u, v) \in K_1 \times K_2$. We say $K_1 \times K_2$ is invex at (u, v) with respect to η_1 and η_2 , if for each $(x, y) \in K_1 \times K_2$ and $t_1, t_2 \in [0, 1]$ we have

$$(u + t_1 \eta_1(x, u), v + t_2 \eta_2(y, v)) \in K_1 \times K_2.$$

Remark 2.1. $K_1 \times K_2$ is said to be an invex set with respect to η_1 and η_2 if $K_1 \times K_2$ is invex at each $(u, v) \in K_1 \times K_2$.

Definition 2.2. [8] Let h_1 and h_2 be non-negative functions on $[0, 1]$, $h_1 \neq 0, h_2 \neq 0$. A non-negative function $f : K_1 \times K_2 \rightarrow \mathbb{R}$ is said to be co-ordinated (h_1, h_2) -preinvex on $K_1 \times K_2$, if the following inequality

$$\begin{aligned} & f(u + \lambda \eta_1(x, u), v + t \eta_2(y, v)) \\ & \leq h_1(1 - \lambda)h_2(1 - t)f(u, v) + h_1(1 - \lambda)h_2(t)f(u, y) \\ & \quad + h_1(\lambda)h_2(1 - t)f(x, v) + h_1(\lambda)h_2(t)f(x, y) \end{aligned}$$

holds for $(x, y), (x, v), (u, y), (u, v) \in K_1 \times K_2$ and $\lambda, t \in [0, 1]$.

3. Main result

In what follows, we assume that K_1, K_2 are two invex set of \mathbb{R} with respect to η_1 and η_2 respectively, where $\eta_1 : K_1 \times K_1 \rightarrow \mathbb{R}, \eta_2 : K_2 \times K_2 \rightarrow \mathbb{R}$ are two continuous bifunctions, $\Delta_1 = [a, a + \eta_1(b, a)] \subset K_1 \subset \mathbb{R}, \Delta_2 = [c, c + \eta_2(d, c)] \subset K_2 \subset \mathbb{R}$ and $\Delta = \Delta_1 \times \Delta_2$ with $a < a + \eta_1(b, a)$ and $c < c + \eta_2(d, c)$, and $f_{\lambda\alpha} = \frac{\partial^2 f}{\partial \lambda \partial \alpha}, h_1, h_2 : [0, 1] \rightarrow [0, \infty)$ such that $h_1 \neq 0, h_2 \neq 0$.

Lemma 3.1. Let $f : \Delta \rightarrow \mathbb{R}$ be a partially differentiable mapping on Δ . If $f_{\lambda\alpha} \in L_1(\Delta)$, then the following identity holds

$$\begin{aligned} & f(x, y) - \frac{1}{\eta_1(b, a)} \int_a^{a + \eta_1(b, a)} f(u, y) du - \frac{1}{\eta_2(d, c)} \int_c^{c + \eta_2(d, c)} f(x, v) dv \\ & + \frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_a^{a + \eta_1(b, a)} \int_c^{c + \eta_2(d, c)} f(u, v) dv du \\ & = \eta_1(b, a)\eta_2(d, c) \int_0^1 \int_0^1 P(x, \lambda)Q(y, \alpha) f_{\lambda\alpha}(a + \lambda \eta_1(b, a), c + \alpha \eta_2(d, c)) d\alpha d\lambda, \end{aligned} \tag{3.1}$$

where $P, Q : \Delta \rightarrow \mathbb{R}$

$$P(x, \lambda) = \begin{cases} \lambda & \text{if } 0 \leq \lambda \leq \frac{x-a}{\eta_1(b, a)} \\ \lambda - 1 & \text{if } \frac{x-a}{\eta_1(b, a)} \leq \lambda \leq 1, \end{cases} \tag{3.2}$$

$$Q(y, \alpha) = \begin{cases} \alpha & \text{if } 0 \leq \alpha \leq \frac{y-c}{\eta_2(d, c)} \\ \alpha - 1 & \text{if } \frac{y-c}{\eta_2(d, c)} \leq \alpha \leq 1. \end{cases} \tag{3.3}$$

