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Musical Isomorphisms on the Semi-Tensor Bundles

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Abstract

We transfer vertical lifts and complete lifts of some tensor fields from the semi-tangent bundle tM to the semi-cotangent bundle t^*M using a musical isomorphism between these bundles. In this article, we also analyze complete lift of vector and affinor (tensor of type $(1,1)$) fields for semi-tangent (pull-back) bundle tM. Finally, we study compatibility of transferring lifts with complete lifts in the semi-cotangent bundle t^*M.

Keywords: *Semi-tensor bundle, complete lift, musical isomorphism, vector field, pull-back bundle.*
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1. Introduction

Let (B_m, g) be a smooth pseudo-Riemannian manifold of dimension m. We denote by $t(B_m)$ and $t^*(B_m)$ the semi-tangent [9], [10], [1] and semi-cotangent bundles [3], [4] over B_m with local coordinates $\left(x^a, x^\alpha, x^{\overline{\alpha}}\right) = (x^a, x^\alpha, y^\alpha)$ and $\left(x^a, x^\alpha, \widetilde{x}^{\overline{\alpha}}\right) = (x^a, x^\alpha, p_\alpha)$, $a, b, \ldots = 1, \ldots, n-m; \alpha, \beta, \ldots = n-m+1, \ldots, n; \overline{\alpha}, \overline{\beta}, \ldots = n+1, \ldots, n+m$, respectively, where $y^x = y^\alpha \frac{\partial}{\partial x^i} \in t_x(B_m)$ and $p_x = p_i dx^i \in t_x^*(B_m)$, $\forall x \in B_m$.
We know that the mappings $g^\flat : t(B_m) \to t^*(B_m)$ and $g^\# : t^*(B_m) \to t(B_m)$ between the semi-tangent and semi-cotangent bundles determine the musical (natural) isomorphisms of any pseudo-Riemannian metric g.
The musical isomorphisms $g^\flat$ and $g^\#$ have respectively components

$$g^\flat : x^I = (x^a, x^\alpha, x^{\overline{\alpha}}) = (x^a, x^\alpha, y^\alpha) \to \widetilde{x}^J = \left(x^b, x^\beta, \widetilde{x}^{\overline{\beta}}\right) = (\delta_a^b x^a, \delta_\alpha^\beta x^\alpha, p_\beta = g_{\beta\alpha} y^\alpha)$$

and

$$g^\# : \widetilde{x}^J = \left(x^b, x^\beta, \widetilde{x}^{\overline{\beta}}\right) = \left(x^b, x^\beta, p_\beta\right) \to x^I = (x^a, x^\alpha, x^{\overline{\alpha}}) = (\delta_b^a x^b, \delta_\beta^\alpha x^\beta, y^\alpha = g^{\alpha\beta} p_\beta)$$

with respect to the local coordinates, where δ is the Kronecker delta. The Jacobian of $g^\flat$ and $g^\#$ are given by

$$(g_*^\flat) = \left(\bar{A}_I^J\right) = \left(\frac{\partial \widetilde{x}^J}{\partial x^I}\right) = \begin{pmatrix} \delta_a^b & 0 & 0 \\ 0 & \delta_\alpha^\beta & 0 \\ 0 & y^\varepsilon \partial_\alpha g_{\beta\varepsilon} & g_{\beta\alpha} \end{pmatrix} \tag{1.1}$$

and

$$(g_*^\#) = \left(A_J^I\right) = \left(\frac{\partial x^I}{\partial \widetilde{x}^J}\right) = \begin{pmatrix} \delta_b^a & 0 & 0 \\ 0 & \delta_\beta^\alpha & 0 \\ 0 & p_\varepsilon \partial_\beta g^{\alpha\varepsilon} & g^{\alpha\beta} \end{pmatrix} \tag{1.2}$$

respectively. Where $I = (a, \alpha, \overline{\alpha})$, $J = \left(b, \beta, \overline{\beta}\right)$.
We denote by $\Im_q^p(t(B_m))$ and $\Im_q^p(t^*(B_m))$ the modules over $F(t(B_m))$ and $F(t^*(B_m))$ of all tensor fields of type (p,q) on $t(B_m)$ and $t^*(B_m)$, respectively, where $F(t(B_m))$ and $F(t^*(B_m))$ denote the rings of real-valued $C^\infty-$functions on $t(B_m)$ and $t^*(B_m)$, respectively. On the

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other hand, if $x^{i'} = \left(x^{a'}, x^{\alpha'}, x^{\overline{\alpha}'}\right)$ is another system of local adapted coordinates in the semi-tangent bundle $t(B_m)$, then we have (see, for details [1])

$$\begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right), \\ x^{\overline{\alpha}'} = \frac{\partial x^{\alpha'}}{\partial x^\beta} y^\beta. \end{cases} \tag{1.3}$$

