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DEGREES OF SOLID VARIETIES OF SEMIRINGS

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Abstract. For any arbitrary variety V, the degree $d_p(V)$ of V with respect to proper *hypersubstitutions* was introduced in [6]. This degree of any variety of bands was determined in [4]. In this paper we characterize the universe of the free algebra of each solid variety of semirings and from this we derive the degree $d_p(V)$ if V is any solid variety of semirings.

1. Introduction

Hypersubstitutions are mappings sending operation symbols to terms and preserving arities. They can be applied to algebras and equations. This gives the concepts of a derived algebra and of a hyperidentity. If every identity of a given variety V of algebras is a hyperidentity, then the variety V is said to be *solid*. More details on the notions of hypersubstitions, hyperidentities and solid varieties can be seen in [9], [8] and [7]. All solid varieties of semirings were determined (see in [3]). Now, we want to investigate some properties of solid varieties of semirings. In Universal Algebra like also in other parts of Mathematics, *numerical invariants* play an important role. An example of a numerical invariant in Universal Algebra is the spectrum of a variety V which is defined as a sequence $spect(V) := (|\mathcal{F}_V(n)|)_{n\geq 1}$ of the cardinality of the n-generated free algebras with respect to V. In this paper, we consider a numerical invariant related to the spectrum, the *degree of a variety.* A hypersubstitution is called *V-proper* if it preserves all identities of the variety V. The set of all V-proper hypersubstitions will be denoted by $P(V)$.
The degree $d_p(V)$ of the variety V with respect to proper hypersubstitutions is the cardinality of the quotient set $P(V)|_{\sim_V}$, where $\sim_V$ is the binary relation on the set $Hyp(\tau)$ (of all hypersubstitutions of type $\tau = (n_i)_{i\in I}$), introduced in [11] and defined by $\sim_V$: $\sigma_1 \sim_V \sigma_2$ iff $\sigma_1(f_i) \approx \sigma_2(f_i) \in IdV$ for $i \in I$.
Then one can ask the following questions:
- What is the degree of a given variety?

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- For a given integer, determine all varieties which have this integer as its degree.
- What structural properties of a given variety V can be derived from informations about the degree?

Some of these questions are already answered for varieties of semigroups. For instance, $d_p(V)$ is known for any variety V of bands (idempotent semigroups) and in this case $d_p(V) \in \{1, 2, 3, 4, 6\}$ (see in [4] and [13]).
In this paper, we want to answer the first question in the case of solid varieties of semirings. To do this we need to recall some basic concepts.

1. A semiring is a non-empty set S together with two binary operations $+$ and $\cdot$ such that the algebras $(S;\ +)$ and $(S;\ \cdot)$ are semigroups which are related by the distributive identities:
$$x_1(x_2 + x_3) \approx x_1x_2 + x_1x_3 \text{ and } (x_1 + x_2)x_3 \approx x_1x_3 + x_2x_3.$$
The variety of all semirings will be denoted by SR.
2. A semiring $(S; +, \cdot)$ is *medial* if
$$x_1 + x_2 + x_3 + x_4 \approx x_1 + x_3 + x_2 + x_4 \in IdS \text{ and}$$
$$x_1x_2x_3x_4 \approx x_1x_3x_2x_4 \in IdS.$$
(IdS is the set of all identities satisfied in the algebra $(S; +, \cdot)$).
3. A semiring $(S; +, \cdot)$ is *idempotent* if
$$x_1 + x_1 \approx x_1 \approx x_1 \cdot x_1 \in IdS.$$
4. A semiring $(S; +, \cdot)$ is *distributive* if
$$x_1x_2 + x_3 \approx (x_1 + x_3)(x_2 + x_3) \in IdS \textit{ and}$$
$$x_1 + x_2x_3 \approx (x_1 + x_2)(x_1 + x_3) \in IdS.$$
5. A variety V of semirings is *medial* if all algebras in V are medial. In a similar way one can define the varieties of distributive semirings and idempotent semirings, respectively.
6. An equation $s \approx t$ is regular if both terms s, t contain the same variables. A variety V is regular if all identities in V are regular.
It is well known that a variety V is regular iff its basis identities are regular.

If V is a variety of semirings and Σ is a set of equations, by $V(\Sigma)$ we denote the subvariety of V which is generated by the set Σ. For reference, below we list some varieties to be used in this paper:
V_{MID} - the variety of all medial, idempotent and distributive semirings,
$V_{BE} := V_{MID}(\{(x_1 + x_2)(x_2 + x_1) \approx x_1x_2 + x_2x_1\})$;
$RA_{2,2} := SR(\{x_1 + x_2 + x_3 \approx x_1 + x_3, x_1x_2x_3 \approx x_1x_3, x_1x_1 \approx x_1 \approx x_1 + x_1\})$;
$\mathcal{T}$ - the trivial variety of semirings, i.e. the class of all one-element semirings.
It is clear that the varieties V_{MID} and V_{BE} are regular since their basis identities are regular.