Proof. Let the quantities

$$I = \int_0^1 \int_0^1 P(x, \lambda)Q(y, \alpha) f_{\lambda\alpha}(a + \lambda \eta_1(b, a), c + \alpha \eta_2(d, c)) d\alpha d\lambda,$$

where P, Q are as in (3.2) and (3.3) respectively,

$$J_1 = \int_0^{\frac{x-a}{\eta_1(b, a)}} \int_0^{\frac{y-c}{\eta_2(d, c)}} \lambda \alpha f_{\lambda\alpha}(a + \lambda \eta_1(b, a), c + \alpha \eta_2(d, c)) d\alpha d\lambda,$$

$$J_2 = \int_0^{\frac{x-a}{\eta_1(b, a)}} \int_{\frac{y-c}{\eta_2(d, c)}}^1 \lambda (\alpha - 1) f_{\lambda\alpha}(a + \lambda \eta_1(b, a), c + \alpha \eta_2(d, c)) d\alpha d\lambda,$$

$$J_3 = \int_{\frac{x-a}{\eta_1(b, a)}}^1 \int_0^{\frac{y-c}{\eta_2(d, c)}} (\lambda - 1) \alpha f_{\lambda\alpha}(a + \lambda \eta_1(b, a), c + \alpha \eta_2(d, c)) d\alpha d\lambda,$$

and

$$J_4 = \int_{\frac{x-a}{\eta_1(b,a)}}^1 \int_{\frac{y-c}{\eta_2(d,c)}}^1 (\lambda-1)(\alpha-1) f_{\lambda\alpha}(a+\lambda\eta_1(b,a), c+\alpha\eta_2(d,c)) d\alpha d\lambda.$$

Clearly

$$I = J_1 + J_2 + J_3 + J_4. \quad (3.4)$$

Integrating by parts J_1, J_2, J_3 and J_4 , we have

$$\begin{aligned} J_1 &= \frac{(x-a)(y-c)}{(\eta_1(b,a))^2(\eta_2(d,c))^2} f(x,y) - \frac{y-c}{\eta_1(b,a)(\eta_2(d,c))^2} \int_0^{\frac{x-a}{\eta_1(b,a)}} f(a+\lambda\eta_1(b,a), y) d\lambda \\ &\quad - \frac{x-a}{\eta_2(d,c)(\eta_1(b,a))^2} \int_0^{\frac{y-c}{\eta_2(d,c)}} f(x, c+\alpha\eta_2(d,c)) d\alpha \\ &\quad + \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_0^{\frac{y-c}{\eta_2(d,c)}} \int_0^{\frac{x-a}{\eta_1(b,a)}} f(a+\lambda\eta_1(b,a), c+\alpha\eta_2(d,c)) d\alpha d\lambda. \end{aligned} \quad (3.5)$$

$$\begin{aligned} J_2 &= \frac{(x-a)(c+\eta_2(d,c)-y)}{(\eta_1(b,a))^2(\eta_2(d,c))^2} f(x,y) - \frac{c+\eta_2(d,c)-y}{\eta_1(b,a)(\eta_2(d,c))^2} \int_0^{\frac{x-a}{\eta_1(b,a)}} f(a+\lambda\eta_1(b,a), y) d\lambda \\ &\quad - \frac{x-a}{(\eta_1(b,a))^2\eta_2(d,c)} \int_0^1 f(x, c+\alpha\eta_2(d,c)) d\alpha \\ &\quad + \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_0^{\frac{x-a}{\eta_1(b,a)}} \int_{\frac{y-c}{\eta_2(d,c)}}^1 f(a+\lambda\eta_1(b,a), c+\alpha\eta_2(d,c)) d\alpha d\lambda. \end{aligned} \quad (3.6)$$

$$\begin{aligned} J_3 &= \frac{(a+\eta_1(b,a)-x)(y-c)}{(\eta_1(b,a))^2(\eta_2(d,c))^2} f(x,y) - \frac{y-c}{\eta_1(b,a)(\eta_2(d,c))^2} \int_{\frac{x-a}{\eta_1(b,a)}}^1 f(a+\lambda\eta_1(b,a), y) d\lambda \\ &\quad - \frac{a+\eta_1(b,a)-x}{(\eta_1(b,a))^2\eta_2(d,c)} \int_0^{\frac{y-c}{\eta_2(d,c)}} f(x, c+\alpha\eta_2(d,c)) d\alpha \\ &\quad + \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_{\frac{x-a}{\eta_1(b,a)}}^1 \int_0^{\frac{y-c}{\eta_2(d,c)}} f(a+\lambda\eta_1(b,a), c+\alpha\eta_2(d,c)) d\alpha d\lambda. \end{aligned} \quad (3.7)$$