The Jacobian of (1.3) has components [1]

$$\bar{A} = \left(A_J^{I'}\right) = \begin{pmatrix} A_b^{a'} & A_\beta^{a'} & 0 \\ 0 & A_\beta^{\alpha'} & 0 \\ 0 & A_{\beta\varepsilon}^{\alpha'} y^\varepsilon & A_\beta^{\alpha'} \end{pmatrix}, \tag{1.4}$$

where

$$A_\beta^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}, A_{\beta\varepsilon}^{\alpha'} = \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\varepsilon}.$$

Let ${}^{cc}\widetilde{X}_t \in \Im_0^1(t(B_m))$ and ${}^{cc}\widetilde{F}_t \in \Im_1^1(t(B_m))$ be complete lifts of tensor fields $\widetilde{X} \in \Im_0^1(M_n)$ and $\widetilde{F} \in \Im_1^1(M_n)$ to the semi-tangent bundle $t(B_m)$, where M_n denotes the fiber bundle [9], [11], [1] over a manifold B_m. In this paper we transfer via the differential $(g_*^\flat)$ the complete lifts (${}^{cc}\widetilde{X}_t \in \Im_0^1(t(B_m))$, ${}^{cc}\widetilde{F}_t \in \Im_1^1(t(B_m))$) and some tensor fields that the γ-operator is applied from the semi-tangent bundle $t(B_m)$ to semi-cotangent bundle $t^*(B_m)$. On the other hand, we know that the semi-tangent $t(B_m)$ and semi-cotangent bundles $t^*(B_m)$ are a pull-back (induced) bundle of $T(B_m)$ and $T^*(B_m)$, respectively [2], [5], [7], [4]. We note that musical isomorphism and its applications were studied in [8]. The main purpose of this paper is to study musical isomorphism between semi-tangent bundles and semi-cotangent bundles. Where $T(B_m) = \bigcup_{x \in B_m} T_x(B_m)$ and $T^*(B_m) = \bigcup_{x \in B_m} T_x^*(B_m)$ respectively denote the tangent and cotangent bundles over B_m [6].

2. Transfer of vertical lifts of vector fields

Let $X \in \Im_0^1(M_n)$, i.e. $X = X^\alpha \partial_\alpha$. On putting

$${}^{vv}X_t = ({}^{vv}X^\alpha)_t = \begin{pmatrix} 0 \\ 0 \\ X^\alpha \end{pmatrix}, \tag{2.1}$$

from (1.4), we easily see that $({}^{vv}X_t)' = \bar{A}({}^{vv}X_t)$. The vector field ${}^{vv}X$ is called the vertical lift of X to the semi-tangent bundle $t(B_m)$. Then, using (1.1) and (2.1)

$$\begin{aligned} g_*^\flat \, {}^{vv}X_t &= \begin{pmatrix} \delta_a^b & 0 & 0 \\ 0 & \delta_\alpha^\beta & 0 \\ 0 & y^\varepsilon \frac{\partial g_{\beta\varepsilon}}{\partial x^\alpha} & g_{\beta\alpha} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ X^\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ g_{\beta\alpha} X^\alpha \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ p_\alpha \end{pmatrix} = ({}^{vv}p_\alpha)_{t^*}, \end{aligned}$$

where $({}^{vv}p_\alpha)_{t^*}$ is a Liouville covector field [4] on the semi-cotangent bundle $t^*(B_m)$.

3. Transfer of complete lifts of vector fields

Let $\widetilde{X} \in \Im_0^1(M_n)$ be a projectable vector field [11] with projection $X = X^\alpha(x^\alpha)\partial_\alpha$ i.e. $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$. Then the complete lift ${}^{cc}\widetilde{X}_t$ of $\widetilde{X}$ to the semi-tangent bundle $t(B_m)$ is given by [1]

$${}^{cc}\widetilde{X}_t = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix} \tag{3.1}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$.
Using (1.1) and (3.1), we have

$$\begin{aligned} g_*^\flat \, {}^{cc}\widetilde{X}_t &= \begin{pmatrix} \delta_a^b & 0 & 0 \\ 0 & \delta_\alpha^\beta & 0 \\ 0 & y^\varepsilon \frac{\partial g_{\beta\varepsilon}}{\partial x^\alpha} & g_{\beta\alpha} \end{pmatrix} \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{X}^b \\ X^\beta \\ X^\alpha y^\varepsilon \partial_\alpha g_{\beta\varepsilon} + g_{\beta\alpha} y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{X}^b \\ X^\beta \\ y^\varepsilon((L_X g)_{\varepsilon\beta} - (\partial_\beta X^\alpha) g_{\alpha\varepsilon} - (\partial_\varepsilon X^\alpha) g_{\beta\alpha}) + g_{\alpha\beta} y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} \widetilde{X}^b \\ X^\beta \\ y^\varepsilon (L_X g)_{\varepsilon\beta} - p_\alpha(\partial_\beta X^\alpha) \end{pmatrix}, \tag{3.2}$$

where L_X is the Lie derivation of g with respect to X:

$$(L_X g)_{\varepsilon\beta} = X^\alpha \partial_\alpha g_{\varepsilon\beta} + (\partial_\varepsilon X^\alpha) g_{\alpha\beta} + (\partial_\beta X^\alpha) g_{\varepsilon\alpha}.$$

