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ON THE RATE OF CONVERGENCE OF THE STANCU TYPE BERNSTEIN OPERATORS FOR FUNCTIONS OF BOUNDED VARIATION

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Abstract. In this paper, we estimate the rate of pointwise convergence of the Stancu type Bernstein operators for functions defined on the interval. To prove our main result, we have used some methods and techniques from probability theory.

1. INTRODUCTION

Let $BV(I)$ denote the class of functions that are of bounded variation on a set $I \subset \mathbb{R}$. Recently, some authors studied some linear positive operators and obtained the rate of convergence for functions in $BV(I)$. For example, Bojanic and Vuilleumier [1] estimated the rate of convergence of Fourier-Legendre series for functions of bounded variation on the interval $[0,1]$, Cheng [2] estimated the rate of convergence of Bernstein polynomials for functions bounded variation on $[0,1]$, and Zeng and Chen [3] and Guo [4] estimated the rate of convergence of Durrmeyer type operators for functions of bounded variation again on $[0,1]$.

For a function defined on the interval $[0,1]$, Bernstein operators $B_n(f)$, $n \geq 1$, are defined by

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{nk}(x), \ n \geq 1 \tag{1}$$

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where $p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis $(0 \leq x \leq 1)$.

The operators defined by (1) were introduced by Bernstein [5] and studied by many authors.

The Stancu polynomials studied in this paper are given [6] by

$$B_n^{(\alpha,\beta)}(f,x) = \sum_{k=0}^{n} p_{nk}(x) f\left(\frac{k+\alpha}{n+\beta}\right) \tag{2}$$

where α, β are real parameters $(0 \leq \alpha \leq \beta)$ and $x \in [0,1]$.

In this paper, by means of techniques of probability theory and methods of Bojanic and Vuilleumier [1], Cheng [2], Zeng and Chen [3] we shall estimate the rate of convergence of the operators $B_n^{(\alpha,\beta)}(f,x)$ for functions of bounded variation.

Theorem 1.1. *Let f be a function of bounded variation on $[0,1]$. Then for every $x \in (0,1)$, we have*

$$\begin{aligned} \left| B_n^{(\alpha,\beta)}(f,x) - \frac{1}{2}\left(f(x+) + f(x-)\right) \right| &\leq 3\frac{(\alpha-\beta x)^2 + nx(1-x)}{(n+\beta)^2 x^2 (1-x)^2} \\ &\quad \times \left\{ \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x) \right\} \\ &\quad + \frac{1}{\sqrt{nx(1-x)}} \left[|f(x+) - f(x-)| \right. \\ &\quad \left. + |f(x) - f(x-)| \, e_n(x) \frac{1}{\sqrt{2e}} \right] \end{aligned} \tag{1.1}$$

where $e_n(x) = \begin{cases} 1, & x = k' \text{ for some } k' \in \mathbb{N} \\ 0, & x \neq k' \text{ for all } k' \in \mathbb{N} \end{cases}$, $e = 2.71...$ *and* $\bigvee_a^b (g_x)$ *is the total variation of g_x on $[a,b]$,*

$$g_x(t) = \begin{cases} f(t) - f(x+) & x < t \leq 1 \\ 0 & t = x \\ f(t) - f(x-) & 0 \leq t < x. \end{cases}$$

2. AUXILIARY RESULT

In this section, we give certain results that are neccessary to prove our main theorems.

Lemma 2.1. *For $B_n^{(\alpha,\beta)}(t^s;x)$, $s = 0,1,2$, one has*

$$B_n^{(\alpha,\beta)}(1,x) = 1$$

$$B_n^{(\alpha,\beta)}(t,x) = x + \frac{\alpha - \beta x}{n+\beta}$$

$$B_n^{(\alpha,\beta)}\left(t^2,x\right) = x^2 + \frac{(\alpha-\beta x)(2nx + \beta x + \alpha) + nx(1-x)}{(n+\beta)^2}. \tag{3}$$

Proof.