$$\begin{aligned} J_4 &= \frac{(a+\eta_1(b,a)-x)(c+\eta_2(d,c)-y)}{(\eta_1(b,a))^2(\eta_2(d,c))^2} f(x,y) - \frac{c+\eta_2(d,c)-y}{\eta_1(b,a)(\eta_2(d,c))^2} \int_{\frac{x-a}{\eta_1(b,a)}}^1 f(a+\lambda\eta_1(b,a), y) d\lambda \\ &\quad - \frac{a+\eta_1(b,a)-x}{(\eta_1(b,a))^2\eta_2(d,c)} \int_{\frac{y-c}{\eta_2(d,c)}}^1 f(x, c+\alpha\eta_2(d,c)) d\alpha \\ &\quad + \frac{1}{\eta_1(b,a)\eta_2(d,c)} \int_{\frac{x-a}{\eta_1(b,a)}}^1 \int_{\frac{y-c}{\eta_2(d,c)}}^1 f(a+\lambda\eta_1(b,a), c+\alpha\eta_2(d,c)) d\alpha d\lambda. \end{aligned} \quad (3.8)$$

Substituting (3.5)-(3.8) in (3.4), and then multiplying both sides of the result by $\eta_1(b,a)\eta_2(d,c)$, we obtain

$$\begin{aligned} \eta_1(b,a)\eta_2(d,c)I &= f(x,y) - \int_0^1 f(a+\lambda\eta_1(b,a), y) d\lambda - \int_0^1 f(x, c+\alpha\eta_2(d,c)) d\alpha \\ &\quad + \int_0^1 \int_0^1 f(a+\lambda\eta_1(b,a), c+\alpha\eta_2(d,c)) d\alpha d\lambda. \end{aligned} \quad (3.9)$$

Putting in (3.9) $u = a + \lambda \eta_1(b, a)$ and $v = c + \alpha \eta_2(d, c)$, we get

$$\begin{aligned} \eta_1(b, a)\eta_2(d, c)I &= f(x, y) - \frac{1}{\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f(u, y) du - \frac{1}{\eta_2(c, d)} \int_c^{c+\eta_2(d, c)} f(x, v) dv \\ &\quad + \frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_a^{a+\eta_1(b, a)c+\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f(u, v) dvdu, \end{aligned}$$

which is the desired result. □

Theorem 3.1. Let $f, g : \Delta \rightarrow \mathbb{R}$ be partially differentiable functions such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ . If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are co-ordinated (h_1, h_2) -preinvex functions with respect to η_1, η_2 respectively, then we have

$$\begin{aligned} |T(f, g)| &\leq \frac{1}{2} \left(\int_0^1 h_1(\lambda) d\lambda \right) \left(\int_0^1 h_2(\alpha) d\alpha \right) \\ &\quad \times \int_a^{a+\eta_1(b, a)c+\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} (N|f(x, y)| + M|g(x, y)|) dydx \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} T(f, g) &= \frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_a^{a+\eta_1(b, a)c+\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f(x, y)g(x, y) dydx \\ &\quad - \frac{1}{(\eta_1(b, a))^2\eta_2(d, c)} \int_a^{a+\eta_1(b, a)c+\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f(x, y) \int_a^{a+\eta_1(b, a)} g(u, y) dudydx \\ &\quad - \frac{1}{\eta_1(b, a)(\eta_2(d, c))^2} \int_a^{a+\eta_1(b, a)c+\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f(x, y) \int_c^{c+\eta_2(d, c)} g(x, v) dvdydx \\ &\quad + \frac{1}{(\eta_1(b, a))^2(\eta_2(d, c))^2} \int_a^{a+\eta_1(b, a)c+\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f(x, y)dydx \int_a^{a+\eta_1(b, a)c+\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} g(x, y)dydx, \end{aligned} \tag{3.11}$$

$$M = \text{ess sup}_{x, t \in \Delta_1, y, s \in \Delta_2} [|f_{\lambda\alpha}(x, y)| + |f_{\lambda\alpha}(x, s)| + |f_{\lambda\alpha}(t, y)| + |f_{\lambda\alpha}(t, s)|], \text{ and}$$

$$N = \text{ess sup}_{x, t \in \Delta_1, y, s \in \Delta_2} [|g_{\lambda\alpha}(x, y)| + |g_{\lambda\alpha}(x, s)| + |g_{\lambda\alpha}(t, y)| + |g_{\lambda\alpha}(t, s)|].$$

