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Comparing the performances of the priors on the Bayesian estimation of change point location in a sequence of normal random variables

Ayten Yiğiter

Hacettepe Üniversitesi
İstatistik Bölümü
06800-Beytepe, Ankara, Türkiye
yigiter@hacettepe.edu.tr
0000-0001-8180-995X

Abstract

In this paper, the problem of one change point occurring simultaneously in both the mean and variance of a sequence of normal random variables is considered. The Bayesian method is used for the estimation of change point location; and the marginal posterior distributions of the change point location are obtained under the assumptions of noninformative and informative prior distributions for the parameters. The performances of the prior distributions on the estimation of the change point location, with different sample sizes, are investigated via extensive simulation studies.

Key words: change point; Bayesian approach; noninformative and informative priors

Normal Dağılımlı Raslantı Değişkenleri Dizisinde Değişme Noktasının Bayesci Tahmini Üzerinde Önsel Dağılımların Performanslarının Karşılaştırılması

Öz

Bu çalışmada, Normal dağılımlı raslantı değişkenlerinin ortalaması ve varyansında değişme noktasının gözlenmesi problemi ele alınmıştır. Değişme noktasının yerinin tahmini için Bayesci yöntem kullanılmış ve parametrelerin bilgi içermeyen ve bilgi içeren önsel dağılımlı olması varsayımı altında değişme noktasının marjinal sonsal dağılımı elde edilmiştir. Değişme noktasının tahmini üzerinde önsel dağılımların performansları geniş benzetim çalışması ile farklı örneklem büyükleri için araştırılmıştır.

Anahtar kelimeler: *Değişme noktası, Bayesci yaklaşım, bilgi içeremeyen ve bilgi içeren önseller.*

1. Introduction

The investigation of the change point in time ordered data or sequential data started in the 1950s [17]. In the literature, a change point is defined as the location τ before or at which the distribution of a sequence of random variables is F(.) and after which the distribution of this sequence of random variables switches to G(.). Generally, in the studies of change point problems, F(.) and G(.) are taken as of the same function forms, but with different parameters (Chen and Gupta [8]). Researchers have conducted plenty of studies on the estimation and the hypothesis testing of single change point problems in normal, binomial, Poisson, exponential, and gamma models. For example, Hinkley [12], Hinkley and Hinkley [13], Worsley [19], Chen *et al.* [4], [5], [7], [8] investigated the change point problem in a sequence of normal random variables; Worsley [20], Haccou, Meelis and Geer [11] studied the change point problem in an exponential model; and Boudjellaba, MacGibbon and Sawyer [3] examinated the change point problem in a sequence of Poisson random variables. In addition, the multiple change points problem was considered by many authors such as Gupta and Chen [10], Chen *et al.* [4], [8] and Yao [21], among others.

The Bayesian approach is widely used in the analysis of change point problems. Smith [18] used the Bayesian approach to solve the inference problems of one change point in the sequence of normal and binomial random variables. Hsu [14] investigated a change point problem in the scale parameter of a sequence of gamma random variables. Menzefricke [16] considered a change point problem in a sequence of normal random variables. Ghorbanzadeh [9] used a Bayesian analysis for detecting a change point in exponential family. Chen *et al.* [8] obtained the joint posterior distribution of both an abrupt and a smooth change in gene expression data.

As it is well known, quite different from the classical (frequentist's) approach, in a Bayesian approach, parameters are not considered as constants; instead, they are assumed to follow some probability distributions, which are called prior distributions. If the investigator has some beliefs about the parameters, it is important to take into account these beliefs in the estimation of the parameters. When the prior and the information from data support each other, the resulting parameter estimations would be better. The estimate of the unknown parameter is obtained by deriving a posterior distribution on the basis of the prior distributions and the likelihood function. There are many discussions on the chooses of the prior distribution in Bayesian analysis such as Bernardo [1], Bernardo, Gutierrcz-Pefia and Smith [2], Lavenda [15]. A Bayesian approach to a change point problem is generally established using the noninformative prior distributions for some unknown parameters, but in the literature, there is no comparison about the performances of various prior distributions on the Bayesian estimation of a change point.

Adding information of the change point location usually improves the Bayesian change point estimation. Therefore, in this study, we have considered a change point problem in both the mean and variance of a sequence of normal random variables, obtained the Bayesian estimate of the change point by using various prior distributions for unknown parameters. The performances of these prior distributions are then investigated through extensive simulation studies.

2. Bayesian Approach

Let X_i, $i=1,2,...,n$, be a sequence of random variables taken from a normal distribution with mean μ_i and variance σ_i^2. For the detection of a change point occurring simultaneously in the mean and variance of the sequence, the null hypothesis

$$H_0: \mu_1 = \mu_2 = \ldots = \mu_n = \mu_0 \text{ and } \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_n^2 = \sigma_0^2$$
$$\text{or } (\mu_i, \sigma_i^2) = (\mu_0, \sigma_0^2), \quad i = 1,2,...,n$$

is tested against the alternative hypothesis:

$$H_1: (\mu_i, \sigma_i^2) = \begin{cases} (\mu_0, \sigma_0^2), & 1 \le i \le \tau \\ (\mu_1, \sigma_1^2), & \tau+1 \le i \le n \end{cases} \tag{1}$$

where μ_0 and σ_0^2 are unknown common mean and variance parameters under the null hypothesis and τ is the unknown location of the change point under the alternative hypothesis. In a Bayesian approach, it is assumed that there is one change point located at τ; and deriving the posterior distribution of τ under different prior distributions for other parameters in the model becomes the major task. Specifically, for the change point model given by Equation (1), three practical scenarios are considered. In each of the three scenarios, the prior distribution for the change point τ is assumed to be the following five mostly commonly used priors in the literature, namely, the uniform, binomial, geometric, Poisson, or double truncated Poisson, given as follows:

$$\pi_{01}(\tau)=\frac{1}{n}, \qquad \tau=1,2,\ldots,n$$
$$=0, \qquad \text{otherwise} \tag{2}$$

$$\pi_{02}(\tau)=\binom{n}{\tau}p_1^{\tau}(1-p_1)^{n-\tau}, \qquad \tau=0,1,\ldots,n$$
$$=0, \qquad \text{otherwise} \tag{3}$$

$$\pi_{03}(\tau)=p_2^{\tau-1}(1-p_2), \qquad \tau=1,2,\ldots$$
$$=0, \qquad \text{otherwise} \tag{4}$$

$$\pi_{04}(\tau)=e^{-\lambda}\frac{\lambda^{\tau}}{\tau!}, \qquad \tau=0,1,\ldots$$
$$=0 \qquad \text{otherwise} \tag{5}$$

$$\pi_{05}(\tau)=\left(\sum_{i=1}^{n}\frac{\lambda^{i}}{i!}\right)^{-1}\frac{\lambda^{\tau}}{\tau!}, \qquad \tau=1,2,\ldots,n$$
$$=0, \qquad \text{otherwise} \tag{6}$$

Three practical scenarios are considered according to the choices of the prior distributions on the mean and variance parameters as follows.