$$B_n^{(\alpha,\beta)}(1,x) = \sum_{k=0}^{n} p_{nk}(x) = B_n(1,x) = 1$$

$$\begin{aligned} B_n^{(\alpha,\beta)}(t,x) &= \sum_{k=0}^{n} \frac{k+\alpha}{n+\beta} p_{nk}(x) \\ &= \frac{n}{n+\beta}\sum_{k=0}^{n}\frac{k}{n}p_{nk}(x) + \frac{\alpha}{n+\beta}\sum_{k=0}^{n}p_{nk}(x) \\ &= \frac{n}{n+\beta}B_n(t,x) + \frac{\alpha}{n+\beta}B_n(1,x) \\ &= \frac{nx}{n+\beta} + \frac{\alpha}{n+\beta} = x + \frac{\alpha-\beta x}{n+\beta} \end{aligned}$$

$$\begin{aligned} B_n^{(\alpha,\beta)}\left(t^2,x\right) &= \sum_{k=0}^{n} (\frac{k+\alpha}{n+\beta})^2 p_{nk}(x) \\ &= \frac{n^2}{(n+\beta)^2}\sum_{k=0}^{n}\left(\frac{k}{n}\right)^2 p_{nk}(x) \\ &\quad + \frac{2\alpha n}{(n+\beta)^2}\sum_{k=0}^{n}\frac{k}{n}p_{nk}(x) + \frac{\alpha^2}{(n+\beta)^2}\sum_{k=0}^{n}p_{nk}(x) \\ &= \frac{n^2}{(n+\beta)^2}B_n\left(t^2,x\right) + \frac{2\alpha n}{(n+\beta)^2}B_n(t,x) \\ &\quad + \frac{\alpha^2}{(n+\beta)^2}B_n(1,x) \\ &= x^2 + \frac{(\alpha-\beta x)(2nx+\beta x+\alpha) + nx(1-x)}{(n+\beta)^2}. \end{aligned}$$

By direct calculation, we also find the following equalities;

$$B_n^{(\alpha,\beta)}\left((t-x)^2,x\right) = \frac{(\alpha-\beta x)^2 + nx(1-x)}{(n+\beta)^2}. \tag{4}$$

□

Lemma 2.2. *For all $x \in (0,1)$, we have*

$$\lambda_n(x,t) := \int_0^t K_n(x,u)\,du \le \frac{1}{(x-t)^2}\frac{(\alpha-\beta x)^2 + nx(1-x)}{(n+\beta)^2}, 0<t<x \tag{5}$$

and

$$1-\lambda_n(x,z) := \int_z^1 K_n(x,u)\,du \le \frac{1}{(z-x)^2}\frac{(\alpha-\beta x)^2 + nx(1-x)}{(n+\beta)^2}, x \le z < 1 \tag{6}$$

where

$$K_n(x,t) = \begin{cases} \sum\limits_{k+\alpha\le(n+\beta)t} P_{nk}(x) & 0<t\le 1 \\ 0 & t=0 \end{cases}. \tag{7}$$

Proof. First we prove (5)

$$\begin{aligned}\lambda_n(x,t) &= \int_0^t K_n(x,u)\,du \\ &\le \int_0^t K_n(x,u)\left(\frac{x-u}{x-t}\right)^2 du \\ &= \frac{1}{(x-t)^2} B_n^{(\alpha,\beta)}\left((u-x)^2, x\right).\end{aligned}$$

By (4), we get

$$\lambda_n(x,t) \le \frac{1}{(x-t)^2}\frac{(\alpha-\beta x)^2 + nx(1-x)}{(n+\beta)^2}.$$

The proof of (6) is similar. □

Lemma 2.3. *Let ξ_1 be a sequence of independent random variables with two point binomial distribution $P(\xi_1 = k) := x^k(1-x)^{1-k}$ ($k = 0, 1,$ and $0 \le x \le 1$ being a parameter). Then*

$$a_1 = E\xi_1 = x,\ E(\xi_1 - a_1)^2 = x(1-x)$$

and

$$E(\xi_1 - a_1)^3 = x(1-x)\left(2x^2 - 2x + 1\right).$$

Proof. Let $\{\xi_1\}_{k=1}^{\infty}$ be a sequence of independent random variables identically distributed with ξ_1, $\eta_n = \sum\limits_{k=1}^{n} \xi_k$. Then the probability distributions of the random variable η_n is

$$P(\mu_n = k) = p_{nk}(x) = \binom{n}{k} x^k (1-x)^{1-k},\ (0 \le k \le n).$$

For $M_i(x) = \sum\limits_{k=0}^{1} k^i x^k (1-x)^{1-k}$, we find that

$$M_0(x) = 1,\ M_1(x) = x,\ M_2(x) = x,\ M_3(x) = x.$$

From the definition of expectation, we get $E(\xi_1) = M_1(x) = x$. Also

$$E(\xi_1 - a_1)^2 = \sum_{j=0}^{2} \binom{2}{j} (-1)^j M_{2-j}[M_1(x)]^j = x(1-x)$$

$$E(\xi_1 - a_1)^3 = \sum_{j=0}^{3} \binom{3}{j} (-1)^j M_{3-j}[M_1(x)]^j = x(1-x)\left(2x^2 - 2x + 1\right).$$