In a manifold (B_m, g), a vector field X is called a Killing vector field if $L_X g = 0$. It is well known that the complete lift ${}^{cc}\widetilde{X_{t^*}}$ of $\widetilde{X}$ to the semi-cotangent bundle $t^*(B_m)$ is given by [4]

$${}^{cc}\widetilde{X_{t^*}} = \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ -p_\varepsilon(\partial_\alpha X^\varepsilon) \end{pmatrix}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$.
We have from (3.2)

$$g_*^\flat\, {}^{cc}\widetilde{X}_t = {}^{cc}\widetilde{X_{t^*}} + \gamma(L_X g),$$

where $\gamma(L_X g)$ is defined by

$$\gamma(L_X g) = \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon (L_X g)_{\varepsilon\beta} \end{pmatrix}.$$

Thus, we have:

Theorem 1. *Let (B_m, g) be a pseudo-Riemannian manifold, and let ${}^{cc}\widetilde{X}_t$ and ${}^{cc}\widetilde{X_{t^*}}$ be complete lifts of a vector field $\widetilde{X} \in \Im_0^1(M_n)$ to the semi-tangent and semi-cotangent bundles, respectively. Then the differential (pushforward) of ${}^{cc}\widetilde{X}_t$ by $g^\flat$ coincides with ${}^{cc}\widetilde{X_{t^*}}$, i.e. $g_*^\flat\, {}^{cc}\widetilde{X}_t = {}^{cc}\widetilde{X_{t^*}}$ if and only if $\widetilde{X}$ is a Killing vector field.*

Theorem 2. *Let $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$. For the Lie product, we have*

$$[{}^{cc}\widetilde{X}_t, {}^{cc}\widetilde{Y}_t] = {}^{cc}\widetilde{[X,Y]}_t$$

in the semi-tangent bundle $t(B_m)$.

Proof. If $\widetilde{X}, \widetilde{Y} \in \Im_0^1(M_n)$ and $\begin{pmatrix} [{}^{cc}\widetilde{X}_t, {}^{cc}\widetilde{Y}_t]^b \\ [{}^{cc}\widetilde{X}_t, {}^{cc}\widetilde{Y}_t]^\beta \\ [{}^{cc}\widetilde{X}_t, {}^{cc}\widetilde{Y}_t]^{\overline{\beta}} \end{pmatrix}$ are components of $[{}^{cc}\widetilde{X}_t, {}^{cc}\widetilde{Y}_t]$ with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t(M_n)$, then we have

$$[{}^{cc}\widetilde{X}_t, {}^{cc}\widetilde{Y}_t]^J = ({}^{cc}\widetilde{X}_t)^I \partial_I ({}^{cc}\widetilde{Y}_t)^J - ({}^{cc}\widetilde{Y}_t)^I \partial_I ({}^{cc}\widetilde{X}_t)^J.$$

Firstly, if $J = b$, we have

$$\begin{aligned}
[{}^{cc}\widetilde{X}_t, {}^{cc}\widetilde{Y}_t]^b &= ({}^{cc}\widetilde{X}_t)^I \partial_I ({}^{cc}\widetilde{Y}_t)^b - ({}^{cc}\widetilde{Y}_t)^I \partial_I ({}^{cc}\widetilde{X}_t)^b \\
&= ({}^{cc}\widetilde{X}_t)^a \partial_a ({}^{cc}\widetilde{Y}_t)^b + ({}^{cc}\widetilde{X}_t)^\alpha \partial_\alpha ({}^{cc}\widetilde{Y}_t)^b + ({}^{cc}\widetilde{X}_t)^{\overline{\alpha}} \partial_{\overline{\alpha}} ({}^{cc}\widetilde{Y}_t)^b \\
&\quad - ({}^{cc}\widetilde{Y}_t)^a \partial_a ({}^{cc}\widetilde{X}_t)^b - ({}^{cc}\widetilde{Y}_t)^\alpha \partial_\alpha ({}^{cc}\widetilde{X}_t)^b - ({}^{cc}\widetilde{Y}_t)^{\overline{\alpha}} \partial_{\overline{\alpha}} ({}^{cc}\widetilde{X}_t)^b \\
&= ({}^{cc}\widetilde{X}_t)^\alpha \partial_\alpha ({}^{cc}\widetilde{Y}_t)^b - ({}^{cc}\widetilde{Y}_t)^\alpha \partial_\alpha ({}^{cc}\widetilde{X}_t)^b \\
&= X^\alpha \partial_\alpha ({}^{cc}\widetilde{Y}_t)^b - Y^\alpha \partial_\alpha ({}^{cc}\widetilde{X}_t)^b \\
&= X^\alpha \partial_\alpha \widetilde{Y}^b - Y^\alpha \partial_\alpha \widetilde{X}^b \\
&= \widetilde{[X,Y]}^b
\end{aligned}$$