2.1. *First Scenario*

For the first scenario, we assume that the joint prior distribution of the means μ_0 and μ_1 is a constant and the joint prior distribution of the variances σ_0^2 and σ_1^2 is noninformative:

$$\omega_{01}(\mu_0,\mu_1\mid \sigma_0^2,\sigma_1^2,\tau)\propto \text{constant},$$
$$\eta_{01}(\sigma_0^2,\sigma_1^2\mid \tau)\propto\frac{1}{\sigma_0^2\sigma_1^2}, \qquad \sigma_0^2,\ \sigma_1^2>0. \tag{7}$$

As the sequence of random variables X_i, $i=1,2,...,n$, follows a normal distribution, the likelihood function of the sample for the model given by Equation (1) is:

$$L_1(\mu_0,\mu_1,\sigma_0^2,\sigma_1^2,\tau)=L_1(\mu_0,\mu_1,\sigma_0^2,\sigma_1^2,\tau\mid x_i, i=1,2,\ldots,n)$$
$$\propto\left(\frac{1}{\sigma_0^2}\right)^{\frac{\tau}{2}}\exp\left\{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{\tau}(x_i-\mu_0)^2\right\}\left(\frac{1}{\sigma_1^2}\right)^{\frac{n-\tau}{2}}\exp\left\{-\frac{1}{2\sigma_1^2}\sum_{i=\tau+1}^{n}(x_i-\mu_1)^2\right\}. \tag{8}$$

Then the joint posterior distribution of the parameters $\mu_0,\mu_1,\sigma_0^2,\sigma_1^2$ and τ can be written as:

$$\pi_k^{(1)}(\mu_0,\mu_1,\sigma_0^2,\sigma_1^2,\tau)$$
$$\propto L_1(\mu_0,\mu_1,\sigma_0^2,\sigma_1^2,\tau)\omega_{01}(\mu_0,\mu_1\mid \sigma_0^2,\sigma_1^2,\tau)\eta_{01}(\sigma_0^2,\sigma_1^2\mid \tau)\pi_{0k}(\tau), \qquad k=1,2,...,5. \tag{9}$$

By taking the integration of Equation (9) with respect to the parameters $\mu_0,\mu_1,\sigma_0^2\ \sigma_1^2$, the marginal posterior distribution of change point is proportionally obtained as follows:

$$\int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \pi_k^{(1)}(\mu_0,\mu_1,\sigma_0^2,\sigma_1^2,\tau)\,d\mu_0\,d\mu_1 d\sigma_0^2 d\sigma_1^2$$

$$\propto \frac{\Gamma\left(\frac{\tau-1}{2}\right)\Gamma\left(\frac{n-\tau-1}{2}\right)}{\sqrt{\tau(n-\tau)}} \left\{\sum_{i=1}^{\tau} x_i^2 - \frac{\left(\sum_{i=1}^{\tau} x_i\right)^2}{\tau}\right\}^{-\left(\frac{\tau-1}{2}\right)} \left\{\sum_{i=\tau+1}^{n} x_i^2 - \frac{\left(\sum_{i=\tau+1}^{n} x_i\right)^2}{n-\tau}\right\}^{-\left(\frac{n-\tau-1}{2}\right)} \pi_{0k}(\tau)$$

$$\stackrel{\Delta}{=} \pi_{1k}^{(1)}(\tau), \qquad k=1,2,...,5 \tag{10}$$

For each of the five priors given in Equations (2)-(6), we obtain the marginal posterior distribution, $\pi_{1k}^{*(1)}(\tau)$, of the parameter τ as:

$$\pi_{1k}^{*(1)}(\tau) = \frac{\pi_{1k}^{(1)}(\tau)}{\sum_{\tau=2}^{n-2} \pi_{1k}^{(1)}(\tau)}, \qquad \tau = 2,3,...,n-2 \text{ and } k=1,2,...,5, \tag{11}$$

where $\pi_{1k}^{(1)}(\tau)$ is defined in Equation (10).

The Bayesian estimate $\hat{\tau}$ of the change point τ is found by the maximization of the marginal posterior distribution given by Equation (11), that is $\hat{\tau}$ is such that:

$$\pi_{1k}^{*(1)}(\hat{\tau}) = \max_{\tau} \pi_{1k}^{*(1)}(\tau), \qquad k=1,2,...,5 \tag{12}$$

where the symbol "$\propto$" means "proportional to" and the symbol "$\stackrel{\Delta}{=}$" means "equal by definition".

2.2. The Second Scenario

For the second scenario, the joint prior distribution of the means μ_0 and μ_1 is taken to be:

$$\omega_{02}(\mu_0,\mu_1 \mid \sigma_0^2,\sigma_1^2,\tau) \propto \frac{1}{\sigma_0} e^{-\frac{1}{2\sigma_0^2}\mu_0^2} \frac{1}{\sigma_1} e^{-\frac{1}{2\sigma_1^2}\mu_1^2} \tag{13}$$

and the joint prior distribution for the variances is still assumed to be given by Equation (7).

