Konuralp Journal of Mathematics, 6 (1) (2018) 144-148

Konuralp Journal of Mathematics

Journal Homepage: *www.dergipark.gov.tr/konuralpjournalmath*
e-ISSN: 2147-625X

Metric Properties of a Space-like Rational Bezier Curve with a Time-like Principal Normal

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Abstract

Rational Bezier curves are one of the most important computer graphics for the computer-aided geometric design (CAGD). This paper presents certain metric properties of a space-like rational Bezier curve with time-like principal normal in Minkowski 3-space. First, the Bezier curve is defined in Minkowski space. Then, Serret-Frenet frames, the curvatures, and the torsions of the curve are obtained for each space-like condition. Finally, the derivative formulas of the curve are calculated, and a numerical example is given.

Keywords: *Bezier curve, CAGD, Minkowski space, space-like curve.*
2010 Mathematics Subject Classification: *53A04, 51B20.*

1. Introduction

The use of Bezier curves is very convenient in computer-aided geometric design (CAGD). These curves have been studied by two Frenchmen engineers (Pierre Bezier at Renault company and Paul de Casteljau at Citröen company) in 1958 – 1960, [1, 2]. Rational Bezier curves are well known polynomial curves that have a particular mathematical representation. Their popularity may be attributed to the ease in which they produce conic sections. Extensive background information on the history of rational Bezier curves and basic preliminaries about the curves can be found in the literature [3, 5, 7, 10].
Minkowski space was introduced by H. Minkowski in 1907. He realized that the special theory of relativity could best be understood in a four-dimensional space possessing a Minkowski metric with the form

$$g(\mathbf{x},\mathbf{y}) = x_1y_1 + x_2y_2 - x_3y_3$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3_1$. Some basic concepts about Minkowski space can be found in [6].
Recently, the special curves have been studied in Minkowski space denoted by $\mathbb{R}^3_1$. In contrast to Euclidean space, in $\mathbb{R}^3_1$, the curves have different properties because of the Minkowski metric. Georgiev presented an original paper on space-like Bezier curves in the three-dimensional Minkowski space in 2008. In his work, he defined the space-like Bezier curve and described some of the conditions required for this type of curve [4]. Pokorná and Chalmovianská have also studied the quadratic and cubic space-like Bezier curves in Minkowski 3-space. Furthermore, Ugail and et al. are considered on Bezier surfaces in three-dimensional Minkowski space, in [11].
Based on the literature and our research, we found that the space-like rational Bezier curves have not been considered in Minkowski space thus far. Therefore, in our paper, we assess the space-like rational Bezier curves with time-like principal normal in Minkowski 3-space.

2. Preliminaries

A rational Bezier curve of degree n with control points $\mathbf{b_0}, \mathbf{b_1}, ..., \mathbf{b_n} \in \mathbb{R}^3$ and the corresponding scalar weight w_i is defined as

$$b^n(t) = \frac{\sum\limits_{i=0}^{n} w_i \mathbf{b}_i B_i^n(t)}{\sum\limits_{i=0}^{n} w_i B_i^n(t)} \quad fort \in [0,1]$$

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where the Bernstein polynomial is $\mathbf{B}_i^n(t) = \begin{pmatrix} \mathrm{n} \\ i \end{pmatrix} t^{\mathrm{i}}(1-t)^{\mathrm{n}-i}$, [3,7].

The weight w_i is typically used as *shape parameters*. For the rational Bezier curve, we assumed that the weights are not zero. In addition, if all weights are equal to 1, the rational Bezier curve reduces to a Bezier curve. Rational Bezier curves have several advantages over Bezier curves. In particular, rational Bezier curves provide more control over the shape of a curve compared with Bezier curves. Furthermore, conic sections such as parabola, hyperbola, and ellipse may be expressed as rational quadratic Bezier curves. Generally, Bezier curves are examined according to their endpoints. De Casteljau's algorithm can be used to examine the points between the endpoints [2, 3, 7].

