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SOME $q-$MATRICES RELATED TO $Z-$TRANSFORM

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Abstract. In a recent paper, Claudio de J. Pita Ruiz V. introduced number arrays of coefficients. In this paper, we study the $Z-$transform of a special $q-$number array. Our goal is to bring together the concepts of convolutions and special $q-$matrices.

1. Introduction and Preliminaries

The $q-$binomial or Gaussian coefficients play an important role in applied mathematics. Basic properties of these coefficients are studied by many researchers. In particular, several matrices are defined by using such coefficients. Among them is the $q-$Pascal matrix, introduced by Ernst in [1], whose umbral and combinatorics properties are studied over years.

Convolutions of sequences of the form n^p with the constant sequence 1 are studied in [4]. The author obtained several results on the convolutions with certain binomial coefficients. In the spirit of this study, we obtain a $q-$matrix by using the convolution of a special $q-$array with the constant sequence of the form q^{kn}. In particular, with the aid of $Z-$transform, we obtain two triangular matrices that look like $q-$Pascal and $q-$Lucas matrices by considering the sequences $[n]_q$ and $[n]_q^2$ by using Pita's methods.

In the following, we briefly summarize basic properties of $q-$binomial coefficients without proofs. Let n be a positive integer and $q \in (0,1)$. The $q-$integer $[n]_q$ and $q-$factorial $[n]_q!$ are respectively defined by

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1}, \tag{1.1}$$

and

$$[n]_q! = \begin{cases} 1, & \text{if } n = 0, \\ [n]_q \cdot [n-1]_q \cdot [n-2]_q \cdots [1]_q\,, & \text{if } n = 1, 2, \ldots \end{cases}$$

In this fashion, the $q-$binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q!\ [k]_q!}\,, \quad n \geq k \geq 1$$

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with $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ for $n < k$ [2]. The $q-$binomial coefficients reduce to the usual binomial coefficients as $q \to 1^-$, that is, $\begin{bmatrix} n \\ k \end{bmatrix}_q \to \binom{n}{k}$. Note that the $q-$binomial coefficients have the following properties:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

Let x and y be two variables. The $q-$analogue of $(x+y)^n$ is defined in [2] by

$$(x+y)_q^n = \begin{cases} 1, & \text{if } n = 0 \\ (x+y)(x+qy)\cdots\left(x+q^{n-1}y\right), & \text{if } n = 1, 2, \ldots \end{cases}$$

The Gauss formula for the $q-$analogue is given by

$$(x+a)_q^n = \sum_{j=0}^{n} q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q a^j x^{n-j}. \tag{1.2}$$

The $q-$derivative $D_q f$ of a function f is given by

$$(D_q\, f)(x) = \frac{f(x) - f(qx)}{(1-q)\, x}$$

if $x \neq 0$.

In this paper, we will mainly be interested in the $Z-$transform of a complex valued function $f : \mathbb{N} \to \mathbb{C}$, defined by the series

$$Z\big(f(n)\big) = \sum_{n=0}^{\infty} \frac{f(n)}{z^n}$$

(see [5], [6]). Each term in this series is a function of one or more variables, and the series is defined for all $z \in \mathbb{C}$ for which it is absolutely convergent. $Z-$transform is a way to solve difference equations [3]. Also $Z-$transform is known in mathematical literature as the method of generating functions.

Property 1. Let f and g be two functions defined on $\mathbb{N}$ with $Z-$transforms $F(z)$ and $G(z)$ for $|z| > R_f$ and $|z| > R_g$, respectively. Then

i) $Z\big(c\, f(n)\big)(z) = c\, F(z), \quad c \in \mathbb{C}.$
ii) $Z\big(f(n) + g(n)\big)(z) = F(z) + G(z), \quad |z| > \max(R_f, R_g).$
iii) $Z\big(a^n\, f(n)\big)(z) = F(\frac{z}{a}), \quad a > 0.$
iv) $Z\big((f * g)(n)\big)(z) = F(z) \cdot G(z), \quad |z| > \max(R_f, R_g).$
v) If $F(z) = G(z)$ for all $|z| > R$ some $R > 0$ then $f(n) = g(n)$, $n = 0, 1, \ldots$

2. Main Results

In the following, we will focus on the $Z-$transform of some $q-$number arrays. We show how to use these computations to solve it. Then we compute $[n]_q * q^{2n}\ [1]_q * q^{3n}\ [1]_q * \cdots * q^{kn}\ [1]_q$ and $[n]_q^2 * q^{3n}\ [1]_q * q^{3n}\ [1]_q * \cdots * q^{kn}\ [1]_q$. In particular, we establish a relation between convolutions and special $q-$matrices.