Now, we turn to the theory of hyperidentities.
Let $W_{(2,2)}(X_2)$ be the set of all binary terms of type $(2, 2)$ built up by variables from the alphabet $X_2 = \{x, y\}$. The operation symbols F and G will be denoted sometimes additively and multiplicatively, respectively.
Hypersubstitutions of type $\tau = (2, 2)$ are mappings
$$\sigma : \{F, G\} \to W_{(2,2)}(X_2)$$

A hypersubstitution σ of type $(2,2)$ can be extended to a mapping $\hat{\sigma}$ defined on the set $W_{(2,2)}(X)$ of all terms of type $(2,2)$, where $X = \{x, y, z, u, v, \cdots\}$ is an arbitrary countably infinite alphabet of variables, by the following steps:

(i) $\hat{\sigma}[t] := t$, if $t \in X$ is a variable, and

(ii) $\hat{\sigma}[f(t_1, t_2)] := \sigma(f)(\hat{\sigma}[t_1], \hat{\sigma}[t_2]), f \in \{F, G\}$ for composed terms.

The right hand side of (ii) must be interpreted as a superposition of term operations of the term algebra of type $(2,2)$.

The hypersubstitution σ of type (2,2) such that $\sigma(F) = t$ and $\sigma(G) = s$ will be denoted by $\sigma_{t,s}$. Together with the hypersubstitution σ_{id} defined by $\sigma_{id}(f) = f(x,y), f \in \{F, G\}$, the set of all hypersubstitutions of type $(2,2)$ forms a monoid, denoted by Hyp.

An identity $s \approx t$ in a variety V of semirings is called a *hyperidentity* in V if for every $\sigma \in Hyp$ the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ belong to the set IdV of all identities satisfied in V.

A variety V is called solid if all identities in V are satisfied as hyperidentities.

Let V be a variety. A hypersubstitution σ is called *V-proper* (or *preserving all identities*) if for all identities $s \approx t \in IdV$, $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$.

Now, we will determine the universe of the free algebra with respect to each element of the lattice of all solid varieties of semirings. This lattice was fully described in [3] as follows:

Theorem 1.1. [3] *The lattice of all solid varieties of semirings is the 4-element chain $\mathcal{T} \subset RA_{(2,2)} \subset V_{BE} \subset V_{MID}$.* ■

2. Free algebras with respect to Solid Varieties of Semirings

In this section we will determine the free algebra with respect to each non-trivial solid variety of semirings.

Definition 2.1. [12] The term algebra of type $\tau = (n_i)_{i \in I}$ is the algebra $\mathcal{F}_\tau(X) := (W_\tau(X); (\overline{f_i})_{i \in I})$, where

$$\overline{f_i}(t_1, \ldots, t_{n_i}) := f_i(t_1, \ldots, t_{n_i}) \text{ for all } t_1, \ldots, t_{n_i} \in W_\tau(X) \text{ and for all } i \in I.$$

For every variety V, it is clear that IdV is a congruence relation on the term algebra $\mathcal{F}_\tau(X)$. Then we can construct the quotient algebra $\mathcal{F}_\tau(X)/IdV$ which is called *free algebra* with respect to V freely generated by the set X and is denoted by $\mathcal{F}_V(X)$. The *free algebra* with respect to V freely generated by the set $X_n = \{x_1, \ldots, x_n\}$ will be denoted by $\mathcal{F}_V(n)$.

The following results and concepts are needed.

Lemma 2.1. [3] *Every subvariety V of V_{MID} satisfies the following identities:*

1.	$x + xy + y \approx x + y,$	*10.*	$xy + yxy + yx \approx xy + yx,$
2.	$x + yx + y \approx x + y,$	*11.*	$xy + yxy + y \approx xy + y,$
3.	$x + xyx + xy \approx x + xy,$	*12.*	$x + xyx + y \approx x + y,$
4.	$x + xyx + yx \approx x + yx,$	*13.*	$x + yxy + y \approx x + y,$
5.	$x + yx + yxy \approx x + yxy,$	*14.*	$xyx + yx + y \approx xyx + y,$
6.	$x + xy + yxy \approx x + yxy,$	*15.*	$xyx + xy + y \approx xyx + y,$
7.	$x + xyx + yxy \approx x + yxy,$	*16.*	$xyx + xy + yxy \approx xyx + yxy,$
8.	$xy + xyx + x \approx xy + x,$	*17.*	$xyx + yx + yxy \approx xyx + yxy,$
9.	$xy + xyx + yx \approx xy + yx,$	*18.*	$xyx + yxy + y \approx xyx + y.$ ■

Definition 2.2. (1) Let V be a variety. Two terms t and s are called V-equivalent if $s \approx t \in IdV$ (see in [10]).
(2) An equation $s \approx t$ is outermost, if the terms s and t start with the same variable and end also with the same variable. A variety V is called outermost, if all identities of IdV are outermost (see in [8]).

It is well known that a variety is outermost iff its basis identities are outermost. Let $Bin_G = \{x, y, xy, yx, xyx, yxy\}$. It is clear that two different elements of Bin_G are not V-equivalent for $V = V_{MID}$ or $V = V_{BE}$ because V_{MID} and V_{BE} are outermost and regular (since the basis identities of V_{MID} and V_{BE} are outermost and regular).

