SOME CHARACTERIZATIONS OF SLANT AND SPHERICAL HELICES DUE TO SABBAN FRAME

BÜLENT ALTUNKAYA AND LEVENT KULA

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ABSTRACT. In this paper, we are investigating that under which conditions of the geodesic curvature of unit speed curve γ that lies on the unit sphere, the curve c which is obtained by using γ, is a spherical helix or slant helix.

1. INTRODUCTION

Izumiya and Takeuchi [4],[2] have defined helices, spherical helices, slant helices and conical geodesic curve and given a classification of special developable surfaces under the condition of the existence of such a special curve as a geodesic [1].

Encheva and Georgiev [3] have used a similar method and determined a *Frenet curve* up to a direct similarity of R^3.

In this paper we have used the the method in [2], [3], and [4] to construct spherical helices and slant helices.

In section 2, we recall some basic concepts of differential geometry of space curves that we will use later. In the next section we will provide some new theorems and proofs about the construction of spherical helices and slant helices.

2. BASIC CONCEPTS

We now recall some basic concepts on classical differential geometry of curves in the Euclidean space E^3. For a regular curve $c : I \subset R \to R^3$ with curvature and torsion, κ and τ, the following Frenet-Serret formulae are given in [5] written in the matrix form for $\kappa > 0$

$$\begin{bmatrix} T^{'} \\ N^{'} \\ B^{'} \end{bmatrix} = \begin{bmatrix} 0 & \kappa\nu & 0 \\ -\kappa\nu & 0 & \tau\nu \\ 0 & -\tau\nu & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where
(2.1)

$$\nu = \nu(t) = \left\| c^{'}(t) \right\|, \kappa = \kappa(t) = \frac{\left\| c^{'}(t) \times c^{''}(t) \right\|}{\left\| c^{'}(t) \right\|^3}, \tau = \tau(t) = \frac{\left\langle c^{'}(t) \times c^{''}(t), c^{'''}(t) \right\rangle}{\left\| c^{'}(t) \times c^{''}(t) \right\|^2}$$

and

$$(2.2) \quad T = T(t) = \frac{c^{'}(t)}{\left\| c^{'}(t) \right\|}, B = B(t) = \frac{c^{'}(t) \times c^{''}(t)}{\left\| c^{'}(t) \times c^{''}(t) \right\|}, N = N(t) = B(t) \times T(t)$$

In the formulae above, we denote unit tangent vector with T, binormal unit vector with B, unit principal normal vector with N, cross product with $\times$, and inner product with $<,>$.

A regular curve c with $\kappa > 0$ is a cylindrical helix if and only if the ratio

$$\frac{\tau}{\kappa}$$

is constant [5].

A regular curve c with $\kappa > 0$ is a slant helix if and only if the geodesic curvature of the spherical image of the principal normal indicatrix of c

$$(2.3) \qquad\qquad \sigma(t) = \left(\frac{\kappa^2}{\nu \left(\kappa^2 + \tau^2 \right)^{3/2}} \left(\frac{\tau}{\kappa} \right)^{'} \right)(t)$$

is constant [4].

A regular curve c with $\kappa > 0$ lies on the surface of a sphere which has a radius r if and only if

$$(2.4) \qquad\qquad r^2 = \left(\frac{1}{\kappa^2} + \left(\frac{1}{\nu \tau} \left(\frac{1}{\kappa} \right)^{'} \right)^2 \right)(t)$$

satisfies [6]. We can easily simplify this equation as follows

$$(2.5) \qquad\qquad \left(\frac{1}{\nu} \left[\frac{1}{\nu \tau} \left(\frac{1}{\kappa} \right)^{'} \right]^{'} + \frac{\tau}{\kappa} \right)(t) = 0.$$