Lemma 2.1. *The $Z-$transform of the $q-$binomial coefficients $\begin{bmatrix} n \\ r \end{bmatrix}_q$ is*

$$\frac{z}{(z-1)_q^{r+1}}. \tag{2.1}$$

Proof. Let $w = \frac{1}{z}$. Then

$$\begin{aligned}
Z\left(\begin{bmatrix} n \\ r \end{bmatrix}_q\right) &= \sum_{n=0}^{\infty} \begin{bmatrix} n \\ r \end{bmatrix}_q z^{-n} \\
&= \sum_{n=0}^{\infty} \frac{[n]_q [n-1]_q \cdots [n-r+1]_q}{[r]_q!}\ w^n \\
&= \frac{1}{[r]_q!} \sum_{n=0}^{\infty} [n]_q [n-1]_q \cdots [n-r+1]_q\ w^n \\
&= \frac{1}{[r]_q!} \sum_{n=0}^{\infty} w^r\ D_q^r\ (w^n) \\
&= \frac{w^r}{[r]_q!}\ D_q^r \left(\sum_{n=0}^{\infty} w^n\right) \\
&= \frac{w^r}{[r]_q!}\ D_q^r \left(\frac{1}{1-w}\right) \\
&= \frac{w^r}{[r]_q!}\ \frac{[r]_q!}{(1-w)_q^{r+1}} \\
&= \frac{w^r}{(1-w)_q^{r+1}} \\
&= \frac{z}{(z-1)(z-q)\cdots(z-q^r)} \\
&= \frac{z}{(z-1)_q^{r+1}}.
\end{aligned}$$

□

Corollary 2.1. *As $q \to 1^-$, we have*

$$Z\left(\begin{bmatrix} n \\ r \end{bmatrix}_q\right) = Z\left(\binom{n}{r}\right) = \frac{z}{(z-1)^{r+1}}.$$

Corollary 2.2. *Let $[n]_q$ be the $q-$integer defined in* (1.1). *Then,* $Z\left([n]_q\right) = \frac{z}{(z-1)_q^2}$.

Corollary 2.3. $Z\left(\begin{bmatrix} n \\ r \end{bmatrix}_q q^n\right) = \dfrac{q^r\ z}{(z-q)_q^{r+1}}.$

Corollary 2.4. $Z\left(\begin{bmatrix} n+r \\ k \end{bmatrix}_q\right) = \dfrac{z^{r+1}}{(z-1)_q^{r+1}}.$

Lemma 2.2. *The $Z-$transform of* $[n]_q * q^{2n}\ [1]_q * q^{3n}\ [1]_q * \cdots * q^{kn}\ [1]_q$ *for* $k \geq 2$ *is*

$$\frac{z^k}{(z-1)_q^{k+1}}. \tag{2.2}$$

Proof. In order to proof the lemma, we use induction on r. We observe for $r = 2$ that

$$\begin{aligned} Z\left([n]_q * q^{2n}\ [1]_q\right) &= Z\left([n]_q\right)\ Z\left(q^{2n}\ [1]_q\right) \\ &= \frac{z}{(z-1)_q^2}\frac{z}{z-q^2} \\ &= \frac{z^2}{(z-1)_q^3}. \end{aligned}$$

We now assume that (2.2) holds for $r > 2$. Then

$$\begin{aligned} Z([n]_q * \cdots * q^{(r+1)n}\ [1]_q) &= Z\left([n]_q * \cdots * q^{rn}\ [1]_q\right) Z\left(q^{(r+1)n}\ [1]_q\right) \\ &= \frac{z^r}{(z-1)_q^{r+1}}\ \frac{z}{z-q^{r+1}} \\ &= \frac{z^{r+1}}{(z-1)_q^{r+2}}. \end{aligned}$$

Therefore, (2.2) also holds for $r+1$. This completes the proof. □

Next, we compute the convolutions of the sequence $[n]_q$ with the constant sequence $q^{kn}\,[1]_q$.