Lemma 2.2. [3] *Let V be a subvariety of V_{MID}. Then every binary term t is V-equivalent to a sum of the form $t_1+t_2+t_3+t_4$, where $t_i \in Bin_G$, $i = 1, 2, 3, 4$.* ■

Now, we can prove:

Lemma 2.3. *Let $s_i, t_i, i \in \{1, 2, 3, 4\}$ be elements from Bin_G.*
If $s_1 + s_2 + s_3 + s_4 \approx t_1 + t_2 + t_3 + t_4 \in IdV$, with $V \in \{V_{MID}, V_{BE}\}$ then $s_1 = t_1$ and $s_4 = t_4$.

Proof: Since V is solid, applying the hypersubstitutions $\sigma_{x,xy}$ (resp. $\sigma_{y,xy}$) to the identity $s_1+s_2+s_3+s_4 \approx t_1+t_2+t_3+t_4 \in IdV$, where $s_i, t_i \in Bin_G, (i \in \{1, 2, 3, 4\})$, we obtain in V the identity $s_1 \approx t_1$ (resp. $s_4 \approx t_4$). This leads to the equalities $s_1 = t_1$ and $s_4 = t_4$ since the binary terms $t_i, s_i, i = 1, 2, 3, 4$ belong to Bin_G and two different elements of Bin_G are not V-equivalents.
The following lemma gives another generating identity of V_{BE}.

Lemma 2.4. [3] *The variety $V_{MID}(xy + xyx + xy \approx xy + x + xy)$ is equal to the variety V_{BE}.* ■

Using the previous lemma, we have:

Lemma 2.5. (1) *The identities $xyx + x + xyx \approx xyx$ and $xy + yx \approx xy + t + yx$, for $t \in \{x, y\}$ hold in V_{BE} but not in V_{MID}.*
(2) *Let $u \in \{x, y\}$ and $t_i \in Bin_G \backslash \{u\}, i = 1, 2, 3$. Then $t_1 + t_2 + t_3 \approx t_1 + u + t_3 \notin IdV_{MID}$.*

Proof:

(1) From $xy+yx \approx xy+x+y+yx \in IdV_{BE}$, it follows the identity $xyx \approx xyx+x+xyx \in IdV_{BE}$ by substituting xyx for y and using the idempotency and the medial laws. Moreover, the variety V_{BE} satisfies the following identities: $xy+x+yx \approx xy+yx+x+yx \approx xy+x+y+yx \approx xy+yx \approx xy+y+yx$, using the idempotency, the medial law and the identity $xy + x + y + yx \approx xy + yx \in IdV_{BE}$.
Assume that the variety V_{MID} satisfies the identity $xyx + x + xyx \approx xyx$. Then we would get in V_{MID} the identities
$xy+xyx+xy \approx xy+xyx+x+xyx+xy \approx xy+x+xy$ by Lemma 2.1 (3). This leads to the equalities $V_{MID} = V_{MID}(xy+xyx+xy \approx xy+x+xy) = V_{BE}$ (Lemma 2.4). This contradicts $V_{MID} \neq V_{BE}$.
Assume that the identity $xy + x + yx \approx xy + yx$ or the identity $xy + y +$

$yx \approx xy + yx$ holds in the variety V_{MID}. Then substituting xyx for y in the first identity and yxy for y in the second one, leads to the equation $xyx + x + xyx \approx xyx$ which is not satisfied as an identity in V_{MID}.

(2) Let $u, v \in \{x, y\}$ such that $u \neq v$ and $t_i \in Bin_G \backslash \{u\}, i = 1, 2, 3$.
Assume that $t_1 + t_2 + t_3 \approx t_1 + u + t_3 \in IdV_{MID}$.
The substitution v by uvu in $t_1 + t_2 + t_3 \approx t_1 + u + t_3 \in IdV_{MID}$
gives $uvu \approx uvu + u + uvu \in IdV_{MID}$. This contracdicts 1. ■

Lemma 2.6. *Let* $(t_1, t_2) \in (\{x, xy, xyx\})^2$ *such that the equation* $t_1 \approx t_2$ *is outermost. Let* $(s, t) \in (Bin_G)^2$. *If the equation* $s \approx t$ *is not outermost, then* $t_1 + s + t_2 \approx t_1 + t + t_2 \notin IdV_{BE}$.

Proof: Assume that $t_1 + s + t_2 \approx t_1 + t + t_2 \in IdV_{BE}$. Since the equation $s \approx t$ is not outermost, without loss of generality, we can assume that s starts with x and t with y. Applying $\sigma_{x+y,x}$ to the equation $t_1 + s + t_2 \approx t_1 + t + t_2$ gives (using the idempotency) the equation $x \approx x + y + x$ which is not satisfied as identity in V_{BE} because V_{BE} is regular. ■

Lemma 2.7. *There are exactly (up to permutation of variables x* and *y) the following binary terms which are V_{MID}-equivalent to sums consisting of three summands from Bin_G, these sums cannot be V_{MID}-equivalent to sums consisting of at most two summands from Bin_G :*

1. $x + xy + t, t \in \{x, xyx, yx\}$,
2. $x + y + t, t \in Bin_G \backslash \{y\}$,
3. $x + yx + t, t \in \{x, xyx, xy\}$,
4. $x + yxy + t, t \in \{x, xyx\}$,
5. $x + xyx + x$,
6. $xyx + x + t, t \in Bin_G \backslash \{x\}$,
7. $xyx + y + t, t \in Bin_G \backslash \{y\}$,
8. $xyx + xy + t, t \in \{x, yx, xyx\}$,
9. $xyx + yxy + t, t \in \{x, xyx\}$,
10. $xyx + yx + t, t \in \{x, xy, xyx\}$,
11. $xy + x + t, t \in Bin_G \backslash \{x\}$,
12. $xy + yx + t, t \in Bin_G \backslash \{yx\}$,
13. $xy + y + t, t \in Bin_G \backslash \{y\}$,
14. $xy + xyx + xy$,
15. $xy + yxy + xy$.