Proof. From Lemma 3.1, we have

$$\begin{aligned} &f(x, y) - \frac{1}{\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} f(u, y) du - \frac{1}{\eta_2(c, d)} \int_c^{c+\eta_2(d, c)} f(x, v) dv \\ &\quad + \frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_a^{a+\eta_1(b, a)c+\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} f(u, v) dvdu \\ &= \eta_1(b, a)\eta_2(d, c) \int_0^1 \int_0^1 P(x, \lambda)Q(y, \alpha) f_{\lambda\alpha}(a + \lambda \eta_1(b, a), c + \alpha \eta_2(d, c)) d\alpha d\lambda, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &g(x, y) - \frac{1}{\eta_1(b, a)} \int_a^{a+\eta_1(b, a)} g(u, y) du - \frac{1}{\eta_2(c, d)} \int_c^{c+\eta_2(d, c)} g(x, v) dv \\ &\quad + \frac{1}{\eta_1(b, a)\eta_2(d, c)} \int_a^{a+\eta_1(b, a)c+\eta_2(d, c)} \int_c^{c+\eta_2(d, c)} g(u, v) dvdu \\ &= \eta_1(b, a)\eta_2(d, c) \int_0^1 \int_0^1 P(x, \lambda)Q(y, \alpha) g_{\lambda\alpha}(a + \lambda \eta_1(b, a), c + \alpha \eta_2(d, c)) d\alpha d\lambda. \end{aligned} \tag{3.13}$$

Multiplying both sides of (3.12) by $\frac{1}{2\eta_1(b,a)\eta_2(d,c)} g(x,y)$ and (3.13) by $\frac{1}{2\eta_1(b,a)\eta_2(d,c)} f(x,y)$ summing the resulting equalities, and then integrating them with respect to x,y over Δ , using the modulus, we obtain

$$\begin{aligned} |T(f,g)| &\leq \frac{1}{2} \int_a^{a+\eta_1(b,a)c+\eta_2(d,c)} \int_c^c \left(|f(x,y)| \int_0^1 \int_0^1 |P(x,t)Q(y,s)| \right. \\ &\quad \times |g_{\lambda\alpha}(a+\lambda\eta_1(b,a),c+\alpha\eta_2(d,c))| d\alpha d\lambda + |g(x,y)| \\ &\quad \left. \times \int_0^1 \int_0^1 |P(x,\lambda)Q(y,\alpha)| |f_{\lambda\alpha}(a+\lambda\eta_1(b,a),c+\alpha\eta_2(d,c))| d\alpha d\lambda \right) dydx, \end{aligned}$$

where $T(f,g)$ is defined as in (3.11).

Using (h_1, h_2) -preinvexity of $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$, we get