□

Lemma 2.4. *(Berry-Esseen). Let $\{\xi_1\}_{k=1}^{\infty}$ be a sequence of independent and identically distributed random variable with finite variance such that the expectation*

$E(\xi_1) := a_1 \in \mathbb{R}$, the variance $Var(\xi_1) := E(\xi_1 - a_1)^2 = b_1^2 > 0$ and $E|\xi_1 - E(\xi_1)|^3 < \infty$. Then there exist a constant C, $\frac{1}{\sqrt{2\pi}} \leq C < 0.82$, such that for all n and t

$$\left| P\left(\frac{1}{b_1\sqrt{n}} \sum_{k=1}^{n} (\xi_1 - a_1) \leq t \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du \right| < C \frac{E|\xi_1 - E(\xi_1)|^3}{b_1^3\sqrt{n}}.$$

Its proof can be found in Shiryayev [7].

Lemma 2.5. *For all $x \in [0,1]$, we have*

$$\left| \left(\sum_{(n+\beta)x < k+\alpha \leq n} p_{nk}(x) \right) - \frac{1}{2} \right| \leq \frac{0,8\left(2x^2 - 2x + 1\right)}{\sqrt{nx(1-x)}} \leq \frac{1}{\sqrt{nx(1-x)}}.$$

Proof. From Lemma 2.3 for $P(\mu_n = k) = p_{nk}(x) = \binom{n}{k} x^k (1-x)^{1-k}$, $(0 \leq k \leq n)$

$$\begin{aligned} \sum_{(n+\beta)x < k+\alpha \leq n} p_{nk}(x) &= P\left((n+\beta)x < \mu_n \leq n\right) \\ &= 1 - P\left(\mu_n \leq (n+\beta)x\right) \\ &= 1 - P\left(\frac{\mu_n - (n+\beta)x}{\sqrt{nx(1-x)}} \leq 0 \right) \end{aligned}$$

then

$$\begin{aligned} \left| \left(\sum_{(n+\beta)x < k+\alpha \leq n} P_{nk}(x) \right) - \frac{1}{2} \right| &= \left| P\left(\frac{\mu_n - (n+\beta)x}{\sqrt{nx(1-x)}} \leq 0 \right) - \frac{1}{2} \right| \\ &< C \frac{E|\xi_1 - E(\xi_1)|^3}{b_1^3\sqrt{n}} \\ &< \frac{0.8\left(2x^2 - 2x + 1\right)}{\sqrt{nx(1-x)}} \leq \frac{1}{\sqrt{nx(1-x)}}. \end{aligned}$$

□

Lemma 2.6. *For all $x \in [0,1]$, we have*

$$B_n^{(\alpha,\beta)}(sgn(t-x), x) = 2 \sum_{(n+\beta)x < k+\alpha \leq n} P_{nk}(x) - 1 + e_n(x) P_{nk}(x).$$

Proof. One has

$$\begin{aligned} B_n^{(\alpha,\beta)}(sgn(t-x), x) &= \sum_{k=0}^{n} sgn\left(\frac{k+\alpha}{n+\beta} - x \right) p_{nk}(x) \\ &= \sum_{(n+\beta)x < k+\alpha \leq n} p_{nk}(x) - \sum_{0 < k+\alpha \leq (n+\beta)x} p_{nk}(x) \end{aligned}$$

and from (3), we can write

$$\begin{aligned} 1 &= B_n^{(\alpha,\beta)}(1, x) = \sum_{(n+\beta)x < k+\alpha \leq n} p_{nk}(x) + \sum_{0 < k+\alpha \leq (n+\beta)x} p_{nk}(x) \\ &\quad + e_n(x) p_{nk}(x). \end{aligned}$$