Let $\gamma : I \to S^2$ be a unit speed spherical curve with an arc length parameter s and denote $\gamma^{'}(s) = t(s)$ where $\gamma^{'}(s) = \frac{d\gamma}{ds}$. If we set a vector $p(s) = \gamma(s) \times t(s)$, by definition we have an orthonormal frame $\{\gamma(s), t(s), p(s)\}$ along γ. This frame is called the $Sabban\ frame$ of γ. Then we have the following $Frenet - Serret$ formulae of γ

$$(2.6) \qquad\qquad \begin{aligned} \gamma^{'}(s) &= t(s) \\ t^{'}(s) &= -\gamma(s) + k_g(s) p(s) \\ p^{'}(s) &= -k_g(s) t(s) \end{aligned}$$

where $k_g(s)$ is the $geodesic\ curvature$ of the curve γ on S^2 which is

$$k_g(s) = det \left(\gamma(s), t(s), t^{'}(s) \right)$$

In [2] Izuyama and Takeuchi showed a way to construct all *Bertrand curves* by the following formula

$$(2.7) \qquad c\left(s\right) = b\int_{s_0}^{s}\gamma\left(\varphi\right)d\varphi + b\cot\theta\int_{s_0}^{s}p\left(\varphi\right)d\varphi + a$$

where b, θ are constant numbers, a is a constant vector, and γ is a unit speed curve on S^2 with the *Sabban frame* above. Also they showed that the spherical curve γ is a circle if and only if the corresponding *Bertrand curves* are circular helices.

In [3] Encheva and Georgiev showed a way to construct all *Frenet curves* $(\kappa > 0)$ by the following formula

$$(2.8) \qquad c\left(s\right) = b\int e^{\int k(s)ds}\gamma\left(s\right)ds + a$$

where b is a constant number, a is a constant vector, γ is a unit speed curve on S^2 with the *Sabban frame* above, and $k : I \rightarrow R$ is a function of class C^1. Also they showed that the spherical curve γ is a circle if and only if the corresponding *Frenet curves* are cylindrical helices.

If we use the equations in (2.1), (2.2), (2.6), (2.7), and (2.8) we can easily see the *Frenet frame* $\{T, N, B\}$ of the curve c and the *Sabban frame* $\{\gamma, t, p\}$ of the curve γ coincides. Therefore we can say the *tangent indicatrix* of the curve c is γ.

3. Characterizations of Helices

Now, we can deduce some results from the equations above. First, we want to show, under which circumstances the equation (2.8) is a spherical helix. As we know before, if γ is a circle, the geodesic curvature of it is constant. Therefore we can write the theorem below.

Theorem 3.1. *Let $\gamma\left(s\right)$ be a unit speed spherical curve on S^2 with the geodesic curvature k_g. If the curve γ is a circle, the curve c defined by (2.8) is a spherical helix if and only if the function $k\left(s\right) = -k_g tan\left[\left(k_g\right)\left(s - b_1\right)\right]$ where $b_1 \in R$.*

Proof. For the curve

$$c\left(s\right) = b\int e^{\int k(s)ds}\gamma\left(s\right)ds + a$$

If we calculate κ, τ, and ν of the curve c by using the equations at (2.1), we will find

$$(3.1) \qquad \begin{aligned} \kappa\left(s\right) &= \frac{1}{be^{\int k(s)ds}} \\ \tau\left(s\right) &= \frac{k_g(s)}{be^{\int k(s)ds}} \\ \nu\left(s\right) &= be^{\int k(s)ds} \end{aligned} .$$

Now, by putting these equations in (2.5), we have

$$\left(\frac{1}{\nu}\left[\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)'\right]' + \frac{\tau}{\kappa}\right)\left(s\right) = 0$$

$$\left(\frac{1}{be^{\int kds}}\left[\frac{1}{be^{\int kds}\frac{k_g}{be^{\int kds}}}\left(\frac{1}{\frac{1}{be^{\int kds}}}\right)'\right]' + \frac{\frac{k_g}{be^{\int kds}}}{\frac{1}{be^{\int kds}}}\right)\left(s\right) = 0$$

$$\left(\frac{1}{be^{\int kds}}\left[\frac{1}{k_g}\left(be^{\int kds}\right)'\right]' + k_g\right)(s) = 0$$

$$\left(\frac{1}{k_g e^{\int kds}}\left[k'e^{\int kds} + k^2 e^{\int kds}\right] + k_g\right)(s) = 0$$

$$k'(s) + k^2(s) = -k_g{}^2.$$

If we solve this differential equation, we will have

$$k(s) = -k_g tan\left[(k_g)(s - b_1)\right]$$

Conversely, if we take $k(s) = -k_g tan\left[(k_g)(s - b_1)\right]$ in (2.8) then

$$\int k(s)\,ds = \int -k_g tan\left[(k_g)(s - b_1)\right]ds.$$

Let $u = k_g(s - b_1) = k_g s - k_g b_1$ then $k_g ds = du$, by using these equations