by virtue of (3.1). Secondly, if $J = \beta$, we have

$$\begin{aligned}
[{}^{cc}\widetilde{X}_t, {}^{cc}\widetilde{Y}_t]^\beta &= ({}^{cc}\widetilde{X}_t)^I \partial_I ({}^{cc}\widetilde{Y}_t)^\beta - ({}^{cc}\widetilde{Y}_t)^I \partial_I ({}^{cc}\widetilde{X}_t)^\beta \\
&= ({}^{cc}\widetilde{X}_t)^a \partial_a ({}^{cc}\widetilde{Y}_t)^\beta + ({}^{cc}\widetilde{X}_t)^\alpha \partial_\alpha ({}^{cc}\widetilde{Y}_t)^\beta + ({}^{cc}\widetilde{X}_t)^{\overline{\alpha}} \partial_{\overline{\alpha}} ({}^{cc}\widetilde{Y}_t)^\beta \\
&\quad - ({}^{cc}\widetilde{Y}_t)^a \partial_a ({}^{cc}\widetilde{X}_t)^\beta - ({}^{cc}\widetilde{Y}_t)^\alpha \partial_\alpha ({}^{cc}\widetilde{X}_t)^\beta - ({}^{cc}\widetilde{Y}_t)^{\overline{\alpha}} \partial_{\overline{\alpha}} ({}^{cc}\widetilde{X}_t)^\beta \\
&= ({}^{cc}\widetilde{X}_t)^\alpha \partial_\alpha ({}^{cc}\widetilde{Y}_t)^\beta - ({}^{cc}\widetilde{Y}_t)^\alpha \partial_\alpha ({}^{cc}\widetilde{X}_t)^\beta \\
&= X^\alpha \partial_\alpha Y^\beta - Y^\alpha \partial_\alpha X^\beta \\
&= [X,Y]^\beta
\end{aligned}$$

by virtue of (3.1). Thirdly, if $J=\overline{\beta}$, then we have

$$\begin{aligned}[^{cc}\widetilde{X}_t, {}^{cc}\widetilde{Y}_t]^{\overline{\beta}} &= ({}^{cc}\widetilde{X}_t)^I\partial_I({}^{cc}\widetilde{Y}_t)^{\overline{\beta}} - ({}^{cc}\widetilde{Y}_t)^I\partial_I({}^{cc}\widetilde{X}_t)^{\overline{\beta}} \\ &= ({}^{cc}\widetilde{X}_t)^a\partial_a({}^{cc}\widetilde{Y}_t)^{\overline{\beta}} + ({}^{cc}\widetilde{X}_t)^\alpha\partial_\alpha({}^{cc}\widetilde{Y}_t)^{\overline{\beta}} + ({}^{cc}\widetilde{X}_t)^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{cc}\widetilde{Y}_t)^{\overline{\beta}} \\ &\quad -({}^{cc}\widetilde{Y}_t)^a\partial_a({}^{cc}\widetilde{X}_t)^{\overline{\beta}} - ({}^{cc}\widetilde{Y}_t)^\alpha\partial_\alpha({}^{cc}\widetilde{X}_t)^{\overline{\beta}} - ({}^{cc}\widetilde{Y}_t)^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{cc}\widetilde{X}_t)^{\overline{\beta}} \\ &= X^\alpha\partial_\alpha(y^\varepsilon\partial_\varepsilon Y^\beta) + y^\varepsilon\partial_\varepsilon X^\alpha\partial_{\overline{\alpha}}y^\sigma\partial_\sigma Y^\beta \\ &\quad -Y^\alpha\partial_\alpha(y^\varepsilon\partial_\varepsilon X^\beta) - y^\varepsilon\partial_\varepsilon Y^\alpha\partial_{\overline{\alpha}}y^\sigma\partial_\sigma X^\beta \\ &= y^\varepsilon X^\alpha\partial_\alpha\partial_\varepsilon Y^\beta + y^\varepsilon(\partial_\varepsilon X^\sigma)\left(\partial_\sigma Y^\beta\right) \\ &\quad -y^\varepsilon Y^\alpha\partial_\alpha\partial_\varepsilon X^\beta - y^\varepsilon(\partial_\varepsilon Y^\sigma)\left(\partial_\sigma X^\beta\right) \\ &= y^\varepsilon\partial_\varepsilon[X,Y]^\beta\end{aligned}$$

by virtue of (3.1). On the other hand, we know that ${}^{cc}[\widetilde{X,Y}]$ have components

$${}^{cc}[\widetilde{X,Y}]_t = \begin{pmatrix} [\widetilde{X,Y}]^b \\ [X,Y]^\beta \\ y^\varepsilon\partial_\varepsilon[X,Y]^\beta \end{pmatrix}$$

with respect to the coordinates $(x^b, x^\beta, x^{\overline{\beta}})$ on $t(M_n)$.
Thus, we have $[{}^{cc}\widetilde{X}_t, {}^{cc}\widetilde{Y}_t] = {}^{cc}[\widetilde{X,Y}]_t$ in $t(B_m)$. □