Proof: To prove that $s \approx t \notin IdV_{MID}$, we will show sometimes that $s \approx t \notin IdV_{BE}$, since $V_{BE} \subset V_{MID}$ so $IdV_{MID} \subseteq IdV_{BE}$.
I) First, we show that each term in the list of Lemma 2.7 cannot be V_{MID}-equivalent to a sum of at most two elements from Bin_G.
• To prove that each term in the list (Lemma 2.7) cannot be V_{MID}-equivalent to an element from Bin_G, using Lemma 2.3, we have to consider the equations
$x + t_1 + x \approx x, t_1 \in Bin_G \backslash \{x\};\ xy + t_2 + xy \approx xy, t_2 \in Bin_G \backslash \{xy\}$;
$xyx + t_3 + x \approx xyx, t_3 \in Bin_G \backslash \{xyx\}$; and to show that the variety V_{MID} does not satisfy one of them as identities.
Indeed, the idempotency and Lemma 2.6 ensure that none of the identities
$x + t_1 + x \approx x, t_1 \in Bin_G \backslash \{x, xyx\};\ xy + t_2 + xy \approx xy, t_2 \in Bin_G \backslash \{xy\})$;
$xyx + t_3 + x \approx xyx, t_3 \in Bin_G \backslash \{xyx, x\}$ is satisfied in V_{BE}. The regularity property of V_{BE} guaranties that $x + xyx + x \approx x \notin IdV_{BE}$.
We conclude that $xyx + x + xyx \approx xyx \notin IdV_{MID}$ [see Lemma 2.5 (1)].
• Now, we prove that none of the terms in the list of Lemma 2.7 can be V_{MID}-equivalent to a sum consisting of two terms of Bin_G.
Lemma 2.3, the idempotency and Lemma 2.6 ensure that we have to consider only

the equations:

i.	$x+xy+yx \approx x+yx$	xiv.	$xyx+yx+xy \approx xyx+xy$
ii.	$x+y+xy \approx x+xy$	xv.	$xy+x+y \approx xy+y$
iii.	$x+y+yx \approx x+yx$	xvi.	$xy+x+yx \approx xy+yx$
iv.	$x+y+yxy \approx x+yxy$	xvii.	$xy+x+xyx \approx xy+xyx$
v.	$x+yx+xy \approx x+xy$	xviii.	$xy+x+yxy \approx xy+yxy$
vi.	$xyx+x+xy \approx xyx+xy$	xix.	$xy+yx+x \approx xy+x$
vii.	$xyx+x+yx \approx xyx+yx$	xx.	$xy+yx+y \approx xy+y$
viii.	$xyx+x+yxy \approx xyx+yxy$	xxi.	$xy+yx+xyx \approx xy+xyx$
ix.	$xyx+x+y \approx xyx+yxy$	xxii.	$xy+yx+yxy \approx xy+yxy$
x.	$xyx+y+xy \approx xyx+xy$	xxiii.	$xy+y+x \approx xy+x$
xi.	$xyx+y+yx \approx xyx+yx$	xxiv.	$xy+y+yx \approx xy+yx$
xii.	$xyx+y+yxy \approx xyx+yxy$	xxv.	$xy+y+xyx \approx xy+xyx$
xiii.	$xyx+xy+yx \approx xyx+yx$	xxvi.	$xy+y+yxy \approx xy+yxy$

and to show that the variety V_{MID} satisfies none of them as identities.
Assume that the previous identities (except iv.,vi., vii., viii.,ix., xii., xvi., xvii.,xxiv., xxvi.) are satisfied in V_{BE}.
Since V_{BE} is solid, using the hypersubstitution $\sigma_{x+y,x}$ or $\sigma_{x+y,y}$, we will get the contradiction $x+y+x \approx x \in IdV_{BE}$. Therefore, we conclude that none of the identities (i.,ii.,iii. v., x.,xi.,xiii., xiv., xv., xviii.,xix.,xx., xxi., xxii.,xxiii. and xxv.) is satisfied in V_{BE}.
Lemma 2.5 (2) guarantees that none of the identities (iv.,vi., vii., viii.,ix., xii., xvi., xvii.,xxiv. and xxvi.) is satisfied in V_{BE}.
II) Secondly, we will prove that each term in the list is V_{MID}-equivalent to itself only.
Using Lemma 2.3 and Lemma 2.6, we have to consider only the equation $t_1+t_2+t_3 \approx t_1+t_4+t_3$ (where both binary terms $t_1+t_2+t_3$ and $t_1+t_4+t_3$ belong to the list of Lemma 2.7, and the terms $t_i, i=1,\cdots,4$ do not satisfy the conditions of Lemma 2.6) and to show that the variety V_{MID} does not satisfy one of them as identity.
1. We consider only the equation $x+xy+yx \approx x+y+yx$.
By Lemma 2.5 (2), one has $x+xy+yx \approx x+y+yx \notin IdV_{MID}$.
2. We consider the equations $x+y+x \approx x+yxy+x$, $x+y+xy \approx x+yx+xy$, and $x+y+xyx \approx x+yxy+xyx$.
By Lemma 2.5 (2), none of these identities holds in V_{MID}.
3. There is no equation to consider.
4. and 5. are similar to 3.
6. We assume that the following identities hold in V_{BE}:

a.	$xyx+x+xy \approx xyx+y+xy$	b.	$xyx+x+xy \approx xyx+yx+xy$
c.	$xyx+x+yx \approx xyx+y+yx$	d.	$xyx+x+yx \approx xyx+xy+yx$
e.	$xyx+x+yxy \approx xyx+y+yxy$		