$$\int k(s)\,ds = \int -\tan u\,du$$
$$= \ln\cos u + \ln b_2$$
$$= \ln\left[b_2 \cos\left(k_g(s - b_1)\right)\right]$$

we have

$$c(s) = b\int e^{\int k(s)ds}\gamma(s)ds + a$$
$$= b\int e^{\int -k_g tan[(k_g)(s-b_1)]ds}\gamma(s)ds + a$$
$$= b\int e^{\ln\left[b_2\cos\left(k_g(s-b_1)\right)\right]}\gamma(s)ds + a$$
$$= b\int b_2\cos\left(k_g(s - b_1)\right)\gamma(s)ds + a.$$

where $b_1, b_2 \in R$.

Now, we must show that curve c is spherical. If we use (2.4) to do it, we will have

$$r^2 = \left(\left(\frac{1}{\kappa^2} + \left(\frac{1}{\nu\tau}\left(\frac{1}{\kappa}\right)' \right) \right)^2 \right)(s)$$

$$= \left(b^2 e^{2\int k ds} + \left(\frac{1}{be^{\int k ds} \frac{k_g}{be^{\int k ds}}} \left(\frac{1}{\frac{1}{be^{\int k ds}}} \right)' \right)^2 \right)(s)$$

$$= \left(b^2 e^{2\int k ds} + \left(\frac{1}{k_g}\left(be^{\int k ds} \right)' \right)^2 \right)(s)$$

$$= \left(b^2 e^{2\int k ds} + \frac{b^2 k^2}{k_g^2} e^{2\int k ds} \right)(s)$$

$$= \left(b^2 e^{2\int k ds} \left(1 + \frac{k^2}{k_g^2} \right) \right)(s)$$

$$= b^2 b_2{}^2 \cos^2\left(k_g\left(s - b_1\right)\right)\left(1 + \frac{\left(-k_g \tan\left[\left(k_g\right)\left(s - b_1\right)\right]\right)^2}{k_g^2} \right)$$

$$= b^2 b_2{}^2 \cos^2\left(k_g\left(s - b_1\right)\right)\left(\frac{1}{\cos^2\left(k_g\left(s - b_1\right)\right)} \right)$$

$$= b^2 b_2{}^2.$$

Therefore, we can say curve c lies on a sphere which has a radius $|bb_2|$ □

Example 3.1. Let's take $\gamma(s) = \left\{ \frac{1}{\sqrt{3}}\cos\left(\frac{s}{(1/\sqrt{3})} \right), \frac{1}{\sqrt{3}}\sin\left(\frac{s}{(1/\sqrt{3})} \right), \sqrt{\frac{2}{3}} \right\}$, we know that γ is a unit speed curve on S^2 with the geodesic curvature $\sqrt{2}$. Then due to Theorem 3.1,

$$k(s) = k_g \tan\left[\left(k_g\right)\left(s - b_1\right)\right]$$

and

$$\alpha(s) = b\int b_2 \cos\left(k_g\left(s - b_1\right)\right)\gamma(s)ds + a$$

where $b, b_1, b_2 \in R$. If we take $b = 2, b_1 = 0, b_2 = 1$ then we have

$$\alpha_1(s) = -2\sqrt{\frac{2}{3}}\cos\left(\sqrt{3}s\right)\sin\left(\sqrt{2}s\right) + 2\cos\left(\sqrt{2}s\right)\sin\left(\sqrt{3}s\right)$$
$$\alpha_2(s) = -\frac{2}{3}\left(3\cos\left(\sqrt{2}s\right)\cos\left(\sqrt{3}s\right) + \sqrt{6}\sin\left(\sqrt{2}s\right)\sin\left(\sqrt{3}s\right) \right)$$
$$\alpha_3(s) = \frac{2\sin\left(\sqrt{2}s\right)}{\sqrt{3}}$$

where $\alpha(s) = \left(\alpha_1(s), \alpha_2(s), \alpha_3(s)\right)$ and $a = (0, 0, 0)$

Now, we can write a new thorem about (2.8) in which we are looking for the *slant helix* condition of the curve c.