Let $\widetilde{X}$ and $\widetilde{Y}$ be a Killing vector fields on M_n. Then we have

$$L_{[\widetilde{X,Y}]_t}g = [L_{\widetilde{X}}, L_{\widetilde{Y}}]g = L_{\widetilde{X}}\circ L_{\widetilde{Y}}g - L_{\widetilde{Y}}\circ L_{\widetilde{X}}g = 0,$$

i.e. $[\widetilde{X,Y}]_t$ is a Killing vector field. Since ${}^{cc}[\widetilde{X,Y}]_t = [{}^{cc}\widetilde{X}_t, {}^{cc}\widetilde{Y}_t]$ and ${}^{cc}[\widetilde{X,Y}]_{t^*} = [{}^{cc}\widetilde{X}_{t^*}, {}^{cc}\widetilde{Y}_{t^*}]$ (see [4]), from Theorem 1. and Theorem 2. we have

Theorem 3. *If $\widetilde{X}$ and $\widetilde{Y}$ be a Killing vector fields on M_n, then*

$$g_*^\flat[{}^{cc}\widetilde{X}_t, {}^{cc}\widetilde{Y}_t] = [{}^{cc}\widetilde{X}_{t^*}, {}^{cc}\widetilde{Y}_{t^*}],$$

where $g_*^\flat$ is a differential (pushforward) of musical isomorphism $g^\flat$.

4. Transfer of $(\gamma F)_t$ and $(\gamma T)_t$

For any $F \in \Im_1^1(B_m)$, if we take account of (1.4), we can prove that $(\gamma F)_t' = \overline{A}(\gamma F)_t$ where $(\gamma F)_t$ is a vector field on the semi-tangent bundle $t(B_m)$ defined by

$$(\gamma F)_t = \left(\gamma F^I\right)_t = \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon F_\varepsilon^\alpha \end{pmatrix} \tag{4.1}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$. On the other hand, vector field $(\gamma F)_{t^*}$ on the semi-cotangent bundle $t^*(B_m)$ is defined by [4]:

$$(\gamma F)_{t^*} = \left(\gamma F^I\right)_{t^*} = \begin{pmatrix} 0 \\ 0 \\ p_\alpha F_\varepsilon^\alpha \end{pmatrix}.$$

Let $T \in \Im_2^1(B_m)$. On putting

$$(\gamma T)_t = \left(\gamma T_J^I\right)_t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & y^\varepsilon T_\varepsilon{}^\alpha{}_\beta & 0 \end{pmatrix}, \tag{4.2}$$

from (1.4), we easily see that $\left(\gamma T_{J'}^{I'}\right)_t = A_I^{I'}A_{J'}^J\left(\gamma T_J^I\right)_t$, where $\left(\overline{A}\right)^{-1} = \left(A_{J'}^J\right)$ is the inverse matrix of $\overline{A}$.

Theorem 4. *If $F \in \Im_1^1(B_m)$ and $T \in \Im_2^1(B_m)$, then*

(i) $g_*^\flat(\gamma F)_t = (\gamma F)_{t^*}$,
(ii) $g_*^\flat(\gamma T)_t = (\gamma T)_{t^*}$.

Proof. (i) From (1.1) and (4.1), we have:

$$g_*^\flat(\gamma F)_t = \begin{pmatrix} \delta_a^b & 0 & 0 \\ 0 & \delta_\alpha^\beta & 0 \\ 0 & y^\varepsilon \frac{\partial g_{\beta\varepsilon}}{\partial x^\alpha} & g_{\beta\alpha} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon F_\varepsilon^\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ g_{\beta\alpha} y^\varepsilon F_\varepsilon^\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p_\alpha F_\varepsilon^\alpha \end{pmatrix} = (\gamma F)_{t^*}.$$

It is well known that $(\gamma F)_{t^*}$ have components [4]:

$$(\gamma F)_{t^*} = \left(\gamma F^I\right)_{t^*} = \begin{pmatrix} 0 \\ 0 \\ p_\alpha F_\varepsilon^\alpha \end{pmatrix}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on the semi-cotangent bundle $t^*(B_m)$. Thus, we have (i) of Theorem 4.
(ii) For simplicity we take $g_*^\flat(\gamma T)_t = \left(\gamma T_J^I\right)_{t^*}$. In fact,