Lemma 2.5 (2) shows that none of the above mentioned identities is satisfied in V_{MID}.
7. can be proved in a similar way as in part 6.
8.,9. and 10. are similar to 3.
11. We have to consider the equations

a.	$xy+x+y \approx xy+yx+y$	b.	$xy+x+xy \approx xy+xyx+xy$
c.	$xy+x+yx \approx xy+y+yx$	d.	$xy+x+xyx \approx xy+yx+xyx$
e.	$xy+x+xyx \approx xy+y+xyx$	f.	$xy+x+yxy \approx xy+y+yxy$.

Lemma 2.5 (2) guarantees that none of the identities a.), b.),c.),e.),f.) is satisfied in V_{MID}.
Assume that the identity d.) holds in V_{BE}. Since V_{BE} is solid, applying the hypersubstitution $\sigma_{x+y,y}$ to d.) we will obtain $y + x + y \approx y \in IdV_{BE}$. This contradicts the regularity property of V_{BE}.
12. is similar to 2.
13. is similar to 2.
14. Nothing to prove.
III) It is left to prove that any binary term that is a sum of three different elements from Bin_G, is V_{MID}-equivalent to one of the terms in our list. This was done in [10].

Lemma 2.8. *There are exactly (up to permutation of variables x and y) the following binary terms which are V_{MID}-equivalent to sums consisting of four different elements from Bin_G, these sums cannot be V_{MID}-equivalent to sums consisting of at most three summands from Bin_G:*

1. $xyx + x + y + t, t \in Bin_G \backslash \{x, y\}$	5. $xy + x + yx + t, t \in \{xy, xyx\}$
2. $xyx + x + xy + t, t \in \{xyx, yx\}$	6. $xy + yx + y + t \in \{xy, yxy\}$
3. $xyx + x + yx + t, t \in \{xyx, xy\}$	7. $xy + x + y + t, t \in Bin_G \backslash \{x, y\}$.
4. $xyx + x + yxy + xyx$	

Proof: In the following lines, the idempotency and the medial laws will be used but for reference, we will not mention this on each occasion.
I) We prove first that any binary term of the list cannot be V_{MID}-equivalent to a sum of at most three elements from Bin_G.
1. Let $t \in Bin_G \backslash \{x, y\}$. Assume that there exists $t_1 \in Bin_G$ such that $xyx + x + y + t \approx xyx + t_1 + t \in IdV_{MID}$.
• If $t_1 \in Bin_G \backslash \{x\}$, substituting $y \mapsto xyx$ in $xyx+x+y+t \approx xyx+t_1+t \in IdV_{MID}$, we would have $xyx + x + xyx \approx xyx \in IdV_{MID}$. This contradicts Lemma 2.5 (1).
• If $t_1 = x$ substituting $x \mapsto yxy$ in $xyx + x + y + t \approx xyx + t_1 + t \in IdV_{MID}$, we would have $yxy + y + yxy \approx yxy \in IdV_{MID}$. This contradicts Lemma 2.5 (1).
2. Let $t \in \{xyx, yx\}$. Assume that there exists $t_2 \in Bin_G$ such that $xyx + x + xy + t \approx xyx + t_2 + t \in IdV_{MID}$.
• If $t_2 \in Bin_G \backslash \{x\}$, substituting $y \mapsto xyx$ in the identity $xyx + x + xy + t \approx xyx + t_1 + t \in IdV_{MID}$, we would have $xyx + x + xyx \approx xyx \in IdV_{MID}$. This contradicts Lemma 2.5 (1)
• If $t_2 = x$, since V_{MID} is solid, applying the hypersubstitution $\sigma_{x+y,y}$ to the identities $xyx+x+xy+xyx \approx xyx+x+xyx$ and $xyx+x+xy+yx \approx xyx+x+yx$, we would get $x \approx x + y + x \in IdV_{MID}$. This contradicts the regularity property of V_{MID}.
3. Let $t \in \{xyx, xy\}$. Assume that there exists $t_3 \in Bin_G$ such that $xyx + x + yx + t \approx xyx + t_3 + t \in IdV_{MID}$.
• If $t_3 \in Bin_G \backslash \{x\}$, the substitution $y \mapsto xyx$, gives a contradiction $xyx+x+xyx \approx xyx \in IdV_{MID}$ [Lemma 2.5 (1)].
• If $t_3 = x$, since V_{MID} is solid, applying the hypersubstitution $\sigma_{x+y,x}$ to the identity $xyx + x + yx + xyx \approx xyx + x + xyx$ and $xyx + x + yx + xy \approx xyx + x + xy$ we would get $x \approx x + y + x \in IdV_{MID}$. This contradicts the regularity property of V_{MID}.
4. The proof is similar to 3.
5. The proof is similar to 3.
6. The proof is similar to 2.
7. The proof is similar to 1.
II) Secondly, we prove that each term in the list of Lemma 2.8 is V_{MID}-equivalent