Several basic concepts of the Minkowski space are known. A vector $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}_1^3$ is called space-like if $\mathrm{g}(\mathbf{x},\mathbf{x}) > 0$ or $\mathbf{x} = 0$; time-like if $\mathrm{g}(\mathbf{x},\mathbf{x}) < 0$; light-like if $\mathrm{g}(\mathbf{x},\mathbf{x}) = 0$ and $\mathbf{x} \neq 0$. The vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if and only if $\mathrm{g}(\mathbf{x},\mathbf{y}) = 0$. The norm of a vector $\mathbf{x}$ in $\mathbb{R}_1^3$ is defined by $\|\mathbf{x}\|_{\mathbb{L}} = \sqrt{|\mathrm{g}(\mathbf{x},\mathbf{x})|}$. If the vector is time-like because $\mathrm{g}(\mathbf{x},\mathbf{x}) < 0$, the norm will be $\|\mathbf{x}\|_{\mathbb{L}} = \sqrt{-\mathrm{g}(\mathbf{x},\mathbf{x})}$. In addition, if the time-like vectors $\mathbf{x}$ and $\mathbf{y}$ are given, then the inner product can be presented as $g(\mathbf{x},\mathbf{y}) = -\|\mathbf{x}\|_{\mathbb{L}} \|\mathbf{y}\|_{\mathbb{L}} \cosh\theta$ where θ is the angle between these vectors. The cross product in $\mathbb{R}_1^3$ is defined by

$$\begin{aligned} &\wedge_{\mathbb{L}} : \mathbb{R}_1^3 \times \mathbb{R}_1^3 \to \mathbb{R}_1^3 \\ &(\mathbf{x},\mathbf{y}) \to \mathbf{x}\wedge_{\mathbb{L}}\mathbf{y} = -\begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}. \end{aligned}$$

If the angle between $\mathbf{x}$ and $\mathbf{y}$ is θ, the cross product can be defined by $\|\mathbf{x}\wedge_{\mathbb{L}}\mathbf{y}\|_{\mathbb{L}} = \|\mathbf{x}\|_{\mathbb{L}} \|\mathbf{y}\|_{\mathbb{L}} \sinh\theta$for the time-like vectors $\mathbf{x},\mathbf{y} \in \mathbb{R}_1^3$. The space-like vectors have three different conditions for using the inner and exterior product. If the space-like vectors $\mathbf{u}$ and $\mathbf{v}$ ensure the condition $|g(\mathbf{u},\mathbf{v})| < \|\mathbf{u}\|_{\mathbb{L}}\|\mathbf{v}\|_{\mathbb{L}}$, then $\mathbf{u}\wedge_{\mathbb{L}}\mathbf{v}$ is a time-like vector, and the equations $|g(\mathbf{u},\mathbf{v})| = \|\mathbf{u}\|_{\mathbb{L}}\|\mathbf{v}\|_{\mathbb{L}}\cos\theta$ and $\|\mathbf{u}\wedge_{\mathbb{L}}\mathbf{v}\|_{\mathbb{L}} = \|\mathbf{u}\|_{\mathbb{L}}\|\mathbf{v}\|_{\mathbb{L}}\sin\theta$ are satisfied, where θ is the space-like angle between the $\mathbf{u}$ and $\mathbf{v}$ space-like vectors. If the $\mathbf{u}$ and $\mathbf{v}$ space-like vectors ensure the condition $|g(\mathbf{u},\mathbf{v})| > \|\mathbf{u}\|_{\mathbb{L}}\|\mathbf{v}\|_{\mathbb{L}}$, $\mathbf{u}\wedge_{\mathbb{L}}\mathbf{v}$ is a time-like vector, and the equations $g(\mathbf{u},\mathbf{v}) = -\|\mathbf{u}\|_{\mathbb{L}}\|\mathbf{v}\|_{\mathbb{L}}\cosh\theta$ and $\|\mathbf{u}\wedge_{\mathbb{L}}\mathbf{v}\|_{\mathbb{L}} = \|\mathbf{u}\|_{\mathbb{L}}\|\mathbf{v}\|_{\mathbb{L}}\sinh\theta$ are satisfied, where, θ is the hyperbolic angle between the $\mathbf{u}$ and $\mathbf{v}$ space-like vectors. If the $\mathbf{u}$ and $\mathbf{v}$ space-like vectors ensure the condition $|g(\mathbf{u},\mathbf{v})|_{\mathbb{L}} = \|\mathbf{u}\|_{\mathbb{L}}\|\mathbf{v}\|_{\mathbb{L}}$, then $\mathbf{u}\wedge_{\mathbb{L}}\mathbf{v}$ is a light-like vector.