Lemma 2.3. *For* $k \geq 2$*, we have*

$$[n]_q * q^{2n}\ [1]_q * q^{3n}\ [1]_q * \cdots * q^{kn}\ [1]_q = \sum_{j=1}^{k} q^{\ j(j-1)} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q. \tag{2.3}$$

Proof. It follows from Lemma 2.2 that

$$Z([n]_q * q^{2n}\ [1]_q \cdots * q^{kn}\ [1]_q) = \frac{z^k}{(z-1)_q^{k+1}}.$$

We can write

$$\frac{z^k}{(z-1)_q^{k+1}} = \frac{A_1 z}{(z-1)_q^2} + \frac{A_2 z}{(z-1)_q^3} + \cdots + \frac{A_{k-1} z}{(z-1)_q^k} + \frac{A_k z}{(z-1)_q^{k+1}}. \tag{2.4}$$

Carrying out calculations along with the Gauss formula, we obtain

$$A_i = q^{i(i-1)} \begin{bmatrix} k-1 \\ i-1 \end{bmatrix}_q, \quad i = 1, 2, \ldots, k.$$

Using (2.1) along with the properties of $Z-$transform, we can write $Z([n]_q * \cdots * q^{kn}\ [1]_q)$ as

$$\begin{bmatrix} k-1 \\ 0 \end{bmatrix}_q Z\left(\begin{bmatrix} n \\ 1 \end{bmatrix}_q\right) + \cdots + q^{i(i-1)} \begin{bmatrix} k-1 \\ i-1 \end{bmatrix}_q Z\left(\begin{bmatrix} n \\ i \end{bmatrix}_q\right) + \cdots + q^{k(k-1)} \begin{bmatrix} k-1 \\ k-1 \end{bmatrix}_q Z\left(\begin{bmatrix} n \\ k \end{bmatrix}_q\right)$$

which is equal to

$$Z\left(\begin{bmatrix} k-1 \\ 0 \end{bmatrix}_q \begin{bmatrix} n \\ 1 \end{bmatrix}_q + \cdots + q^{i(i-1)} \begin{bmatrix} k-1 \\ i-1 \end{bmatrix}_q \begin{bmatrix} n \\ i \end{bmatrix}_q + \cdots + q^{k(k-1)} \begin{bmatrix} k-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q\right).$$

The above equation simplifies to

$$Z\left(\sum_{j=1}^{k} q^{\ j(j-1)} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q\right).$$

Finally, it follows from Property 1 (v) that

$$[n]_q * q^{2n}\ [1]_q * q^{3n}\ [1]_q * \cdots * q^{kn}\ [1]_q = \sum_{j=1}^{k} q^{\ j(j-1)} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q.$$

□

Using the above convolution formula, we obtain a special matrix with entries given by

$$p_{kj} = q^{\ j(j-1)} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q.$$

Since

$$\begin{aligned}
[n]_q &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q \cdot \begin{bmatrix} n \\ 1 \end{bmatrix}_q \\
[n]_q * q^{2n}\ [1]_q &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q \begin{bmatrix} n \\ 1 \end{bmatrix}_q + q^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q \cdot \begin{bmatrix} n \\ 2 \end{bmatrix}_q \\
[n]_q * q^{2n}\ [1]_q * q^{3n}\ [1]_q &= \begin{bmatrix} 2 \\ 0 \end{bmatrix}_q \cdot \begin{bmatrix} n \\ 1 \end{bmatrix}_q + q^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \cdot \begin{bmatrix} n \\ 2 \end{bmatrix}_q + q^6 \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q \cdot \begin{bmatrix} n \\ 3 \end{bmatrix}_q \\
&\vdots
\end{aligned}$$

we obtain

$$\mathscr{P} = \begin{bmatrix}
\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q & 0 & 0 & 0 & 0 & \cdots \\
\begin{bmatrix} 1 \\ 0 \end{bmatrix}_q & q^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q & 0 & 0 & 0 & \cdots \\
\begin{bmatrix} 2 \\ 0 \end{bmatrix}_q & q^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q & q^6 \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q & 0 & 0 & \cdots \\
\begin{bmatrix} 3 \\ 0 \end{bmatrix}_q & q^2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q & q^6 \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q & q^{12} \begin{bmatrix} 3 \\ 3 \end{bmatrix}_q & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.$$

This matrix looks like a $q-$Pascal matrix. In particular, it converges to a Pascal matrix in [4] as $q \to 1^-$.