to itself only. Using Lemma 2.3, we have to consider the identities:

a.	$xyx + x + y + xy \approx xyx + x + yx + xy$	*b.*	$xyx + x + y + xyx \approx xyx + x + xy + xyx$
c.	$xyx + x + y + xyx \approx xyx + x + yx + xyx$	*d.*	$xyx + x + y + xyx \approx xyx + x + yxy + xyx$
e.	$xyx + x + y + yx \approx xyx + x + xy + yx$	*f.*	$xyx + x + xy + xyx \approx xyx + x + yx + xyx$
g.	$xyx + x + xy + xyx \approx xyx + x + yxy + xyx$	*h.*	$xyx + x + yx + xyx \approx xyx + x + yxy + xyx$
i.	$xy + x + yx + xy \approx xy + yx + y + xy$	*k.*	$xy + x + yx + xy \approx xy + x + y + xy$
l.	$xy + x + yx + xyx \approx xy + x + y + xyx$	*n.*	$xy + x + yx + yxy \approx xy + x + y + yxy$

and have to show that the variety V_{MID} satisfies none of them.
Assume that these identities are satisfied in V_{MID}. The substitution $x \mapsto yxy$ (resp. $y \mapsto xyx$) in a.,b.,c.,d.,e.,k.,l.,n.,(resp. i.) gives the identity $xyx + x + xyx \approx xyx$ which is not satisfied in V_{MID}.
Since V_{MID} is solid, applying the hypersubstitution $\sigma_{x+y,x}$ (resp.to $\sigma_{x+y,y}$) to f. and g. (resp. h), one obtains $x + y + x \approx x \in IdV_{MID}$. This contradicts the regularity property of V_{MID}.
We conclude that none of these identities are satisfied in V_{MID}.
III) The last step of this proof is to show that any binary term which is a sum of four different elements from Bin_G, is V_{MID}-equivalent to one of the terms in our list. This was done in [10]. ■

Using Lemma 2.2, Lemma 2.3, Lemma 2.7, Lemma 2.8 and the idempotent laws, we have the following result:

Theorem 2.1. *The universe of the free algebra with respect to V_{MID} freely generated by two variables x, y consists of $[t]_{IdV_{MID}}, t \in \{x, xy, xyx\}$, the terms $x + t$* with *$t \in Bin_G \backslash \{x\}, xy + t$* with *$t \in Bin_G \backslash \{xy\}, xyx + t$* with *$t \in Bin_G \backslash \{xyx\}$, the terms from Lemma 2.7 and Lemma 2.8 and all of the terms arising from the aforementioned terms by permuting the variables x* and *y.* ■

Now, we turn to the variety V_{BE}. The following result gives a description of all binary terms over the variety V_{BE}.

Theorem 2.2. *Every binary term is V_{BE}-equivalent to a sum of the form $t_1 + t_2 + t_3$ $(t_j \in Bin_G, j = 1, 2, 3)$, consisting of at most three summands.*

Proof: Let V be a subvariety of V_{MID}. Lemma 2.2 shows that every binary term is V-equivalent to a sum consisting of at most four summands. Some of these sums were reduced (over the variety V_{MID}) to sums consisting of at most three summands ([3]). The same goes for the variety V_{BE}, since $IdV_{MID} \subset IdV_{BE}$. Thus, it is left to show that the terms in the list of Lemma 2.8 can be shortened (over the variety V_{BE}) to sums consisting of at most three summands from Bin_G. This is clear by using Lemma 2.5 (1), Lemma 2.1 and the identity $xy + yx \approx xy + x + y + yx$. ■

Lemma 2.9. *There are exactly (up to permutation of variables x* and *y) the following binary terms which are V_{BE}-equivalent to sums consisting of three different elements from Bin_G, these sums cannot be V_{BE}-equivalent to sums consisting of at most two summands from Bin_G:*

1. $x+xy+t,\ t\in\{x,xyx\}$
2. $x+y+t,\ t\in Bin_G\backslash\{y,yxy\}$
3. $x+yx+t,\ t\in\{x,xyx\}$
5. $x+xyx+x$
7. $xyx+y+t,\ t\in Bin_G\backslash\{y,yxy\}$
8. $xyx+xy+t,\ t\in\{x,xyx\}$.
10. $xyx+yx+t,\ t\in\{x,xyx\}$
11. $xy+x+t,\ t\in\{y,xy,yxy\}$
12. $xy+yx+xy$
13. $xy+y+t,\ t\in\{x,xy,xyx\}$

Notice that this enumeration corresponds exactly to that used in Lemma 2.7.