3. Main Results

Definition 3.1. *Let $X = \{b_0, b_1, ..., b_n\}$ be a set of space-like control points in $\mathbb{R}_1^n$. The*

$$SCH\{X\} = \left\{ \lambda_0 b_0 + ... + \lambda_n b_n | \sum_{i=0}^{n} \lambda_i = 1, \lambda_i \geq 0 \right\}$$

set formed by these space-like points is called ***Spacelike convex hull*** *, [4].*

Definition 3.2. *Let $\mathbf{b_0}, \mathbf{b_1}, ..., \mathbf{b_n}$ be the control points in the same space-like region in the three-dimensional Minkowski space $\mathbb{R}_1{}^3$. A space-like rational Bezier curve of degree n can be defined as*

$$b^n(t) = \frac{\sum\limits_{i=0}^{n} w_i \mathbf{b}_i B_i^n(t)}{\sum\limits_{i=0}^{n} w_i B_i^n(t)} \quad for\ t \in [0,1]$$

where w_i is the scalar weight.

Theorem 3.3. *Let $b^n(t)$ be a rational Bezier curve. If all vectors of the space-like convex hull SCHX are space-like, then the curve $b^n(t)$ is space-like, in [4].*

Lemma 3.4. *The rth degree derivative of the space-like rational Bezier curve $\mathbf{b^n}(t)$ is*

$$\frac{db^n(t)}{dt} = \frac{\left[\sum\limits_{i=0}^{n} w_i \mathbf{b}_i B_i^n(t)\right]' - \left[\sum\limits_{i=0}^{n} w_i B_i^n(t)\right]' . b^n(t)}{\sum\limits_{i=0}^{n} w_i B_i^n(t)}$$

for space-like control points $\mathbf{b_i} \in \mathbb{R}_1^3$, $\mathrm{i} = 0, 1, ..., n$. Hence, the first derivatives at the end points $t = 0$ and $t = 1$ can be obtained by

$$\begin{aligned} \left.\frac{db^n(t)}{dt}\right|_{t=0} &= \frac{nw_1}{w_0}[\mathbf{b}_1 - \mathbf{b}_0] = \frac{nw_1}{w_0}\Delta\mathbf{b}_0, \\ \left.\frac{db^n(t)}{dt}\right|_{t=1} &= \frac{nw_{n-1}}{w_n}[\mathbf{b}_{n-1} - \mathbf{b}_{n-1}] = \frac{nw_{n-1}}{w_n}\Delta\mathbf{b}_{n-1}. \end{aligned}$$

Theorem 3.5. *If the rational Bezier $\mathbf{b^n}(t)$ is space-like, then the tangent vectors of the endpoints $t = 0$ and $t = 1$ are space-like vectors, too.*

Proof Since $\mathbf{b^n}(t)$ is a space-like rational Bezier spline $g\left(\frac{d\mathbf{b^n}(t)}{dt}, \frac{d\mathbf{b^n}(t)}{dt}\right) > 0$ for $\forall t \in [0,1]$, then $\Delta\mathbf{b}_0$ is a time-like vector. At the starting point $t = 0$, $g\left(\frac{nw_1}{w_0}.\Delta\mathbf{b}_0, \frac{nw_1}{w_0}.\Delta\mathbf{b}_0\right) > 0$, $\mathrm{n} \in \mathbb{R}$, the inequality $g(\Delta\mathbf{b}_0, \Delta\mathbf{b}_0) > 0$ can be obtained. As a result, the tangent vector of the starting point $\Delta\mathbf{b}_0$ is a space-like vector. Similarly, the ending point $t = 1$ can be obtained.