$$\begin{aligned} \left(\gamma T_\beta^{\overline{\alpha}}\right)_{t^*} &= g_{\alpha\sigma}\delta_\beta^\theta y^\varepsilon T_\varepsilon^\sigma{}_\theta = g_{\alpha\sigma} y^\varepsilon T_\varepsilon^\sigma{}_\beta = g_{\alpha\sigma}\delta_\alpha^\varepsilon \delta_\varepsilon^\alpha y^\varepsilon T_\varepsilon^\sigma{}_\beta = g_{\alpha\sigma}\delta_\alpha^\varepsilon \delta_\varepsilon^\alpha y^\varepsilon T_\varepsilon^\sigma{}_\beta \\ &= g_{\alpha\sigma}\delta_\varepsilon^\alpha y^\varepsilon T_\alpha^\sigma{}_\beta = g_{\alpha\sigma} y^\varepsilon T_\alpha^\sigma{}_\beta = g_{\alpha\sigma} y^\alpha T_\alpha^\sigma{}_\beta = g_{\alpha\sigma}\delta_\sigma^\varepsilon \delta_\varepsilon^\sigma y^\alpha T_\alpha^\sigma{}_\beta \\ &= g_{\alpha\sigma}\delta_\varepsilon^\sigma y^\alpha T_\alpha^\varepsilon{}_\beta = g_{\alpha\varepsilon} y^\alpha T_\alpha^\varepsilon{}_\beta = p_\varepsilon T_\alpha^\varepsilon{}_\beta = p_\varepsilon \delta_\beta^\alpha \delta_\alpha^\beta T_\alpha^\varepsilon{}_\beta = p_\varepsilon T_{\beta\alpha}^\varepsilon \end{aligned}$$

Thus, we have $\left(\gamma T_\beta^{\overline{\alpha}}\right)_{t^*} = p_\varepsilon T_{\beta\alpha}^\varepsilon$. Similarly, from (1.1) and (4.2), we can easily find all other components of $\left(\gamma T_J^I\right)_{t^*}$ equal to zero, where $I = (a, \alpha, \overline{\alpha})$, $J = \left(b, \beta, \overline{\beta}\right)$. We know that $(\gamma T)_{t^*}$ have components on $t^*(B_m)$ [4]:

$$(\gamma T)_{t^*} = \left(\gamma T_J^I\right)_{t^*} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p_\varepsilon T_{\beta\alpha}^\varepsilon & 0 \end{pmatrix}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$. Thus, we have $g_*^\flat(\gamma T)_t = (\gamma T)_{t^*}$. □

5. Complete lift of affinor fields

Let $\widetilde{F} \in \Im_1^1(M_n)$ be a projectable affinor field [10] with projection $F = F_\beta^\alpha(x^\alpha)\partial_\alpha \otimes dx^\beta$, i.e. $\widetilde{F}$ has components

$$\widetilde{F} = \left(\widetilde{F}_j^i\right) = \begin{pmatrix} \widetilde{F}_b^a(x^\alpha, x^\alpha) & \widetilde{F}_\beta^a(x^\alpha, x^\alpha) \\ 0 & F_\beta^\alpha(x^\alpha) \end{pmatrix}$$

with respect to the coordinates (x^a, x^α). On putting

$$\left({}^{cc}\widetilde{F}\right)_t = \left({}^{cc}\widetilde{F_B^A}\right)_t = \begin{pmatrix} \widetilde{F_b^a} & \widetilde{F_\beta^a} & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & y^\varepsilon \partial_\varepsilon F_\beta^\alpha & F_\beta^\alpha \end{pmatrix} \tag{5.1}$$

we easily see that $\left({}^{cc}\widetilde{F_{J'}^{I'}}\right)_t = A_J^{I'} A_{J'}^I \left({}^{cc}\widetilde{F_J^I}\right)_t$.
We call $\left({}^{cc}\widetilde{F_{J'}^{I'}}\right)_t$ the complete lift of the tensor field $\widetilde{F}$ of type (1,1) to the semi-tangent bundle $t(B_m)$.

Proof. For simplicity, we put $I' = \overline{\alpha}'$, $J' = \beta'$ in ${}^{cc}F_{J'}^{I'}$ and take account of (1.4) and (5.1), we obtain

$$\begin{aligned} \left({}^{cc}\widetilde{F_{\beta'}^{\overline{\alpha}'}}\right)_t &= A_{\overline{\alpha}}^{\overline{\alpha}'} A_{\beta'}^{\overline{\beta}}\, {}^{cc}F_{\overline{\beta}}^{\overline{\alpha}} + A_{\overline{\alpha}}^{\overline{\alpha}'} A_{\beta'}^{\beta}\, {}^{cc}F_{\beta}^{\overline{\alpha}} + A_\alpha^{\overline{\alpha}'} A_{\beta'}^{\beta}\, {}^{cc}F_\beta^\alpha \\ &= A_\alpha^{\alpha'} A_{\beta'\sigma'}^{\beta} y^{\sigma'} F_\beta^\alpha + A_\alpha^{\alpha'} A_{\beta'}^{\beta} y^{\sigma'} \partial_{\sigma'} F_\beta^\alpha + A_\alpha^{\alpha'}{}_{\sigma'} y^{\sigma'} A_{\beta'}^{\beta} F_\beta^\alpha \\ &= A_\alpha^{\alpha'} y^{\sigma'} \partial_{\sigma'} A_{\beta'}^{\beta} F_\beta^\alpha + A_\alpha^{\alpha'} A_{\beta'}^{\beta} y^{\sigma'} \left(\partial_{\sigma'} F_\beta^\alpha\right) + y^{\sigma'} \left(\partial_{\sigma'} A_\alpha^{\alpha'}\right) A_{\beta'}^{\beta} F_\beta^\alpha \\ &= y^{\sigma'} A_\alpha^{\alpha'} \left(\partial_{\sigma'} A_{\beta'}^{\beta}\right) F_\beta^\alpha + y^{\sigma'} A_\alpha^{\alpha'} A_{\beta'}^{\beta} \left(\partial_{\sigma'} F_\beta^\alpha\right) \\ &\quad + y^{\sigma'} \left(\partial_{\sigma'} A_\alpha^{\alpha'}\right) A_{\beta'}^{\beta} F_\beta^\alpha \\ &= y^{\sigma'} \partial_{\sigma'} \left(A_\alpha^{\alpha'} A_{\beta'}^{\beta} F_\beta^\alpha\right) \\ &= y^{\varepsilon'} \partial_{\varepsilon'} F_{\beta'}^{\alpha'}. \end{aligned}$$