Thus there follows

$$\begin{aligned} B_n^{(\alpha,\beta)}(sgn(t-x),x) &= \sum_{(n+\beta)x<k+\alpha\le n} p_{nk}(x) \\ &\quad - \left[1 - \sum_{(n+\beta)x<k+\alpha\le n} p_k(x) - e_n(x)\, p_{nk}(x)\right] \\ &= 2\sum_{(n+\beta)x<k+\alpha\le n} P_{nk}(x) - 1 + e_n(x)\, P_{nk}(x). \end{aligned}$$

□

Lemma 2.7. *There holds the inequality*

$$\begin{aligned} &\left|\frac{f(x+)-f(x-)}{2} B_n^{(\alpha,\beta)}(sgn(t-x),x) \right. \\ &\left. + \left[f(x) - \frac{1}{2}f(x+) - \frac{1}{2}f(x-)\right] B_n^{(\alpha,\beta)}(\delta_x, x)\right| \\ \text{(2.1)} \quad \le\ & \frac{1}{\sqrt{nx(1-x)}}\left[|f(x+)-f(x-)| + |f(x)-f(x-)|\, e_n(x)\frac{1}{\sqrt{2e}}\right]. \end{aligned}$$

Proof. We have

$$\begin{aligned} &\left|\frac{f(x+)-f(x-)}{2} B_n^{\alpha,\beta}(sgn(t-x),x) \right. \\ &\left. + \left[f(x) - \frac{1}{2}f(x+) - \frac{1}{2}f(x-)\right] B_n^{\alpha,\beta}(\delta_x, x)\right| \\ = &\left|\frac{f(x+)-f(x-)}{2}\left[2\sum_{(n+\beta)x<k+\alpha\le n} P_{nk}(x) - 1 + e_n(x)\, P_{nk}(x)\right]\right. \\ &\left. + \left[f(x) - \frac{1}{2}f(x+) - \frac{1}{2}f(x-)\right] e_n(x)\, p_{nk}(x)\right| \\ \le &\left|\frac{f(x+)-f(x-)}{2}\left[2\sum_{(n+\beta)x<k+\alpha\le n} P_{nk}(x) - 1\right]\right| \\ &+ |[f(x) - f(x-)]\, e_n(x)\, p_{nk}(x)|. \end{aligned}$$

□

3. PROOF OF THE THEOREM

Now we can establish the theorem.

For any $f(t) \in BV[0,1]$, we decompose $f(t)$ into four parts as

$$\begin{aligned} f(t) &= \frac{f(x+)+f(x-)}{2} + g_x(t) \\ &\quad + \frac{f(x+)-f(x-)}{2} sgn(t-x) \\ \text{(3.1)} \qquad &\quad + \delta_x(t)\left(f(x) - \frac{f(x+)+f(x-)}{2}\right) \end{aligned}$$

where

$$\delta_x(t) = \begin{cases} 1, & x = t \\ 0, & x \neq t \end{cases}.$$

If we apply $B_n^{(\alpha,\beta)}$ to both sides of (3.1), we have

$$\begin{aligned} B_n^{(\alpha,\beta)}(f,x) &= \frac{1}{2}\left(f(x+)+f(x-)\right)B_n^{(\alpha,\beta)}(1,x) \\ &\quad + B_n^{(\alpha,\beta)}(g_x,x) + \frac{f(x+)-f(x-)}{2}B_n^{(\alpha,\beta)}(sgn(t-x),x) \\ &\quad + \left[f(x) - \frac{1}{2}\left(f(x+)+f(x-)\right)\right]B_n^{(\alpha,\beta)}(\delta_x,x). \end{aligned} \tag{3.2}$$

If we take the absolute values of the inequality (3.2) and note $B_n^{(\alpha,\beta)}(1,x) = 1$ by (4), we have

$$\begin{aligned} &\left|B_n^{(\alpha,\beta)}(f,x) - \frac{1}{2}\left(f(x+)+f(x-)\right)B_n^{(\alpha,\beta)}(1,x)\right| \\ \leq &\left|B_n^{(\alpha,\beta)}(g_x,x)\right| \\ &+ \left|\frac{f(x+)-f(x-)}{2}B_n^{(\alpha,\beta)}(sgn(t-x),x)\right. \\ &+ \left.\left(f(x) - \frac{1}{2}\left(f(x+)+f(x-)\right)\right)B_n^{(\alpha,\beta)}(\delta_x,x)\right|. \end{aligned} \tag{3.3}$$