Lemma 2.4. *The $Z-$transform of $[n]_q^2$ is given by* $\dfrac{z\,(z+q)}{(z-1)_q^3}$.

The following lemma can be proven by induction on k:

Lemma 2.5. *Let $k \geq 3$. Then the $Z-$transform of $[n]_q^2 * q^{3n}\ [1]_q * \cdots * q^{kn}\ [1]_q$ is*

$$\frac{z^{k-1}(z+q)}{(z-1)_q^{k+1}}.$$

The following Lemma can be handled in much the same way with Lemma 2.3:

Lemma 2.6.

$$[n]_q^2 * q^{3n}\ [1]_q * q^{3n}\ [1]_q * \cdots * q^{kn}\ [1]_q = \sum_{j=1}^{k} \left\{ q^{\ j(j-1)} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q + q^{\ (j-1)^2} \begin{bmatrix} k-2 \\ j-2 \end{bmatrix}_q \right\} \begin{bmatrix} n \\ j \end{bmatrix}_q .$$

The coefficients in the above convolution convolution formula gives a triangular array with entries $\ell_{kj} = q^{\ j(j-1)} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q + q^{\ (j-1)^2} \begin{bmatrix} k-2 \\ j-2 \end{bmatrix}_q$. In particular, this triangular array looks like the $q-$Lucas like triangle. Since

$$\begin{aligned}
[n]_q^2 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q \begin{bmatrix} n \\ 1 \end{bmatrix}_q + \left(q^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q + q \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q \right) \begin{bmatrix} n \\ 2 \end{bmatrix}_q \\
[n]_q^2 * q^{3n}\ [1]_q &= \begin{bmatrix} 2 \\ 0 \end{bmatrix}_q \begin{bmatrix} n \\ 1 \end{bmatrix}_q + \left(q^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q + q \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q \right) \begin{bmatrix} n \\ 2 \end{bmatrix}_q + \left(q^6 \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q + q^4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q \right) \begin{bmatrix} n \\ 3 \end{bmatrix}_q \\
[n]_q^2 * q^{3n}\ [1]_q * q^{4n}\ [1]_q &= \begin{bmatrix} 3 \\ 0 \end{bmatrix}_q \begin{bmatrix} n \\ 1 \end{bmatrix}_q + \left(q^2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q + q \begin{bmatrix} 2 \\ 0 \end{bmatrix}_q \right) \begin{bmatrix} n \\ 2 \end{bmatrix}_q + \left(q^6 \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q + q^4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \right) \begin{bmatrix} n \\ 3 \end{bmatrix}_q \\
&\quad + \left(q^{12} \begin{bmatrix} 3 \\ 3 \end{bmatrix}_q + q^9 \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q \right) \begin{bmatrix} n \\ 4 \end{bmatrix}_q \\
&\ \vdots
\end{aligned}$$

we obtain

$$\mathcal{L} = \begin{bmatrix}
\begin{bmatrix} 1 \\ 0 \end{bmatrix}_q & q^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q + q \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q & 0 & 0 & 0 & \cdots \\
\begin{bmatrix} 2 \\ 0 \end{bmatrix}_q & q^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q + q \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q & q^6 \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q + q^4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q & 0 & 0 & \cdots \\
\begin{bmatrix} 3 \\ 0 \end{bmatrix}_q & q^2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q + q \begin{bmatrix} 2 \\ 0 \end{bmatrix}_q & q^6 \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q + q^4 \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q & q^{12} \begin{bmatrix} 3 \\ 3 \end{bmatrix}_q + q^9 \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.$$

Note that this matrix converges to the Lucas matrix given in [4] as $q \to 1^-$.

3. Conclusion

In this paper, we studied some properties of special $q-$number arrays related to $Z-$transform. It might be probable to extend these results for $[n]_q^p$ where p is a nonnegative integer. It also might be probable to obtain other useful properties and special matrices by using the $q-$analogue of the $Z-$transform.

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