Similarly, we can easily find another components of $\left({}^{cc}\widetilde{F_{J'}^{I'}}\right)_t$. □

6. Transfer of complete lifts of affinor fields

Let $\widetilde{F}$ be projectable affinor fields [10] on M_n with projection F on B_m. Using (1.1), (1.2) and (5.1), we have

$$\begin{aligned} g_*^{\flat}\left({}^{cc}\widetilde{F_J^I}\right)_t &= A_K^I A_J^{L cc}\widetilde{F_L^K} \\ &= \begin{pmatrix} \widetilde{F_b^a} & \widetilde{F_\beta^a} & 0 \\ 0 & F_\alpha^\beta & 0 \\ 0 & y^\varepsilon(\partial_\theta g_{\beta\varepsilon})F_\alpha^\theta + g_{\beta\theta}y^\varepsilon\partial_\varepsilon F_\alpha^\theta + g_{\beta\theta}p_\varepsilon(\partial_\alpha g^{\sigma\varepsilon})F_\sigma^\theta & g_{\beta\theta}g^{\sigma\alpha}F_\sigma^\theta \end{pmatrix}. \end{aligned} \tag{6.1}$$

Since $g = (g_{\alpha\beta})$ and $g^{-1} = \left(g^{\alpha\beta}\right)$ are pure tensor fields with respect to F, we find

$$g_{\beta\theta}g^{\sigma\alpha}F_\sigma^\theta = g_{\beta\theta}g^{\theta\sigma}F_\sigma^\alpha = \delta_\beta^\sigma F_\sigma^\alpha = F_\beta^\alpha \tag{6.2}$$

and

$$\begin{aligned} &= y^\varepsilon(\partial_\theta g_{\beta\varepsilon})F_\alpha^\theta + g_{\beta\theta}y^\varepsilon\partial_\varepsilon F_\alpha^\theta + g_{\beta\theta}p_\varepsilon(\partial_\alpha g^{\sigma\varepsilon})F_\sigma^\theta \\ &= y^\varepsilon(\phi_\alpha g_{\beta\varepsilon} + \partial_\alpha(g\circ F)_{\beta\varepsilon} - g_{\theta\varepsilon}\partial_\beta F_\alpha^\theta) + g_{\beta\theta}p_\varepsilon(\partial_\alpha g^{\sigma\varepsilon})F_\sigma^\theta \\ &= y^\varepsilon\phi_\alpha g_{\varepsilon\beta} + y^\varepsilon\partial_\alpha(g\circ F)_{\beta\varepsilon} - p_\theta\partial_\beta F_\alpha^\theta + g_{\beta\theta}p_\varepsilon(\partial_\alpha g^{\sigma\varepsilon})F_\sigma^\theta \\ &= y^\varepsilon\phi_\alpha g_{\varepsilon\beta} - p_\theta\partial_\beta F_\alpha^\theta + y^\varepsilon\partial_\alpha(g\circ F)_{\beta\varepsilon} + g_{\beta\theta}p_\varepsilon(\partial_\alpha g^{\sigma\varepsilon})F_\sigma^\theta \\ &= y^\varepsilon\phi_\alpha g_{\varepsilon\beta} - p_\theta\partial_\beta F_\alpha^\theta + y^\varepsilon\partial_\alpha(g_{\varepsilon\gamma}F_\beta^\gamma) + g_{\beta\theta}p_\varepsilon(\partial_\alpha g^{\sigma\varepsilon})F_\sigma^\theta \\ &= y^\varepsilon\phi_\alpha g_{\varepsilon\beta} - p_\theta\partial_\beta F_\alpha^\theta + y^\varepsilon(\partial_\alpha g_{\varepsilon\gamma})F_\beta^\gamma + y^\varepsilon(\partial_\alpha F_\beta^\gamma)g_{\varepsilon\gamma} \\ &\quad + g_{\beta\gamma}p_\varepsilon(\partial_\alpha g^{\sigma\varepsilon})F_\sigma^\gamma \\ &= y^\varepsilon\phi_\alpha g_{\varepsilon\beta} - p_\theta\partial_\beta F_\alpha^\theta + y^\varepsilon(\partial_\alpha g_{\varepsilon\gamma})F_\beta^\gamma + y^\varepsilon(\partial_\alpha F_\beta^\gamma)g_{\varepsilon\gamma} \\ &\quad + g_{\gamma\sigma}p_\varepsilon(\partial_\alpha g^{\sigma\varepsilon})F_\beta^\gamma \\ &= y^\varepsilon\phi_\alpha g_{\varepsilon\beta} - p_\theta\partial_\beta F_\alpha^\theta + y^\varepsilon(\partial_\alpha g_{\varepsilon\gamma})F_\beta^\gamma + y^\varepsilon(\partial_\alpha F_\beta^\gamma)g_{\varepsilon\gamma} \\ &\quad - g^{\sigma\varepsilon}p_\varepsilon(\partial_\alpha g_{\gamma\sigma})F_\beta^\gamma \\ &= y^\varepsilon\phi_\alpha g_{\varepsilon\beta} - p_\theta\partial_\beta F_\alpha^\theta + y^\varepsilon(\partial_\alpha g_{\varepsilon\gamma})F_\beta^\gamma + p_\gamma(\partial_\alpha F_\beta^\gamma) \\ &\quad - y^\sigma(\partial_\alpha g_{\gamma\sigma})F_\beta^\gamma \\ &= y^\varepsilon\phi_\alpha g_{\varepsilon\beta} + p_\varepsilon(\partial_\alpha F_\beta^\varepsilon - \partial_\beta F_\alpha^\varepsilon). \end{aligned} \tag{6.3}$$