First we estimate $B_n^{(\alpha,\beta)}(g_x,x)$ as follows:

$$\left|B_n^{(\alpha,\beta)}(g_x,x)\right| = \left|\int_0^1 g_x(t)\,d_tK_n(x,t)\,dt\right|, \tag{8}$$

with the kernel $K_n(x,t)$ of (7).To estimate the integral of (8), we decompose it into three parts, as follows

$$\begin{aligned} &\int_0^1 g_x(t)\,d_tK_n(x,t)\,dt \\ = &\left|\left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} + \int_{x+(1-x)/\sqrt{n}}^1\right)g_x(t)\,d_tK_n(x,t)\,dt\right| \\ \leq &\left|\int_0^{x-x/\sqrt{n}} g_x(t)\,d_tK_n(x,t)\,dt\right| \\ &+ \left|\int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} g_x(t)\,d_tK_n(x,t)\,dt\right| + \left|\int_{x+(1-x)/\sqrt{n}}^1 g_x(t)\,d_tK_n(x,t)\,dt\right| \\ : &= \left|I_1(n,x)\right| + \left|I_2(n,x)\right| + \left|I_3(n,x)\right|. \end{aligned}$$

$$|I_1(n,x)| = \left| \int_0^{x-x/\sqrt{n}} g_x(t)\, d_t(\lambda_n(x,t))\, dt \right|,$$

$$|I_2(n,x)| = \left| \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} g_x(t)\, d_t(\lambda_n(x,t))\, dt \right|$$

and

$$|I_3(n,x)| = \left| \int_{x+(1-x)/\sqrt{n}}^{1} g_x(t)\, d_t(\lambda_n(x,t))\, dt \right|,$$

$\lambda_n(x,t)$ being defined in (5).First we estimate $I_2(n,x)$. For $t \in [x - x/\sqrt{n}, x + (b_n - x)/\sqrt{n}]$, we have, as $g_x(x) = 0$

$$\begin{aligned} |I_2(n,x)| &= \left| \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} (g_x(t) - g_x(x))\, d_t(\lambda_n(x,t))\, dt \right| \\ &\le \bigvee_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} (g_x) \le \frac{1}{n-1} \sum_{k=2}^{n} \bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x). \qquad (3.4) \end{aligned}$$

Next, we estimate $I_1(n,x)$. Using partial Lebesgue-Stieltjes integration, we obtain

$$\begin{aligned} I_1(n,x) &= \int_0^{x-x/\sqrt{n}} g_x(t)\, d_t(\lambda_n(x,t))\, dt \\ &= g_x\left(x - \frac{x}{\sqrt{n}}\right) \lambda_n\left(x, x - x/\sqrt{n}\right) - g_x(0)\, \lambda_n(x,0) \\ &\quad - \int_0^{x-x/\sqrt{n}} \lambda_n(x,t)\, d_t(g_x(t))\, dt. \end{aligned}$$

Because$\left|g_x\left(x - \frac{x}{\sqrt{n}}\right)\right| = \left|g_x\left(x - \frac{x}{\sqrt{n}}\right) - g_x(x)\right| \le \bigvee_{x-x/\sqrt{n}}^{x} (g_x)$, it follows that

$$\begin{aligned} |I_1(n,x)| &\le \bigvee_{x-x/\sqrt{n}}^{x} (g_x) \left|\lambda_n\left(x, x - x/\sqrt{n}\right)\right| \\ &\quad + \int_0^{x-x/\sqrt{n}} \lambda_n(x,t)\, d_t\left(-\bigvee_t^x (g_x)\right) dt \\ &\le \bigvee_{x-x/\sqrt{n}}^{x} (g_x) \frac{(\alpha - \beta x)^2 + nx(1-x)}{(n+\beta)^2 \left(\frac{x}{\sqrt{n}}\right)^2} \\ &\quad + \frac{(\alpha - \beta x)^2 + nx(1-x)}{(n+\beta)^2} \int_0^{x-x/\sqrt{n}} \frac{1}{(x-t)^2} d_t\left(-\bigvee_t^x (g_x)\right) dt. \end{aligned}$$

Furthermore, again by partial integration,

$$\int_0^{x-x/\sqrt{n}} \frac{1}{(x-t)^2} d_t\left(-\bigvee_t^x (g_x)\right) dt = -\frac{1}{\left(\frac{x}{\sqrt{n}}\right)^2} \bigvee_{x-x/\sqrt{n}}^x (g_x) + \frac{1}{x^2}\bigvee_0^x (g_x) + \int_0^{x-x/\sqrt{n}} \frac{2}{(x-t)^3} \bigvee_t^x (g_x)\, dt.$$