From Equations (7), (8) and (13), the joint posterior distribution of the parameters $\mu_0,\mu_1,\sigma_0^2,\sigma_1^2$ and τ can be written as:

$$\pi_k^{(2)}(\mu_0,\mu_1,\sigma_0^2,\sigma_1^2,\tau) \propto L_1(\mu_0,\mu_1,\sigma_0^2,\sigma_1^2,\tau)\omega_{02}(\mu_0,\mu_1 \mid \sigma_0^2,\sigma_1^2,\tau)\eta_{01}(\sigma_0^2,\sigma_1^2 \mid \tau)\pi_{0k}(\tau), \quad k=1,2,...,5. \tag{14}$$

Similarly, by taking the integration of Equation (14) with respect to the parameters μ_0, μ_1, σ_0^2 σ_1^2, the marginal posterior distribution of change point is proportionally obtained as follows:

$$\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty \pi_k^{(2)}(\mu_0,\mu_1,\sigma_0^2,\sigma_1^2,\tau)\,d\mu_0\,d\mu_1 d\sigma_0^2 d\sigma_1^2$$
$$\propto \frac{\Gamma\left(\frac{\tau-1}{2}\right)\Gamma\left(\frac{n-\tau-1}{2}\right)}{\sqrt{(\tau+1)(n-\tau+1)}} A^{-\left(\frac{\tau-1}{2}\right)} B^{-\left(\frac{n-\tau-1}{2}\right)} \pi_{0k}(\tau), \qquad k=1,2,\ldots,5$$
$$\stackrel{\Delta}{=} \pi_{1k}^{(2)}(\tau) \tag{15}$$

where A and B are given by:

$$A=\sum_{i=1}^{\tau} X_i^2-\frac{\left(\sum_{i=1}^{\tau} X_i\right)^2}{\tau+1},\ \ B=\sum_{i=\tau+1}^{n} X_i^2-\frac{\left(\sum_{i=\tau+1}^{n} X_i\right)^2}{n-\tau+1}.$$

From Equation (15), the marginal posterior distribution, $\pi_{1k}^{*(2)}(\tau)$, of the parameter τ, for each of the five priors given in Equations (2)-(6), is given as:

$$\pi_{1k}^{*(2)}(\tau)=\frac{\pi_{1k}^{(2)}(\tau)}{\sum_{\tau=2}^{n-2}\pi_{1k}^{(2)}(\tau)}, \qquad \tau=2,3,\ldots,n-2, \text{ and } k=1,2,\ldots,5, \tag{16}$$

where $\pi_{1k}^{(2)}(\tau)$ is defined in Equation (15).

The Bayesian estimate $\hat{\tau}$ of the change point τ is obtained by the maximization of the marginal posterior distribution of the change point τ, that is, $\hat{\tau}$ is such that:

$$\pi_{1k}^{*(2)}(\hat{\tau})=\max_{\tau}\pi_{1k}^{*(2)}(\tau), \qquad k=1,2,\ldots,5. \tag{17}$$

2.3. The Third Scenario

For the third scenario, the joint prior distribution of means μ_0 and μ_1 is taken as Equation (13) and for the variances, the joint prior distribution is assumed to be the inverse gamma distribution:

$$\eta_{02}(\sigma_0^2,\sigma_1^2\,|\,\tau)\propto\left(\frac{1}{\sigma_0^2}\right)^{\alpha_0+1}\left(\frac{1}{\sigma_1^2}\right)^{\alpha_1+1} e^{-\frac{\beta_0}{\sigma_0^2}} e^{-\frac{\beta_1}{\sigma_1^2}}, \qquad \sigma_0^2,\sigma_1^2>0 \tag{18}$$

where the parameters $\alpha_0,\alpha_1,\beta_0$, β_1 are greater than zero.

The likelihood function of the sample from the normal distribution under the model (1) is given by Equation (8) and then the joint posterior distribution of the parameters $\mu_0,\mu_1,\sigma_0^2,\sigma_1^2$ and τ can be written as:

$$\pi_k^{(3)}(\mu_0,\mu_1,\sigma_0^2,\sigma_1^2,\tau)\propto L_1(\mu_0,\mu_1,\sigma_0^2,\sigma_1^2,\tau)\omega_{02}(\mu_0,\mu_1|\ \sigma_0^2,\sigma_1^2,\tau)\eta_{02}(\sigma_0^2,\sigma_1^2|\ \tau)\pi_{0k}(\tau) \tag{19}$$

Similarly, by taking the integration of Equation (19) with respect to the parameters μ_0,μ_1,σ_0^2, σ_1^2, the marginal posterior distribution of change point is proportionally obtained as follows:

$$\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty \pi_k^{(3)}(\mu_0,\mu_1,\sigma_0^2,\sigma_1^2,\tau)\,d\mu_0\,d\mu_1 d\sigma_0^2 d\sigma_1^2$$

$$\propto \frac{\Gamma\left(\frac{\tau-1+2\alpha_0}{2}\right)\Gamma\left(\frac{n-\tau-1+2\alpha_1}{2}\right)}{\sqrt{(\tau+1)(n-\tau+1)}}(A+2\beta_0)^{-\left(\frac{\tau-1+2\alpha_0}{2}\right)}(B+2\beta_1)^{-\left(\frac{n-\tau-1+2\alpha_1}{2}\right)}\pi_{0k}(\tau),$$

$$k=1,\ldots,5$$

$$\stackrel{\Delta}{=}\pi_{1k}^{(3)}(\tau) \tag{20}$$

Where A and B are given above in Equation (16), and we then obtain the marginal posterior distribution for each of the five priors given in Equations (2)-(6), $\pi_{1k}^{*(3)}(\tau)$, of the parameter τ to be:

$$\pi_{1k}^{*(3)}(\tau)=\frac{\pi_{1k}^{(3)}(\tau)}{\sum_{\tau=2}^{n-2}\pi_{1k}^{(3)}(\tau)}, \qquad \tau=2,3,\ldots,n-2 \text{, and } k=1,2,\ldots,5, \tag{21}$$

where $\pi_{1k}^{(3)}(\tau)$ is defined in Equation (20).

The Bayesian estimate $\hat{\tau}$ of the change point τ is found by the maximization of the marginal posterior distribution given by Equation (21), or $\hat{\tau}$ is such that:

$$\pi_{1k}^{*(3)}(\hat{\tau})=\max_{\tau}\pi_{1k}^{*(3)}(\tau), \qquad k=1,2,\ldots,5. \tag{22}$$

3. Simulation Study

To evaluate the performances of the prior distributions on the Bayesian estimation, $\hat{\tau}$ (given by Equations (12), (17), and (22), respectively), of the change point location τ, we have carried out extensive simulation studies with various sample sizes.