$$\begin{aligned} |T(f,g)| &\leq \frac{1}{2} \int_a^{a+\eta_1(b,a)c+\eta_2(d,c)} \int_c^c \left(|f(x,y)| \int_0^1 \int_0^1 |P(x,\lambda)Q(y,\alpha)| \right. \\ &\quad \times (h_1(1-\lambda)h_2(1-\alpha)|g_{\lambda\alpha}(a,c)| + h_1(1-\lambda)h_2(\alpha)|g_{\lambda\alpha}(a,d)| \\ &\quad + h_1(\lambda)h_2(1-\alpha)|g_{\lambda\alpha}(b,c)| + h_1(\lambda)h_2(\alpha)|g_{\lambda\alpha}(b,d)|) d\alpha d\lambda \\ &\quad + |g(x,y)| \int_0^1 \int_0^1 |P(x,\lambda)Q(y,\alpha)| (h_1(1-\lambda)h_2(1-\alpha)|f_{\lambda\alpha}(a,c)| \\ &\quad + h_1(1-\lambda)h_2(\alpha)|f_{\lambda\alpha}(a,d)| + h_1(\lambda)h_2(1-\alpha)|f_{\lambda\alpha}(b,c)| \\ &\quad + h_1(\lambda)h_2(\alpha)|f_{\lambda\alpha}(b,d)|) d\alpha d\lambda \Big) dydx \\ &\leq \frac{1}{2} \int_a^{a+\eta_1(b,a)c+\eta_2(d,c)} \int_c^c (N|f(x,y)| + M|g(x,y)|) \\ &\quad \times \left(\left(\int_0^1 h_1(1-\lambda)|P(x,\lambda)| d\lambda \right) \left(\int_0^1 h_2(1-\alpha)|Q(y,\alpha)| d\alpha \right) \right. \\ &\quad + \left(\int_0^1 h_1(1-\lambda)|P(x,\lambda)| d\lambda \right) \left(\int_0^1 h_2(\alpha)|Q(y,\alpha)| d\alpha \right) \\ &\quad + \left(\int_0^1 h_1(\lambda)|P(x,\lambda)| d\lambda \right) \left(\int_0^1 h_2(1-\alpha)|Q(y,\alpha)| d\alpha \right) \\ &\quad \left. + \left(\int_0^1 h_1(\lambda)|P(x,\lambda)| d\lambda \right) \left(\int_0^1 h_2(\alpha)|Q(y,\alpha)| d\alpha \right) \right) dydx \\ &= \frac{1}{2} \int_a^{a+\eta_1(b,a)c+\eta_2(d,c)} \int_c^c (N|f(x,y)| + M|g(x,y)|) \\ &\quad \times \left(\left(\int_0^1 h_1(\lambda)|P(x,1-\lambda)| d\lambda \right) \left(\int_0^1 h_2(\alpha)|Q(y,1-\alpha)| d\alpha \right) \right. \\ &\quad + \left(\int_0^1 h_1(\lambda)|P(x,1-\lambda)| d\lambda \right) \left(\int_0^1 h_2(\alpha)|Q(y,\alpha)| d\alpha \right) \\ &\quad + \left(\int_0^1 h_1(\lambda)|P(x,\lambda)| d\lambda \right) \left(\int_0^1 h_2(\alpha)|Q(y,1-\alpha)| d\alpha \right) \\ &\quad \left. + \left(\int_0^1 h_1(\lambda)|P(x,\lambda)| d\lambda \right) \left(\int_0^1 h_2(\alpha)|Q(y,\alpha)| d\alpha \right) \right) dydx \\ &= \frac{1}{2} \int_a^{a+\eta_1(b,a)c+\eta_2(d,c)} \int_c^c (N|f(x,y)| + M|g(x,y)|) \end{aligned}$$

$$\begin{aligned} & \times \left(\left(\int_0^1 h_1(\lambda) (|P(x, 1-\lambda)| + |P(x, \lambda)|) d\lambda \right) \right. \\ & \left. \times \left(\int_0^1 h_2(\alpha) (|Q(y, 1-\alpha)| + |Q(y, \alpha)|) d\alpha \right) \right) dydx. \end{aligned} \tag{3.14}$$

Clearly for all $\lambda, \alpha \in [0, 1]$, we have

$$|P(x, 1-\lambda)| + |P(x, \lambda)| = |Q(y, 1-\alpha)| + |Q(y, \alpha)| = 1. \tag{3.15}$$

Substituting (3.15) in (3.14) we get the desired result. □

Theorem 3.2. Assume that all the hypotheses of Theorem 3.1 are satisfied, then we have

$$|T(f, g)| \leq \eta_1^2(b, a) \eta_2^2(d, c) MN \left(\int_0^1 h_1(\lambda) d\lambda \right)^2 \left(\int_0^1 h_2(\alpha) d\alpha \right)^2,$$

where $T(f, g)$ is defined as in (3.11) and M, N as in Theorem 3.1.