Proof of Lemma 2.9: In Lemma 2.7, we already showed that over the variety V_{MID} some sums consisting of three summands can be reduced. The same reductions are valid over V_{BE} since $IdV_{MID}\subseteq IdV_{BE}$. Therefore, using the list of all binary terms in Lemma 2.7, we have first to check whether reductions can be made by additional identities valid in V_{BE}.
1. One has $x+xy+yx\approx x+xy+y+yx\approx x+y+yx$ by Lemma 2.5 (1) and by Lemma 2.1 (1). The last term is considered in part 2 .
2. The identity $x+y+yxy\approx x+yxy$ arises from Lemma 2.5 (1).
3. In a similar way as we did in part 1, we can prove $x+yx+xy\approx x+y+xy$.
4. One has $x+yxy+x\approx x+yxy+y+x\approx x+y+x$ by Lemma 2.5 (1) and Lemma 2.1 (13).
One has also $x+yxy+xyx\approx x+y+yxy+xyx\approx x+y+xy$ by using the identity $xyx+x+xyx\approx xyx$ and Lemma 2.5 (1).
5. There is nothing to prove.
6. The identity $xyx+x+t\approx xyx+t$ is clear by Lemma 2.5(1).
7. Can be proved in a similar way as in part 2.
8. In a similar way as we did in part 2., we have $xyx+xy+yx\approx xyx+y+yx$. The last term is already considered in part 7.
9. We have $xyx+yxy+x\approx xyx+yxy+y+x\approx xyx+y+x$ by Lemma 2.5 (1) and Lemma 2.1 (18). The last term already occurs in part 7. Also we have $xyx+yxy+xyx\approx xyx+y+yxy+xyx\approx xyx+y+xyx$ by Lemma 2.5 (1) and Lemma 2.1 (7). The last term is considered in part 7.
10. In a similar way as we did in part 1., one has $xyx+yx+xy\approx xyx+y+xy$. The last term is considered in part 7.
11. Lemma 2.5 (1) shows that $xy+x+yx\approx xy+yx$ and $xy+x+xyx\approx xy+xyx$.
12. In a similar way as we did in part 1., we get $xy+yx+y\approx xy+x+y$ and $xy+yx+x\approx xy+y+x$. The last terms already occur in part 11. As we did in part 8, we get $xy+yx+xyx\approx xy+y+xyx$ and $xy+yx+yxy\approx xy+x+yxy$. The last terms are already considered in part 11.
13. Can be shown in a similar way as 11.
14. One has $xy+xyx+xy\approx xy+xyx+x+xy\approx xy+x+xy$ by using the identity $xyx+x+xyx\approx xyx$ and Lemma 2.1 (8). see 11. for the last term.
15. On has $xy+yxy+xy\approx xy+yxy+y+xy\approx xy+y+xy$ by Lemma 2.5 (1) and Lemma 2.1 (11). The last term occurs in 13.
It is left to prove that each term in the list of Lemma 2.9 is only V_{BE}-equivalent to itself. This was already proved in Lemma 2.7. ■

From Theorem 2.2, every binary term is V_{BE}-equivalent to a sum consisting of at most three summands. Using Lemma 2.3, Lemma 2.9 and the idempotency, we have the following results:

Theorem 2.3. *The universe of the free algebra with respect to V_{BE} freely generated by two variables x, y is $\{[t]_{IdV_{BE}}, t \in N^{W_{(2,2)}(X_2)}(V_{BE})\}$, with $N^{W_{(2,2)}(X_2)}(V_{BE})$ is the set consisting of elements from $\{x, xy, xyx\}$, the terms $x + t$* with *$t \in Bin_G \backslash \{x\}, xy + t$* with *$t \in Bin_G \backslash \{xy\}, xyx + t$* with *$t \in Bin_G \backslash \{xyx\}$, the terms $x+t$* with *$t \in Bin_G \backslash \{x\}, xy+t$* with *$t \in Bin_G \backslash \{xy\}, xyx+t$* with *$t \in Bin_G \backslash \{xyx\}$, the terms of the list of Lemma 2.9 and all of the terms arising from the aforementioned terms by permuting the variables x* and *y.*

The last step of this section will be devoted to the solid variety $RA_{(2,2)}$.

Lemma 2.10. *If $s_1 + s_2 \approx t_1 + t_2 \ \in IdRA_{(2,2)}$,* where *$s_i, t_i \in Bin_G$, $i \in \{1, 2\}$,* then *$s_1 \approx t_1 \in IdRA_{(2,2)}$* and *$s_2 \approx t_2 \in IdRA_{(2,2)}$.*

Proof: Assume that $s_1 + s_2 \approx t_1 + t_2 \in IdRA_{(2,2)}$, where $s_i, t_i \in Bin_G$ with $i \in \{1, 2\}$. Since $RA_{(2,2)}$ is solid, applying the hypersubstitutions $\sigma_{x,xy}$ and $\sigma_{y,xy}$ to the previous identity, we obtain in $RA_{(2,2)}$ the identities $s_1 \approx t_1$ and $s_2 \approx t_2$. ■

Theorem 2.4. *The universe of the free algebra $\mathcal{F}_{RA_{(2,2)}}(X_2)$ with respect to $RA_{(2,2)}$ freely generated by two variables x, y is $\{[t]_{IdRA_{(2,2)}}, t \in N^{W_{(2,2)}(X_2)}(RA_{(2,2)})\}$, with $N^{W_{(2,2)}(X_2)}(RA_{(2,2)}) = \{x, y, xy, yx, x + y, x + xy, x + yx, y + x, y + xy, y + yx, xy + x, xy + y, xy + yx, yx + x, yx + y, yx + xy\}$.*