Specifically, we simulated random samples X_i, $i=1,2,\ldots,n$, from the normal distributions, with sample sizes 8, 12, 20, 40, 60, 100, respectively, for the cases of the true change point being located at the front (the $\frac{n}{4}$th observation), at the center (the $\frac{n}{2}$th observation), and at the end (the $\frac{3n}{4}$th observation) of the sequence. For each sample, the values of parameters (μ_0,σ_0) and (μ_1,σ_1) are generated from the prior distributions defined in Section 2. The number of iterations is taken to be 1000 times for each sample size.

It is assumed that the probabilities of changing the parameters (μ_0,σ_0^2) to the parameters (μ_1,σ_1^2) in each observation X_i, $i=1,2,\ldots,n$, are $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$. Therefore p_1 in Equation (3) are taken to be $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$. Similarly, p_2 in Equation (4) are taken to be $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$. The values of λ in Equations (5) and (6) are taken to be $\frac{n}{4}$, $\frac{n}{2}$, and $\frac{3n}{4}$ by using the Poisson approximation of the binomial distribution**.** For the first scenario, the values of variance σ_0^2 are generated form uniform distributions on the interval (0, 1), (0, 2), (0, 3), and (0, 3.5), respectively, and the values of variance σ_1^2 are generated from uniform distributions on the interval (0, 1.5), (0, 3.5), (0, 1.5), and (0, 2) respectively, and for each pair of the variances aforementioned, the values of means (μ_0, μ_1) are chosen to be constant such as (μ_0, μ_1) = (.5, .15), (.8, .24), (1, 3), (1.5, 4.5), (.5, .2), (.8, .3), (1, 4) ,(1.5, 5.5). For the second scenario, the values μ_0 and μ_1 are generated sequentially from normal distributions $N(0,\sigma_0^2)$ and $N(0,\sigma_1^2)$, after the values of the variances (σ_0^2, σ_1^2) are generated from uniform distribution as given for the first scenario. For the third scenario, the values of the variances (σ_0^2, σ_1^2) are generated from the inverse gamma distributions with the parameters taken to be (β_0, β_1)=(.1, 2), (2, .1), (2, 3), (3, 2) for each the values of α_0 and α_1 ($\alpha_0 = \alpha_1$=1,2,…,8). The values μ_0 and μ_1 are then generated respectively from normal distributions $N(0,\sigma_0^2)$ and $N(0,\sigma_1^2)$.

The marginal posterior distributions of the change point given by Equations (11), (16), and (21) are obtained with respect to the five prior distributions of the parameters under the three scenarios and the estimates of the change point are calculated for each parameter value, sample size and change point location in the sample. The relative frequencies of the Bayesian estimate $\hat{\tau}$ being equal to the true change point location are calculated as well for each scenario. The mean of the estimates $\hat{\tau}$, $\bar{\hat{\tau}}$, and the estimated mean square errors (MSEs) are calculated for each scenario. The MSEs are categorized into three groups such as MSE≤1, 1<MSE≤3 and MSE>3. The relative frequencies of these categories in 1000 trials are tabulated in Tables 1-3. For example, in Table 1, when the prior distribution of τ is uniform, n=100 and $\sigma_0, \sigma_1 \le 1$ then the percentage of trials where MSE≤1 in 1000 repeats is 0.7778.

According to our simulation results, the Bayesian estimates of the change point location are the same when the Poisson and double truncated Poisson are used as the prior distributions for the change point location.

In Figures 1, 2, 3 and 4, the left panel displays the relative frequency of the Bayesian estimate $\hat{\tau}$ versus the sample size and the right panel displays the estimated MSE versus the sample size. These figures are plotted for the case of the change point location being at the center of the sequence.

Table 1. The comparison of the performances of the prior distributions on the estimation of the change point for the first scenario, in terms of the frequencies of the MSEs being in different ranges

Prior	n	$\sigma_0,\sigma_1 \le 1$			$\sigma_0 \le 1, \sigma_1 > 1$			$\sigma_0 > 1, \sigma_1 \le 1$			$\sigma_0,\sigma_1 > 1$		
		MSE≤1	1<MSE≤3	MSE>3	MSE≤1	1<MSE≤3	MSE>3	MSE≤1	1<MSE≤3	MSE>3	MSE≤1	1<MSE≤3	MSE>3
Uniform	8	0.7037	0.1852	0.1111	0.4242	0.3939	0.1818	0.5417	0.2500	0.2083	0.0000	0.5000	0.5000
	12	0.7037	0.0741	0.2222	0.2424	0.3333	0.4242	0.4167	0.2500	0.3333	0.0000	0.0833	0.9167
	20	0.7407	0.0370	0.2222	0.3030	0.2727	0.4242	0.4583	0.2083	0.3333	0.0000	0.0000	1.0000
	40	0.7778	0.0000	0.2222	0.4545	0.2121	0.3333	0.5833	0.1250	0.2917	0.0000	0.0833	0.9167
	60	0.7778	0.0000	0.2222	0.5455	0.2424	0.2121	0.6250	0.1667	0.2083	0.0000	0.2500	0.7500
	100	0.7778	0.0000	0.2222	0.5758	0.1818	0.2424	0.7500	0.1250	0.1250	0.0000	0.2500	0.7500
Binomial	8	0.9259	0.0741	0.0000	0.9091	0.0909	0.0000	0.9583	0.0417	0.0000	0.7500	0.2500	0.0000
	12	0.8519	0.1111	0.0370	0.8182	0.1818	0.0000	0.8750	0.1250	0.0000	0.4167	0.4167	0.1667
	20	0.7778	0.1111	0.1111	0.6364	0.3636	0.0000	0.7500	0.2500	0.0000	0.2500	0.2500	0.5000
	40	0.7778	0.1111	0.1111	0.6364	0.3333	0.0303	0.7500	0.1250	0.1250	0.1667	0.3333	0.5000
	60	0.7778	0.1111	0.1111	0.6364	0.3636	0.0000	0.7500	0.1250	0.1250	0.1667	0.0833	0.7500
	100	0.7778	0.1111	0.1111	0.6061	0.3333	0.0606	0.7500	0.1250	0.1250	0.0000	0.2500	0.7500
Geometric	8	0.7407	0.1481	0.1111	0.5455	0.3636	0.0909	0.5000	0.2917	0.2083	0.3333	0.1667	0.5000
	12	0.7037	0.1481	0.1481	0.5152	0.0000	0.4848	0.4583	0.0833	0.4583	0.2500	0.0833	0.6667
	20	0.5926	0.1111	0.2963	0.2727	0.0000	0.7273	0.2917	0.2083	0.5000	0.0000	0.0000	1.0000
	40	0.6667	0.0741	0.2593	0.2121	0.0606	0.7273	0.4167	0.0833	0.5000	0.0000	0.0000	1.0000
	60	0.7407	0.0000	0.2593	0.2727	0.0909	0.6364	0.5000	0.0000	0.5000	0.0000	0.0000	1.0000
	100	0.7407	0.0000	0.2593	0.3636	0.1212	0.5152	0.5000	0.1667	0.3333	0.0000	0.0000	1.0000
Poisson or double truncated Poisson	8	0.8148	0.1852	0.0000	0.7273	0.2727	0.0000	0.6667	0.2917	0.0417	0.4167	0.4167	0.1667
	12	0.8148	0.1111	0.0741	0.5758	0.3939	0.0303	0.6250	0.2917	0.0833	0.1667	0.4167	0.4167
	20	0.7778	0.1111	0.1111	0.5152	0.4545	0.0303	0.6667	0.2083	0.1250	0.0000	0.3333	0.6667
	40	0.7778	0.0741	0.1481	0.5758	0.3333	0.0909	0.6667	0.2083	0.1250	0.0000	0.2500	0.7500
	60	0.7778	0.0741	0.1481	0.6364	0.3030	0.0606	0.7083	0.1667	0.1250	0.0000	0.2500	0.7500
	100	0.7778	0.0741	0.1481	0.6061	0.3030	0.0909	0.7500	0.1250	0.1250	0.0000	0.2500	0.7500