Where $I = (a,\alpha,\overline{\alpha})$, $J = \left(b,\beta,\overline{\beta}\right)$, $K = \left(c,\theta,\overline{\theta}\right)$, $L = (d,\sigma,\overline{\sigma})$. Also, the component $\left({}^{cc}\widetilde{F_\beta^{\overline{\alpha}}}\right)_t$ of $\left({}^{cc}\widetilde{F_J^I}\right)_t$ is defined as Tachibana operator $\phi_F g$ of F, i.e.,

$$\phi_\sigma g_{\theta\beta} = F_\sigma^\gamma\partial_\gamma g_{\theta\beta} - \partial_\sigma(g\circ F)_{\theta\beta} + g_{\gamma\beta}\partial_\theta F_\sigma^\gamma + g_{\theta\gamma}\partial_\beta F_\sigma^\gamma.$$

Substituting (6.2) and (6.3) into (6.1), we obtain

$$g_*^{\flat}\left({}^{cc}\widetilde{F_J^I}\right)_t = \begin{pmatrix} \widetilde{F_b^a} & \widetilde{F_\beta^a} & 0 \\ 0 & F_\alpha^\beta & 0 \\ 0 & y^\varepsilon\phi_\alpha g_{\varepsilon\beta} + p_\varepsilon(\partial_\alpha F_\beta^\varepsilon - \partial_\beta F_\alpha^\varepsilon) & F_\beta^\alpha \end{pmatrix}. \tag{6.4}$$

It is well known that the complete lift $\left({}^{cc}\widetilde{F}\right)_{t^*}$ of $\widetilde{F} \in \Im_1^1(M_n)$ to the semi-cotangent bundle $t^*(B_m)$ is given by [4]

$$\left({}^{cc}\widetilde{F}\right)_{t^*} = \left({}^{cc}\widetilde{F_J^I}\right)_{t^*} = \begin{pmatrix} \widetilde{F_b^a} & \widetilde{F_\beta^a} & 0 \\ 0 & F_\alpha^\beta & 0 \\ 0 & p_\varepsilon(\partial_\alpha F_\beta^\varepsilon - \partial_\beta F_\alpha^\varepsilon) & F_\beta^\alpha \end{pmatrix} \tag{6.5}$$

with respect to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t^*(B_m)$. From (6.4) and (6.5), we easily obtain

$$g_*^{\flat}\left({}^{cc}\widetilde{F}\right)_t = \left({}^{cc}\widetilde{F}\right)_{t^*} + \gamma(\phi_F g),$$

where

$$\gamma(\phi_F g) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & y^\varepsilon\phi_\alpha g_{\varepsilon\beta} & 0 \end{pmatrix}.$$

Finally, we can prove

Theorem 5. *Let $\left({}^{cc}\widetilde{F}\right)_t$ and $\left({}^{cc}\widetilde{F}\right)_{t^*}$ be complete lifts of $\widetilde{F} \in \Im_1^1(M_n)$ to the semi-tangent and semi-cotangent bundles, respectively. Then the differential of $\left({}^{cc}\widetilde{F}\right)_t$ by $g^\flat$ coincides with $\left({}^{cc}\widetilde{F}\right)_{t^*}$, i.e. $g_*^\flat\left({}^{cc}\widetilde{F}\right)_t = \left({}^{cc}\widetilde{F}\right)_{t^*}$ if and only if $\phi_F g = 0$.*

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