Definition 3.6. *Let us take the non-unit speed space-like rational Bezier curve $b^n(t)$ for the space-like control points $\mathbf{b_0}, \mathbf{b_1}, ..., \mathbf{b_{n-1}} \in \mathbb{R}^3_1$. The space-like curve indicates that the tangents of the curve must also be space-like. If the principle normal $\mathbf{T}'$ is time-like, the space-like rational Bezier curve is called "a space-like rational Bezier curve with time-like principal normal".*

Let us take the non-unit speed space-like rational Bezier curve with the time-like principal $\mathrm{b^n}(\mathrm{t})$ for the space-like control points $\mathbf{b_0}, \mathbf{b_1}, ..., \mathbf{b_{n-1}} \in \mathbb{R}^3_1$. If $\mathrm{b^n}(\mathrm{t})$ is a space-like curve, then the tangent vector $\mathbf{T}$ must be space-like. Let us define the orthonormal frame of $\mathrm{b^n}(\mathrm{t})$ as $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}|_{\mathrm{t}=0}$ at the starting point $t = 0$, where the tangent vector $\mathbf{T}$ is space-like, the principal normal vector $\mathbf{N}$ is time-like, and the binormal vector $\mathbf{B}$ is space-like. Hence, the orthonormal frame conditions in Minkowski 3-space $g(\mathbf{T}, \mathbf{T}) = 1, g(\mathbf{N}, \mathbf{N}) = -1, g(\mathbf{B}, \mathbf{B}) = 1, g(\mathbf{T}, \mathbf{N}) = 0, g(\mathbf{T}, \mathbf{B}) = 0$, and $g(\mathbf{N}, \mathbf{B}) = 0$ are satisfied. Furthermore, the exterior product for the orthonormal frame base vectors satisfy the conditions $\mathbf{T} \wedge_{\mathbb{L}} \mathbf{N} = -\mathbf{B}$, $\mathbf{N} \wedge_{\mathbb{L}} \mathbf{B} = -\mathbf{T}$, and $\mathbf{B} \wedge_{\mathbb{L}} \mathbf{T} = \mathbf{N}$.

Theorem 3.7. *Let $b^n(t)$ be a rational space-like Bezier curve with time-like principal normal for the space-like control points $b_0, b_1, ..., b_n$. If the condition $|g(\Delta\mathbf{b_0}, \Delta\mathbf{b_1})|_{\mathbb{L}} < \|\Delta\mathbf{b_0}\|_{\mathbb{L}} . \|\Delta\mathbf{b_1}\|_{\mathbb{L}}$ is satisfied, the curvature and torsion of the curve can be defined as*

$$\begin{aligned}
\kappa|_{t=0} &= \frac{n-1}{n}\frac{w_0w_2}{w_1^2}\frac{\|\Delta\mathbf{b_1}\|_{\mathbb{L}}}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}{}^2}.\sin\theta \\
\tau|_{t=0} &= -\frac{n-2}{n}\frac{w_0w_3}{w_1w_2}\frac{\det(\Delta\mathbf{b_0}, \Delta\mathbf{b}_1, \Delta\mathbf{b_2})}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}{}^2\|\Delta\mathbf{b_1}\|_{\mathbb{L}}{}^2\sin^2\theta}.
\end{aligned}$$

Proof: If the space-like vectors $\Delta\mathbf{b_0}$ and $\Delta\mathbf{b_1}$ satisfy the condition $|g(\Delta\mathbf{b_0}, \Delta\mathbf{b_1})|_{\mathbb{L}} < \|\Delta\mathbf{b_0}\|_{\mathbb{L}} . \|\Delta\mathbf{b_1}\|_{\mathbb{L}}$, then the curvature at the starting point can be calculated by

$$\begin{aligned}
\kappa|_{t=0} &= \left.\frac{\left\|\frac{db^n(t)}{dt}\wedge_{\mathbb{L}}\frac{d^2b^n(t)}{dt^2}\right\|_{\mathbb{L}}}{\left\|\frac{db^n(t)}{dt}\right\|^3_{\mathbb{L}}}\right|_{t=0} \\
&= \frac{n-1}{n}\frac{w_0w_2}{w_1^2}\frac{\|\Delta\mathbf{b_0}\wedge_{\mathbb{L}}(\Delta\mathbf{b_1}-\Delta\mathbf{b_0})\|_{\mathbb{L}}}{\|\Delta\mathbf{b_0}\|^3_{\mathbb{L}}} \\
&= \frac{n-1}{n}\frac{w_0w_2}{w_1^2}\frac{\|\Delta\mathbf{b_1}\|_{\mathbb{L}}}{\|\Delta\mathbf{b_0}\|^2_{\mathbb{L}}}.\sin\theta.
\end{aligned}$$