Table 2. The comparison of the performances of the prior distributions on the estimation of the change point for the second scenario, in terms of the frequencies of the MSEs being in different ranges

Prior	n	$\sigma_0,\sigma_1 \le 1$			$\sigma_0 \le 1, \sigma_1 > 1$			$\sigma_0 > 1, \sigma_1 \le 1$			$\sigma_0,\sigma_1 > 1$		
		MSE≤1	1<MSE≤3	MSE>3	MSE≤1	1<MSE≤3	MSE>3	MSE≤1	1<MSE≤3	MSE>3	MSE≤1	1<MSE≤3	MSE>3
Uniform	8	0.1667	0.3889	0.4444	0.1905	0.4286	0.3810	0.0000	0.3810	0.6190	0.0833	0.2500	0.2917
	12	0.1111	0.1389	0.7500	0.1905	0.0476	0.7619	0.0000	0.0952	0.9048	0.0833	0.0417	0.5000
	20	0.1111	0.1389	0.7500	0.1429	0.1429	0.7143	0.0000	0.0476	0.9524	0.0833	0.0417	0.5000
	40	0.2222	0.0278	0.7500	0.2381	0.0476	0.7143	0.0000	0.3333	0.6667	0.1250	0.0000	0.5000
	60	0.2500	0.0000	0.7500	0.2857	0.0000	0.7143	0.0952	0.3810	0.5238	0.1250	0.0000	0.5000
	100	0.2500	0.0000	0.7500	0.2857	0.0000	0.7143	0.1905	0.4762	0.3333	0.1250	0.0000	0.5000
Binomial	8	0.7778	0.2222	0.0000	0.8571	0.1429	0.0000	0.4762	0.3810	0.1429	0.4583	0.1667	0.0000
	12	0.6944	0.3056	0.0000	0.7143	0.2857	0.0000	0.4762	0.2381	0.2857	0.3333	0.2917	0.0000
	20	0.2500	0.6389	0.1111	0.2857	0.5714	0.1429	0.3810	0.2857	0.3333	0.1250	0.2500	0.2500
	40	0.2500	0.1389	0.6111	0.2857	0.4286	0.2857	0.2857	0.5238	0.1905	0.1250	0.1250	0.3750
	60	0.2500	0.0556	0.6944	0.2857	0.2857	0.4286	0.3333	0.6190	0.0476	0.1250	0.0417	0.4583
	100	0.2500	0.0000	0.7500	0.2857	0.0000	0.7143	0.3810	0.4762	0.1429	0.1250	0.0000	0.5000
Geometric	8	0.3333	0.1389	0.5278	0.5238	0.0952	0.3810	0.3333	0.3333	0.3333	0.2500	0.0417	0.3333
	12	0.3333	0.0000	0.6667	0.4286	0.0952	0.4762	0.3333	0.0000	0.6667	0.2083	0.0417	0.3750
	20	0.0000	0.0556	0.9444	0.0952	0.1429	0.7619	0.0000	0.0952	0.9048	0.0000	0.0417	0.5833
	40	0.0000	0.0000	1.0000	0.1429	0.0476	0.8095	0.0000	0.0000	1.0000	0.0000	0.0833	0.5417
	60	0.0000	0.0000	1.0000	0.1905	0.0000	0.8095	0.0000	0.0000	1.0000	0.0000	0.0833	0.5417
	100	0.0000	0.0000	1.0000	0.1905	0.0476	0.7619	0.0000	0.1429	0.8571	0.1250	0.0000	0.5000
Poisson or double truncated Poisson	8	0.5833	0.4167	0.0000	0.6667	0.3333	0.0000	0.3333	0.3810	0.2857	0.2917	0.2917	0.0417
	12	0.2500	0.5278	0.2222	0.3810	0.5238	0.0952	0.2381	0.2381	0.5238	0.2083	0.1250	0.2917
	20	0.2500	0.1944	0.5556	0.2857	0.2857	0.4286	0.0952	0.2857	0.6190	0.1250	0.0417	0.4583
	40	0.2500	0.0278	0.7222	0.2857	0.0952	0.6190	0.0952	0.5714	0.3333	0.1250	0.0000	0.5000
	60	0.2500	0.0000	0.7500	0.2857	0.0476	0.6667	0.2381	0.5714	0.1905	0.1250	0.0417	0.4583
	100	0.2500	0.0000	0.7500	0.2857	0.0000	0.7143	0.2381	0.5714	0.1905	0.1250	0.0000	0.5000

Table 3. The comparison of the performances of the prior distributions on the estimation of the change point for the third scenario, in terms of the frequencies of the MSEs being in different ranges