Proof: Let $t \in W_{(2,2)}(X_2)$. Then, it is easy to verify that t is $RA_{(2,2)}$-equivalent to $t_1 + t_2$ where $t_i \in \{x, xy, yx, y\}, \ i = 1, 2$ because of Lemma 2.10 and the identities $xyz \approx xz$ as well as $x + y + z \approx x + z$ satisfied in $RA_{(2,2)}$.
Thus we have only to consider the binary terms that are sums consisting of exactly two binary terms from Bin_G. Since the variety $RA_{(2,2)}$ is outermost, for $r, s, t, u \in \{x, y, xy, yx\}$ with $r \neq s$ and $u \neq t$ the equation $s + t \approx r + u$ is not an identity in $RA_{(2,2)}$. We conclude that the universe of the free algebra $\mathcal{F}_{X_2}(RA_{(2,2)})$ is $\{[t]_{IdRA_{(2,2)}}, t \in \{x, y, xy, yx, x+y, x+xy, x+yx, y+x, y+xy, y+yx, xy+x, xy+y, xy + yx, yx + x, yx + y, yx + xy\}\}$. ■

Now, we will determine the degree of each solid variety of semirings.

3. Degrees of Solid Varieties of Semirings

First, we give a characterisation of the trivial variety of any type τ.

Theorem 3.1. *Let V be a variety of type $\tau = (n_i)_{i \in I}$ such that $n_j \geq 2$ for some $j \in I$. Then V is a solid variety and $d_p(V) = 1$ if and only if V is trivial.*

Proof: It is clear that the trivial variety $\mathcal{T}$ is solid and $dp(\mathcal{T}) = 1$. Let V be a variety of type $\tau = (n_i)_{i \in I}$ such that $n_j \geq 2$ for some $j \in I$. Assume that V is solid and $d_p(V) = 1$. Then $|P(V)|_{\sim V}| = |Hyp(\tau)|_{\sim V}| = 1$. Moreover, because of $n_j \geq 2$ for some $j \in I$, there exist at least two hypersubstitutions σ_1 and σ_2, belonging to the class of σ_{id}, such that $\sigma_1(f_j) = x_{n_{j1}}$ and $\sigma_2(f_j) = x_{n_{j2}}$, with $x_{n_{j1}} \neq x_{n_{j2}}$. Thus we have $x_{n_{j1}} \approx f_j(x_1, \cdots, n_j) \approx x_{n_{j2}} \in IdV$. It follows that $x_{n_{j1}} \approx x_{n_{j2}} \in IdV$. Therefore V is trivial. ■

The following result shows a connection between $W_{(2,2)}(\{x, y\})|_{IdV}$ and $Hyp(2, 2)|_{\sim V}$

Lemma 3.1. *Let V be a variety of type $(2,2)$. The map*

$$\begin{array}{llcl} \Phi: & (W_{(2,2)}(\{x,y\})|_{IdV})^2 & \longrightarrow & Hyp(2,2)|_{\sim_V} \\ & ([s]_{IdV},[t]_{IdV}) & \longmapsto & [\sigma_{s,t}]_{\sim_V} \end{array}$$

is bijective.

Proof: Let $s_1, t_1, s,$ and t four binary terms.
We have the following equivalences

$$\begin{array}{rcl} ([s_1]_{IdV},[t_1]_{IdV}) = ([s]_{IdV},[t]_{IdV}) & \Longleftrightarrow & \left\{\begin{array}{l} s_1 \approx s \in IdV \\ t_1 \approx t \in IdV \end{array}\right. \\ & \Longleftrightarrow & \left\{\begin{array}{l} \sigma_{s_1,t_1}(F) \approx \sigma_{s,t}(F) \in IdV \\ \sigma_{s_1,t_1}(G) \approx \sigma_{s,t}(G) \in IdV \end{array}\right. \\ & \Longleftrightarrow & \sigma_{s_1,t_1} \sim_V \sigma_{s,t} \end{array}$$

It follows that Φ is well defined and is injective. By the definition Φ is onto. Therefore Φ is bijective. ■

Now, we have all tools to prove:

Theorem 3.2. *The degrees of non trivial solid varieties of semirings with respect to proper hypersubstitution are*

(1) $d_p(V_{MID}) = 168^2$
(2) $d_p(V_{BE}) = 82^2$
(3) $d_p(RA_{(2,2)}) = 16^2$

Proof: By Theorem 1.1, there are three non trivial solid varieties of semirings, namely V_{MID}, V_{BE} and $RA_{(2,2)}$. For any solid variety V of semirings, we have $dp(V) = |P(V)|_{\sim_V}| = |Hyp(2,2)|_{\sim_V}|$.
But Lemma 3.2 ensures that
$|Hyp(2,2)|_{\sim_V}| = \left|\left(W_{(2,2)}(\{x,y\})|_{IdV}\right)^2\right| = \left(\left|W_{(2,2)}(\{x,y\})|_{IdV}\right|\right)^2.$
Therefore, using Theorem 2.1, Theorem 2.3 and Theorem 2.4 we have the results. This means, if V is a solid variety of semirings, then $d_p(V) \in \{1, 168^2, 82^2, 16^2\}$.

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