Now, let us find the torsion at the starting point.

$$\begin{aligned}
\tau|_{t=0} &= \frac{g\left(\frac{db^n(t)}{dt}\wedge_{\mathbb{L}}\frac{d^2b^n(t)}{dt^2}, \frac{d^3b^n(t)}{dt^3}\right)}{\left\|\frac{db^n(t)}{dt}\wedge_{\mathbb{L}}\frac{d^2b^n(t)}{dt^2}\right\|^2_{\mathbb{L}}} \\
&= \frac{n-2}{n}\frac{w_0w_3}{w_1w_2}\frac{g(\Delta\mathbf{b_0}\wedge_{\mathbb{L}}\Delta\mathbf{b_1}, \Delta\mathbf{b_2}-2\Delta\mathbf{b_1}+\Delta\mathbf{b_0})}{\|\Delta\mathbf{b_0}\wedge_{\mathbb{L}}(\Delta\mathbf{b_1}-\Delta\mathbf{b_0})\|^2_{\mathbb{L}}} \\
&= -\frac{n-2}{n}\frac{w_0w_3}{w_1w_2}\frac{\det(\Delta\mathbf{b_0}, \Delta\mathbf{b}_1, \Delta\mathbf{b_2})}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}{}^2\|\Delta\mathbf{b_1}\|_{\mathbb{L}}{}^2\sin^2\theta}.
\end{aligned}$$

Theorem 3.8. *If the space-like vectors ensure the condition $|g(\Delta\mathbf{b_0}, \Delta\mathbf{b_1})| > \|\Delta\mathbf{b_0}\|_{\mathbb{L}} . \|\Delta\mathbf{b_1}\|_{\mathbb{L}}$, then the curvature and torsion are*

$$\begin{aligned}
\kappa|_{t=0} &= \frac{n-1}{n}\frac{w_0w_2}{w_1^2}\frac{\|\Delta\mathbf{b_1}\|_{\mathbb{L}}}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}{}^2}.\sinh\theta \\
\tau|_{t=0} &= -\frac{n-2}{n}\frac{w_0w_3}{w_1w_2}\frac{\det(\Delta\mathbf{b_0}, \Delta\mathbf{b}_1, \Delta\mathbf{b_2})}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}{}^2\|\Delta\mathbf{b_1}\|_{\mathbb{L}}{}^2\sinh^2\theta}.
\end{aligned}$$

Proof: If the $\Delta\mathbf{b_0}$ and $\Delta\mathbf{b_1}$ vectors satisfy the $|g(\Delta\mathbf{b_0}, \Delta\mathbf{b_1})| > \|\Delta\mathbf{b_0}\|_{\mathbb{L}} . \|\Delta\mathbf{b_1}\|_{\mathbb{L}}$ condition, the curve at the starting point for space-like vectors can be calculated by

$$\begin{aligned}
\kappa|_{t=0} &= \left.\frac{\left\|\frac{db^n(t)}{dt}\wedge_{\mathbb{L}}\frac{d^2b^n(t)}{dt^2}\right\|_{\mathbb{L}}}{\left\|\frac{db^n(t)}{dt}\right\|^3_{\mathbb{L}}}\right|_{t=0} \\
&= \frac{n-1}{n}\frac{w_0w_2}{w_1^2}\frac{\|\Delta\mathbf{b_0}\wedge_{\mathbb{L}}(\Delta\mathbf{b_1}-\Delta\mathbf{b_0})\|_{\mathbb{L}}}{\|\Delta\mathbf{b_0}\|^3_{\mathbb{L}}} \\
&= \frac{n-1}{n}\frac{w_0w_2}{w_1^2}\frac{\|\Delta\mathbf{b_1}\|_{\mathbb{L}}}{\|\Delta\mathbf{b_0}\|^2_{\mathbb{L}}}.\sinh\theta.
\end{aligned}$$