Proof. Clearly from Lemma 3.1, the equalities (3.12) and (3.13) are holds. Multiplying (3.12) by (3.13) by $\frac{1}{\eta_1(b, a)\eta_2(d, c)}$, integrating the resulting equality with respect to x, y over Δ , and then using the modulus in both sides, we get

$$\begin{aligned} |T(f, g)| & \leq \eta_1(b, a) \eta_2(d, c) \int_a^{a+\eta_1(b, a)c+\eta_2(d, c)} \int_c^{c+\eta_1(b, a)c+\eta_2(d, c)} \left(\int_0^1 \int_0^1 |P(x, \lambda)| |Q(y, \alpha)| \right. \\ & \times |f_{\lambda\alpha}(a + \lambda\eta_1(b, a), c + \alpha\eta_2(d, c))| d\alpha d\lambda \left(\int_0^1 \int_0^1 |P(x, \lambda)| \right. \\ & \left. \left. \times |Q(y, \alpha)| |g_{\lambda\alpha}(a + \lambda\eta_1(b, a), c + \alpha\eta_2(d, c))| d\alpha d\lambda \right) dydx. \end{aligned}$$

Using the (h_1, h_2) -preinvex on the co-ordinates of $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ and (3.15), we get

$$\begin{aligned}
|T(f, g)| &\leq \eta_1(b, a)\eta_2(d, c) \\
&\quad \times \int_a^{a+\eta_1(b, a)c+\eta_2(d, c)} \int_c^c \left(\int_0^1 \int_0^1 |P(x, \lambda)| |Q(y, \alpha)| \right. \\
&\quad \times (h_1(1-\lambda)h_2(1-\alpha)|f_{\lambda\alpha}(a, c)| + h_1(1-\lambda)h_2(\alpha)|f_{\lambda\alpha}(a, d)| \\
&\quad \left. + h_1(\lambda)h_2(1-\alpha)|f_{\lambda\alpha}(b, c)| + h_1(\lambda)h_2(\alpha)|f_{\lambda\alpha}(b, d)| \right) d\alpha d\lambda \\
&\quad \times \left(\int_0^1 \int_0^1 |P(x, \lambda)| |Q(y, \alpha)| \right. \\
&\quad \times (h_1(1-\lambda)h_2(1-\alpha)|g_{\lambda\alpha}(a, c)| + h_1(1-\lambda)h_2(\alpha)|g_{\lambda\alpha}(a, d)| \\
&\quad \left. + h_1(\lambda)h_2(1-\alpha)|g_{\lambda\alpha}(b, c)| + h_1(\lambda)h_2(\alpha)|g_{\lambda\alpha}(b, d)| \right) d\alpha d\lambda \, dydx \\
&\leq \eta_1(b, a)\eta_2(d, c)MN \\
&\quad \times \int_a^{a+\eta_1(b, a)c+\eta_2(d, c)} \int_c^c \left(\int_0^1 h_1(1-\lambda)|P(x, \lambda)| d\lambda \int_0^1 h_2(1-\alpha)|Q(y, \alpha)| d\alpha \right. \\
&\quad \left. + \int_0^1 h_1(1-\lambda)|P(x, \lambda)| d\lambda \int_0^1 h_2(\alpha)|Q(y, \alpha)| d\alpha \right. \\
&\quad \left. + \int_0^1 h_1(\lambda)|P(x, \lambda)| d\lambda \int_0^1 h_2(1-\alpha)|Q(y, \alpha)| d\alpha \right. \\
&\quad \left. + \int_0^1 h_1(\lambda)|P(x, \lambda)| d\lambda \int_0^1 h_2(\alpha)|Q(y, \alpha)| d\alpha \right)^2 dydx \\
&= \eta_1(b, a)\eta_2(d, c)MN \\
&\quad \times \int_a^{a+\eta_1(b, a)c+\eta_2(d, c)} \int_c^c \left(\left(\int_0^1 h_1(\lambda)(|P(x, 1-\lambda)| + |P(x, \lambda)|) d\lambda \right) \right. \\
&\quad \left. \times \left(\int_0^1 h_2(\alpha)(|Q(y, 1-\alpha)| + |Q(y, \alpha)|) d\alpha \right) \right)^2 dydx \\
&= \eta_1^2(b, a)\eta_2^2(d, c)MN \left(\int_0^1 h_1(\lambda) d\lambda \right)^2 \left(\int_0^1 h_2(\alpha) d\alpha \right)^2,
\end{aligned}$$

which is the desired result. \square

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