Prior	n	$\sigma_0,\sigma_1 \le 1$			$\sigma_0 \le 1, \sigma_1 > 1$			$\sigma_0 > 1, \sigma_1 \le 1$			$\sigma_0,\sigma_1 > 1$		
		MSE≤1	1<MSE≤3	MSE>3	MSE≤1	1<MSE≤3	MSE>3	MSE≤1	1<MSE≤3	MSE>3	MSE≤1	1<MSE≤3	MSE>3
Uniform	8	0.3333	0.0952	0.5714	0.0833	0.5833	0.3333	0.0000	0.3333	0.6667	0.1111	0.5556	0.3333
	12	0.1667	0.1667	0.6667	0.0417	0.2917	0.6667	0.0000	0.0000	1.0000	0.0000	0.4444	0.5556
	20	0.0000	0.0952	0.9048	0.0000	0.0833	0.9167	0.0000	0.0000	1.0000	0.0000	0.2222	0.7778
	40	0.0000	0.0476	0.9524	0.0000	0.0833	0.9167	0.0000	0.0000	1.0000	0.0000	0.2222	0.7778
	60	0.0000	0.0952	0.9048	0.0417	0.2083	0.7500	0.0000	0.0000	1.0000	0.0000	0.3333	0.6667
	100	0.0000	0.2857	0.7143	0.0417	0.5000	0.4583	0.0000	0.0000	1.0000	0.0000	0.3333	0.6667
Binomial	8	1.0000	0.0000	0.0000	0.4167	0.3333	0.2500	0.3333	0.5000	0.1667	0.7778	0.2222	0.0000
	12	0.9048	0.0952	0.0000	0.3750	0.1250	0.5000	0.2500	0.3333	0.4167	0.6667	0.3333	0.0000
	20	0.4286	0.3810	0.1905	0.0000	0.5417	0.4583	0.0000	0.5833	0.4167	0.3333	0.6667	0.0000
	40	0.4048	0.2381	0.3571	0.0000	0.7500	0.2500	0.0000	0.6667	0.3333	0.1111	0.6667	0.2222
	60	0.1905	0.3571	0.4524	0.0417	0.6250	0.3333	0.0000	0.5000	0.5000	0.1111	0.3333	0.5556
	100	0.0714	0.4286	0.5000	0.1667	0.4583	0.3750	0.0000	0.4167	0.5833	0.2222	0.1111	0.6667
Geometric	8	0.4286	0.1667	0.4048	0.0833	0.4583	0.4583	0.3333	0.1667	0.5000	0.3333	0.3333	0.3333
	12	0.2381	0.1667	0.5952	0.0833	0.0833	0.8333	0.3333	0.0000	0.6667	0.3333	0.0000	0.6667
	20	0.0000	0.0000	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000	1.0000	0.0000	0.1111	0.8889
	40	0.0000	0.0000	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000	1.0000
	60	0.0000	0.0000	1.0000	0.0417	0.0417	0.9167	0.0000	0.0000	1.0000	0.0000	0.0000	1.0000
	100	0.0000	0.0000	1.0000	0.0833	0.0833	0.8333	0.0000	0.0000	1.0000	0.0000	0.0000	1.0000
Poisson or double truncated Poisson	8	0.8333	0.1667	0.0000	0.3750	0.3750	0.2500	0.1667	0.5000	0.3333	0.5556	0.4444	0.0000
	12	0.6190	0.2143	0.1667	0.3333	0.1250	0.5417	0.1667	0.1667	0.6667	0.4444	0.2222	0.3333
	20	0.1905	0.2381	0.5714	0.0000	0.2500	0.7500	0.0000	0.1667	0.8333	0.1111	0.5556	0.3333
	40	0.1429	0.2143	0.6429	0.0000	0.4167	0.5833	0.0000	0.1667	0.8333	0.1111	0.2222	0.6667
	60	0.0952	0.3333	0.5714	0.0417	0.4583	0.5000	0.0000	0.2500	0.7500	0.0000	0.3333	0.6667
	100	0.0238	0.4286	0.5476	0.0833	0.5000	0.4167	0.0000	0.1667	0.8333	0.1111	0.2222	0.6667

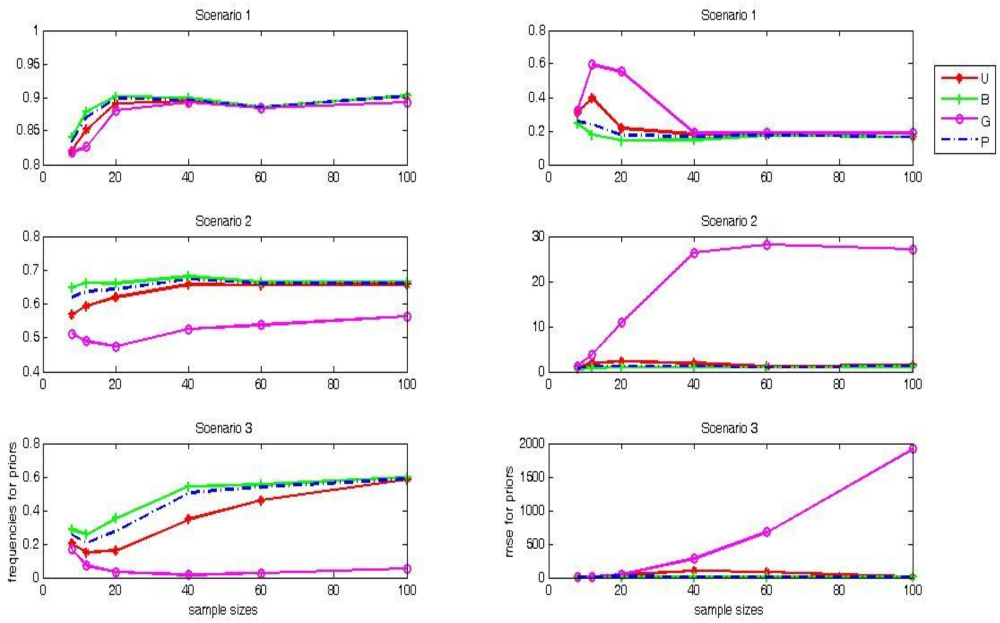

Figure 1. The relative frequency of the Bayesian estimate $\hat{\tau}$ and the estimated MSEs versus the sample size with the parameter values $(\mu_0,\sigma_0)=(.5,.0377)$, $(\mu_1,\sigma_1)=(.15,.2048)$ for the first scenario, $(\mu_0,\sigma_0)=(.0567,.1734)$, $(\mu_1,\sigma_1)=(.1721,.9853)$ for the second scenario, and $(\mu_0,\sigma_0)=(.7127,\ .8738)$, $(\mu_1,\sigma_1)=(.1613,\ .2266)$ $\alpha_0=\alpha_1=8$, $(\beta_0,\beta_1)=(.1,\ 2)$ for the third scenario. (U: Uniform, B: Binomial, G: Geometric, P: Poisson or double truncated Poisson)