Now, let us find the torsion at the starting point.

$$\begin{aligned}
\tau|_{t=0} &= \frac{g\left(\frac{db^n(t)}{dt}\wedge_{\mathbb{L}}\frac{d^2b^n(t)}{dt^2}, \frac{d^3b^n(t)}{dt^3}\right)}{\left\|\frac{db^n(t)}{dt}\wedge_{\mathbb{L}}\frac{d^2b^n(t)}{dt^2}\right\|^2_{\mathbb{L}}} \\
&= \frac{n-2}{n}\frac{w_0w_3}{w_1w_2}\frac{g(\Delta\mathbf{b_0}\wedge_{\mathbb{L}}\Delta\mathbf{b_1}, \Delta\mathbf{b_2}-2\Delta\mathbf{b_1}+\Delta\mathbf{b_0})}{\|\Delta\mathbf{b_0}\wedge_{\mathbb{L}}(\Delta\mathbf{b_1}-\Delta\mathbf{b_0})\|^2_{\mathbb{L}}} \\
&= -\frac{n-2}{n}\frac{w_0w_3}{w_1w_2}\frac{\det(\Delta\mathbf{b_0}, \Delta\mathbf{b}_1, \Delta\mathbf{b_2})}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}{}^2\|\Delta\mathbf{b_1}\|_{\mathbb{L}}{}^2\sinh^2\theta}.
\end{aligned}$$

Theorem 3.9. *The Serret-Frenet frame derivative formula of the space-like rational Bezier curve $b^n(t)$ with time-like principal normal at the $t=0$ can be calculated by*

$$\begin{aligned}
\mathbf{T}' &= (n-1)\frac{w_0w_2}{w_1^2}\frac{\|\Delta\mathbf{b_1}\|_{\mathbb{L}}}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}}.\sin\theta.\mathbf{N}\\
\mathbf{N}' &= (n-1)\frac{w_0w_2}{w_1^2}\frac{\|\Delta\mathbf{b_1}\|_{\mathbb{L}}}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}}.\sin\theta.\mathbf{T}\\
&\quad -(n-2)\frac{w_0w_3}{w_1w_2}\frac{\det(\Delta\mathbf{b_0},\Delta\mathbf{b_1},\Delta\mathbf{b_2})}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}\|\Delta\mathbf{b_1}\|_{\mathbb{L}}^2\sin^2\theta}.\mathbf{B}\\
\mathbf{B}' &= -(n-2)\frac{w_0w_3}{w_1w_2}\frac{\det(\Delta\mathbf{b_0},\Delta\mathbf{b_1},\Delta\mathbf{b_2})}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}\|\Delta\mathbf{b_1}\|_{\mathbb{L}}^2\sin^2\theta}.\mathbf{N}
\end{aligned}$$

for the condition $|g(\Delta\mathbf{b_0},\Delta\mathbf{b_1})| < \|\Delta\mathbf{b_0}\|_{\mathbb{L}}.\|\Delta\mathbf{b_1}\|_{\mathbb{L}}$.

Proof: The Serret-Frenet derivative matrix for a non-unit space-like curve with time-like principal normal is

$$\begin{pmatrix}\mathbf{T}'\\ \mathbf{N}'\\ \mathbf{B}'\end{pmatrix} = \begin{pmatrix}0 & \kappa v_1 & 0\\ \kappa v_1 & 0 & \tau v_1\\ 0 & \tau v_1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}$$