Putting $t = x - \frac{x}{\sqrt{u}}$ in the last integral, we obtain

$$\int_0^{x-x/\sqrt{n}} \frac{2}{(x-t)^3} \bigvee_t^x (g_x)\, dt = \frac{1}{x^2}\int_1^n \bigvee_{x-x/\sqrt{u}}^x (g_x)\, du = \frac{1}{x^2}\sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^x (g_x).$$

Consequently,

$$\begin{aligned}
|I_1(n,x)| &\le \bigvee_{x-x/\sqrt{n}}^x (g_x) \frac{(\alpha-\beta x)^2 + nx(1-x)}{(n+\beta)^2\left(\frac{x}{\sqrt{n}}\right)^2} \\
&\quad + \frac{(\alpha-\beta x)^2 + nx(1-x)}{(n+\beta)^2} \\
&\quad \times \left\{ -\frac{1}{\left(\frac{x}{\sqrt{n}}\right)^2} \bigvee_{x-x/\sqrt{n}}^x (g_x) + \frac{1}{x^2}\bigvee_0^x (g_x) \right. \\
&\quad \left. + \frac{1}{x^2}\sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^x (g_x) \right\} \\
&\le \frac{(\alpha-\beta x)^2 + nx(1-x)}{(n+\beta)^2 x^2} \left\{ \bigvee_0^x (g_x) + \sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^x (g_x) \right\}. \qquad (3.5)
\end{aligned}$$

Using a similar method as for estimating $|I_3(n,x)|$, we get

$$|I_3(n,x)| \le \frac{(\alpha-\beta x)^2 + nx(1-x)}{(n+\beta)^2 (1-x)^2} \left\{ \bigvee_x^1 (g_x) + \sum_{k=1}^n \bigvee_x^{x+(1-x)/\sqrt{n}} (g_x) \right\}. \qquad (3.6)$$

Hence from (3.4)-(3.6), it follows that

$$\begin{aligned}\left|B_n^{(\alpha,\beta)}(g_x,x)\right| &\leq |I_1(n,x)|+|I_2(n,x)|+|I_3(n,x)| \\ &\leq \frac{1}{n-1}\sum_{k=2}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x) \\ &\quad+\frac{(\alpha-\beta x)^2+nx(1-x)}{(n+\beta)^2x^2}\left\{\bigvee_0^x(g_x)+\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x}(g_x)\right\} \\ &\quad+\frac{(\alpha-\beta x)^2+nx(1-x)}{(n+\beta)^2(1-x)^2} \\ &\quad\times\left\{\bigvee_x^1(g_x)+\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)\right\}.\end{aligned}$$

Because $\frac{1}{x^2}+\frac{1}{(1-x)^2}\leq\frac{1}{x^2(1-x)^2}$, $x\in[0,1]$

$$\begin{aligned}\left|B_n^{(\alpha,\beta)}(g_x,x)\right| &\leq \frac{(\alpha-\beta x)^2+nx(1-x)}{(n+\beta)^2(1-x)^2x^2}\left\{\bigvee_0^1(g_x)+\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)\right\} \\ &\quad+\frac{1}{n-1}\sum_{k=2}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x). \qquad (3.7)\end{aligned}$$

On the other hand, note that $\bigvee_0^1(g_x)\leq\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)$, so that

$$\begin{aligned}\left|B_n^{(\alpha,\beta)}(g_x,x)\right| &\leq 2\left[\frac{(\alpha-\beta x)^2+nx(1-x)}{(n+\beta)^2(1-x)^2x^2}\right]\left\{\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)\right\} \\ &\quad+\frac{1}{n-1}\sum_{k=2}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x).\end{aligned}$$

Noting that $\frac{1}{n-1}\leq\frac{(\alpha-\beta x)^2+nx(1-x)}{(n+\beta)^2(1-x)^2x^2}$, for $n>1$, we have

$$\left|B_n^{(\alpha,\beta)}(g_x,x)\right|\leq 3\frac{(\alpha-\beta x)^2+nx(1-x)}{(n+\beta)^2(1-x)^2x^2}\left\{\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x)\right\}. \qquad (9)$$

Putting (2.1) and (3.6) in (3.3), we deduce the required result (1.1). Thus the proof of our theorem is finally complete.

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