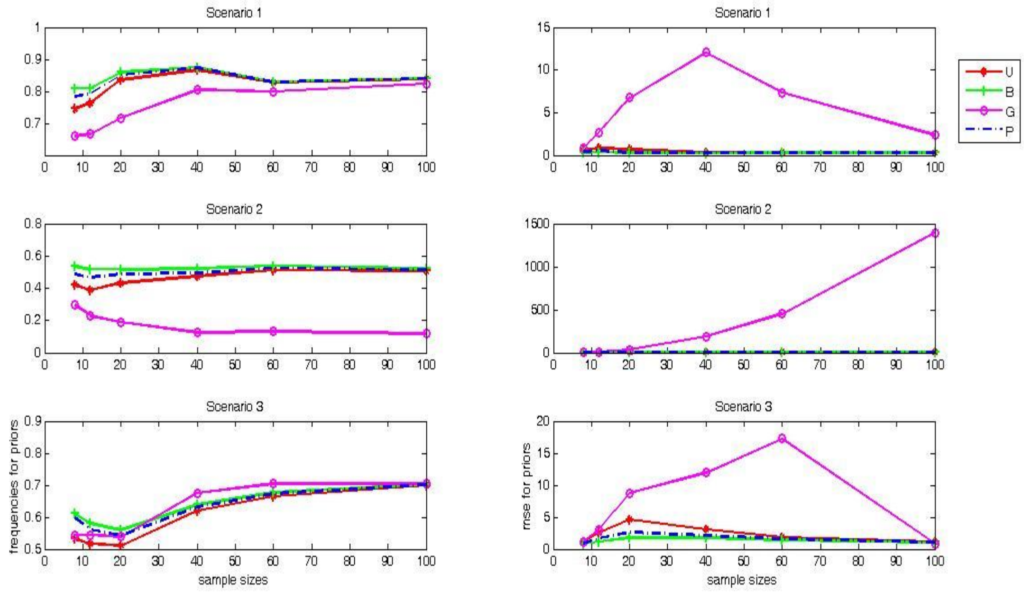

Figure 2. The relative frequency of the Bayesian estimate $\hat{\tau}$ and the estimated MSEs versus the sample size with the parameter values $(\mu_0,\sigma_0)=(1.5,\ .9880)$, $(\mu_1,\sigma_1)=(4.5,1.0196)$ for the first scenario, $(\mu_0,\sigma_0)=(.1307,.3993)$, $(\mu_1,\sigma_1)=(.2578,1.4761)$ for the second scenario, and $(\mu_0,\sigma_0)=(-.2465,\ .2440)$, $(\mu_1,\sigma_1)=(.7093,1.1544)$, $\alpha_0=\alpha_1=8$, $(\beta_0,\beta_1)=(2,\ 3)$ for the third scenario. (U: Uniform, B: Binomial, G: Geometric, P: Poisson or double truncated Poisson)

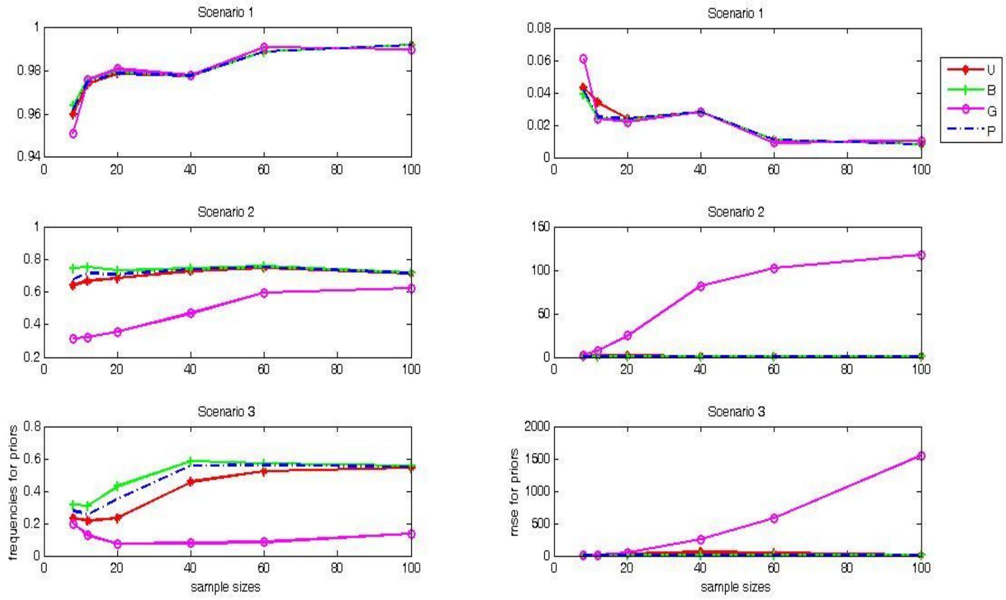

Figure 3. The relative frequency of the Bayesian estimate $\hat{\tau}$ and the estimated MSEs versus the sample size with the parameter values $(\mu_0, \sigma_0) = (1, 1.4697)$, $(\mu_1, \sigma_1) = (4, .0636)$ for the first scenario, $(\mu_0, \sigma_0) = (-.9515, 1.6174)$, $(\mu_1, \sigma_1) = (1.8378, .8418)$ for the second scenario, and $(\mu_0, \sigma_0) = (.8751, 1.0730)$, $(\mu_1, \sigma_1) = (.2111, .2965)$, $\alpha_0 = \alpha_1 = 7$, $(\beta_0, \beta_1) = (.1, 2)$ for the third scenario. (U: Uniform, B: Binomial, G: Geometric, P: Poisson or double truncated Poisson)