where the speed of the curve is $v_1 = n\|b_1-b_0\|_{\mathbb{L}} = n\|\Delta\mathbf{b_0}\|_{\mathbb{L}}$, [6]. If the space-like vectors $\Delta\mathbf{b_0}$ and $\Delta\mathbf{b_1}$ that ensure the condition $|g(\Delta\mathbf{b_0},\Delta\mathbf{b_1})|_{\mathbb{L}} < \|\Delta\mathbf{b_0}\|_{\mathbb{L}}.\|\Delta\mathbf{b_1}\|_{\mathbb{L}}$, then the exterior product is $\|\Delta\mathbf{b_0}\wedge_{\Delta}\mathbf{b_1}\| = \|\Delta\mathbf{b_0}\|_{\mathbb{L}}\|\Delta\mathbf{b_1}\|_{\mathbb{L}}\sin\theta$. Hence, the derivative formulas of the Frenet frame can be calculated by

$$\begin{aligned}
\mathbf{T}' &= \kappa v_1.\mathbf{N}\\
&= \frac{(n-1)}{n}\frac{w_0w_2}{w_1^2}\frac{\|\Delta\mathbf{b_1}\|_{\mathbb{L}}}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}^2}\sin\theta.n\|\Delta\mathbf{b_0}\|_{\mathbb{L}}.\mathbf{N}\\
&= (n-1)\frac{w_0w_2}{w_1^2}\frac{\|\Delta\mathbf{b_1}\|_{\mathbb{L}}}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}}\sin\theta.\mathbf{N}\\
\mathbf{N}' &= \kappa v_1\mathbf{T}+\tau v_1\mathbf{B}\\
&= (n-1)\frac{w_0w_2}{w_1^2}\frac{\|\Delta\mathbf{b_1}\|_{\mathbb{L}}}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}}.\sin\theta.\mathbf{T}\\
&\quad -(n-2)\frac{w_0w_3}{w_1w_2}\frac{\det(\Delta\mathbf{b_0},\Delta\mathbf{b_1},\Delta\mathbf{b_2})}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}\|\Delta\mathbf{b_1}\|_{\mathbb{L}}^2\sin^2\theta}.\mathbf{B}\\
\mathbf{B}' &= \tau v_1\mathbf{N}\\
&= -(n-2)\frac{w_0w_3}{w_1w_2}\frac{\det(\Delta\mathbf{b_0},\Delta\mathbf{b_1},\Delta\mathbf{b_2})}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}\|\Delta\mathbf{b_1}\|_{\mathbb{L}}^2\sin^2\theta}.\mathbf{N}.
\end{aligned}$$

Theorem 3.10. *If the space-like vectors $\Delta\mathbf{b_0}$ and $\Delta\mathbf{b_1}$ satisfy the condition $|g(\Delta\mathbf{b_0},\Delta\mathbf{b_1})| > \|\Delta\mathbf{b_0}\|_{\mathbb{L}}.\|\Delta\mathbf{b_1}\|_{\mathbb{L}}$, then the Serret-Frenet derivative formulas can be calculated by*

$$\begin{aligned}
\mathbf{T}' &= (n-1)\frac{w_0w_2}{w_1^2}\frac{\|\Delta\mathbf{b_1}\|_{\mathbb{L}}}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}}.\sinh\theta.\mathbf{N}\\
\mathbf{N}' &= (n-1)\frac{w_0w_2}{w_1^2}\frac{\|\Delta\mathbf{b_1}\|_{\mathbb{L}}}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}}.\sinh\theta.\mathbf{T}\\
&\quad -(n-2)\frac{w_0w_3}{w_1w_2}\frac{\det(\Delta\mathbf{b_0},\Delta\mathbf{b_1},\Delta\mathbf{b_2})}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}\|\Delta\mathbf{b_1}\|_{\mathbb{L}}^2\sinh^2\theta}.\mathbf{B}\\
\mathbf{B}' &= -(n-2)\frac{w_0w_3}{w_1w_2}\frac{\det(\Delta\mathbf{b_0},\Delta\mathbf{b_1},\Delta\mathbf{b_2})}{\|\Delta\mathbf{b_0}\|_{\mathbb{L}}\|\Delta\mathbf{b_1}\|_{\mathbb{L}}^2\sinh^2\theta}.\mathbf{N}.
\end{aligned}$$

Proof: If the space-like vectors $\Delta\mathbf{b_0}$ and $\Delta\mathbf{b_0}$ ensure the condition $|g(\Delta\mathbf{b_0},\Delta\mathbf{b_1})| > \|\Delta\mathbf{b_0}\|_{\mathbb{L}}.\|\Delta\mathbf{b_1}\|_{\mathbb{L}}$, then the exterior product is $\|\Delta\mathbf{b_0}\wedge_{\mathbb{L}}\mathbf{b_1}\|_{\mathbb{L}} = \|\Delta\mathbf{b_0}\|\|\Delta\mathbf{b_1}\|_{\mathbb{L}}\sinh\theta$. Therefore, the derivative formulas of the Frenet frame can be calculated as proof, similar to Theorem 3.5.