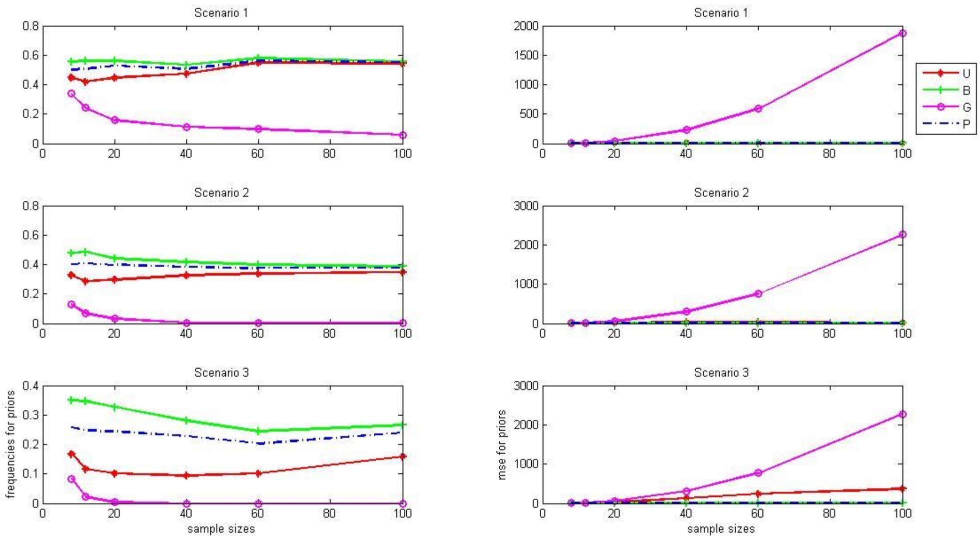

Figure 4. The relative frequency of the Bayesian estimate $\hat{\tau}$ and the estimated MSEs versus the sample size with the parameter values $(\mu_0, \sigma_0) = (1, 1.3975)$, $(\mu_1, \sigma_1) = (4, 2.1618)$ for the first scenario, $(\mu_0, \sigma_0) = (-.2755, 1.4757)$, $(\mu_1, \sigma_1) = (1.7810, 2.4538)$ for the second scenario, and $(\mu_0, \sigma_0) = (1.8367, 2.2518)$, $(\mu_1, \sigma_1) = (1.0143, 1.4247)$, $\alpha_0 = \alpha_1 = 1$, $(\beta_0, \beta_1) = (.1, 2)$ for the third scenario. (U: Uniform, B: Binomial, G: Geometric, P: Poisson or double truncated Poisson)

4. Conclusion

Figures 1-4 show the results with the true change point location being at the center of the sequence, the results for the other true change point locations (at the beginning and at end of the sequence) are more or less the same. In general, as it is seen from Figures 1-4, when the sample size is below 40, as the sample size increases, the relative frequency of the Bayesian estimate $\hat{\tau}$ also increases, in the mean time, the estimated MSEs decrease as desired for the some prior distributions of the change point. When $\sigma_0, \sigma_1 \leq 1$, the relative frequencies of the Bayesian estimate $\hat{\tau}$ being equal to the true change location are high, with approximately 90%, 70%, 60%, respectively, for the three scenarios. From Figures 1-4, one can observe that generally the behavior of the prior distributions of the change point is more or less the same except for the geometric prior distribution. The difference is clearly seen especially in Figure 4. When $\sigma_0, \sigma_1 > 1$, the relative frequencies of the Bayesian estimate $\hat{\tau}$ being equal to the true change location are seen decreasing to, approximately 58% and 40%, respectively, for the first and the second scenario with the uniform, binomial, and Poisson priors, but for the third scenario, the relative frequencies of the Bayesian estimate $\hat{\tau}$ respectively are less than 40%.

Tables 1-3 display the comparisons of performances of the prior distributions with respect to the classes of the standard deviations (σ_0, σ_1) and the sample sizes in terms of the frequencies of the estimated MSEs for each scenario. The performance of the uniform prior distribution for each scenario improves as the sample size increases, whereas the other prior distributions have a good performance for the small sample sizes being below 20. For the first scenario, when $\sigma_0, \sigma_1 \leq 1$, the performances of the posterior distributions of the change point are at least 77% according to MSE ≤ 1, whereas for the second and third scenarios the performances of the posterior distributions of the change point where especially binomial and Poisson prior distributions are above 58% according to MSE ≤ 1, when the sample size 8. The posterior distributions of the change point lose their ability to detect the change point location when both of the standard deviations are greater than 1.

Table 4 displays the frequencies of the estimated MSEs of the change point estimates for each prior distribution regardless of the sample size. This table shows that for all three scenarios, the performance of the binomial prior distribution on the Bayesian estimation of the change point has the best success rate of 90%, 60%, and 70%, respectively, the performance of the Poisson prior distribution shows a good success rate of 85%, 53%, and 52%, respectively, and then the success rates of the uniform prior distribution are 68%, 30%, and 24%, respectively. The performance of the geometric prior distribution has the worst success among of others.

Table 4. The comparison of the performances of the prior distributions on the estimation of the change point for all three scenarios, in terms of the frequencies of the MSEs being in different ranges

Scenario	Prior	MSE≤1	1<MSE≤3	MSE>3	MSE≤3
First	Uniform	0.4965	0.1788	0.3247	0.6753
	Binomial	0.7101	0.1910	0.0990	0.9010
	Geometric	0.4444	0.0990	0.4566	0.5434
	Poisson or double truncated Poisson	0.6128	0.2413	0.1458	0.8542
Second	Uniform	0.1595	0.1487	0.6918	0.3082
	Binomial	0.4050	0.2849	0.3100	0.6900
	Geometric	0.1523	0.0627	0.7849	0.2151
	Poisson or double truncated Poisson	0.2903	0.2437	0.4659	0.5341

Third	Uniform	0.0517	0.1877	0.7605	0.2395
	Binomial	0.3391	0.3602	0.3008	0.6992
	Geometric	0.0939	0.0690	0.8372	0.1628
	Poisson or double truncated Poisson	0.2222	0.2931	0.4847	0.5153

Combining all the simulation results, it is fair to state that the first scenario has the best ability to detect the change point in both the mean and the variance of the sequence of normal random variables.

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