A Numerical Example

Let $b^n(t)$ be a space-like rational Bezier curve with time-like principal normal for the space-like control points $b_0=(1,2,3)$, $b_1=(2,5,6)$, $b_2=(5,7,6)$, and $b_3=(6,9,7)$. In this example, we will take the weights $w_0=w_1=w_2=w_3=1$. Then, the space-like convex hull is given by the space-like vectors $\Delta\mathbf{b_0}=(1,3,3)$, $\Delta\mathbf{b_1}=(3,2,0)$, and $\Delta\mathbf{b_2}=(1,2,1)$. The derivative of $b^n(t)$ at $t=0$ are $\left.\frac{db^n(t)}{dt}\right|_{t=0}=(3,9,9)$, $\left.\frac{d^2b^n(t)}{dt^2}\right|_{t=0}=(12,-6,-18)$, $\left.\frac{d^3b^n(t)}{dt^3}\right|_{t=0}=(-24,6,24)$. The principal normal $N=\frac{1}{\sqrt{68}}(6,25,27)$ is a time-like vector. The inequality $|g(\Delta\mathbf{b_0},\Delta\mathbf{b_1})| > \|\Delta\mathbf{b_0}\|_{\mathbb{L}}.\|\Delta\mathbf{b_1}\|_{\mathbb{L}}$ is also satisfied. Furthermore, the curvature and torsion of $b^n(t)$ are $\kappa|_{t=0}=\frac{2\sqrt{68}}{3}$ and $\tau|_{t=0}=-\frac{5}{204}$. Consequently, the Serret-Frenet derivative matrix can be defined as

$$\begin{pmatrix}\mathbf{T}'\\ \mathbf{N}'\\ \mathbf{B}'\end{pmatrix} = \begin{pmatrix}0 & 2\sqrt{68} & 0\\ 2\sqrt{68} & 0 & -\frac{5}{68}\\ 0 & -\frac{5}{68} & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}.$$

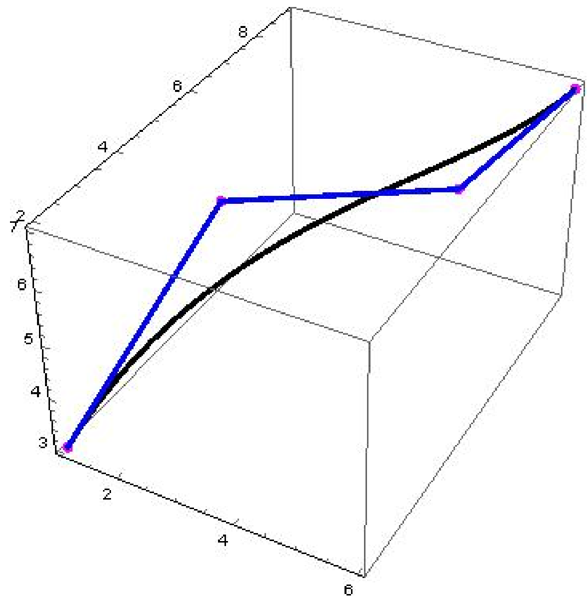

Figure 1: Space-like rational Bezier curve
with time-like principal normal

4. Conclusion

In this paper, we assess the n-degree space-like rational Bezier curve with time-like principal normal in Minkowski 3-space. This work aims to prove certain metric properties for space-like conditions. As a proposal for researchers in future studies, it can be considered how the curve formulation can be extended to the surface.

5. Acknowledgement

The author is grateful to anonymous referees for their careful reading of the manuscript, which helped improve it greatly.

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