Theorem 3.2. *Let $\gamma(s)$ be a unit speed spherical curve on S^2; b, m, n be constant numbers; and a be a constant vector. The geodesic curvature of $\gamma(s)$ satisfies*

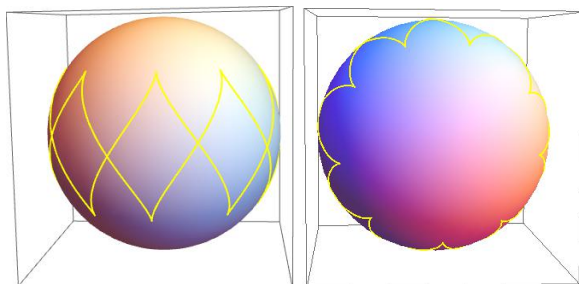$$k_g{}^2(s) = \frac{\left(ms + n\right)^2}{1 - \left(ms + n\right)^2}$$

FIGURE 1. Spherical Helice

if and only if

$$c(s) = b \int e^{\int k(s)ds} \gamma(s)ds + a$$

is a slant helix.

Proof. Let, for γ

(3.2)
$$k_g^2(s) = \frac{(ms+n)^2}{1-(ms+n)^2}.$$

For the curve $c(s)$ we have

$$c'(s) = be^{\int k(s)ds}\gamma(s)$$
$$c''(s) = be^{\int k(s)ds}\left(k(s)\gamma(s) + \gamma'(s)\right)$$
$$c'''(s) = be^{\int k(s)ds}\left(\left(k^2(s) + k'(s)\right)\gamma(s) + 2k(s)\gamma'(s) + \gamma''(s)\right)$$
$$\kappa(s) = \frac{1}{be^{\int k(s)ds}}$$
$$\tau(s) = \frac{k_g(s)}{be^{\int k(s)ds}}$$
$$\nu(s) = be^{\int k(s)ds}.$$

The geodesic curvature of the spherical image of the principal normal indicatrix of c is as follows

$$\sigma(s) = \left(\frac{\kappa^2}{\nu(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'\right)(s)$$

$$= \left(\frac{\frac{1}{\nu^2}}{\nu\left(\frac{1}{\nu^2}+\frac{k_g^2}{\nu^2}\right)^{3/2}}k_g'\right)(s).$$

So we have

(3.3)
$$\sigma(s) = \frac{k_g'(s)}{\left(k_g^2(s)+1\right)^{3/2}}$$

Now, let's take $u(s) = ms + n$ then we have (3.2) as follows

(3.4)
$$k_g^2(s) = \frac{u^2(s)}{1-u^2(s)}.$$

If we take the derivates of the both sides of (3.4) for s we have

$$2k_g(s) k_g^{'}(s) = \left(\frac{2uu^{'}\left(1-u^2\right) - \left(-2uu^{'}\right)u^2}{\left(1-u^2\right)^2} \right)(s)$$

$$k_g(s) k_g^{'}(s) = \left(\frac{uu^{'}}{\left(1-u^2\right)^2} \right)(s)$$

(3.5)
$$k_g^{'}(s) = \left(\left(\frac{uu^{'}}{\left(1-u^2\right)^2} \right) \left(\varepsilon\sqrt{\frac{1-u^2}{u^2}} \right) \right)(s)$$

where $\varepsilon = \pm 1$. Putting (3.4) and (3.5) in (3.3), we have

$$\sigma(s) = \frac{k_g^{'}(s)}{\left(k_g^{\,2}(s)+1\right)^{3/2}}$$

$$= \left(\varepsilon \frac{\frac{\sqrt{1-u^2}uu^{'}}{|u|(1-u^2)^2}}{\left(\frac{u^2}{1-u^2}+1\right)^{3/2}} \right)(s)$$

$$= \left(\varepsilon \frac{\sqrt{1-u^2}uu^{'}}{|u|\left(1-u^2\right)^2}\left(1-u^2\right)^{3/2} \right)(s)$$

$$= \left(\varepsilon \frac{\left(1-u^2\right)^2}{\left(1-u^2\right)^2}\frac{u}{|u|}u^{'} \right)(s)$$

$$= \varepsilon \frac{ms+n}{|ms+n|}m$$

$$= \varepsilon m$$

which is constant.

Conversely, let $c(s)$ be a *slant helix*, then the geodesic curvature of the spherical image of the principal normal indicatrix of c is a constant function. So we can take

$$\sigma(s) = \left(\frac{\kappa^2}{\nu\left(\kappa^2+\tau^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)^{'} \right)(s) = m$$

where $m \in R$. Therefore, from (3.3)

$$m = \left(\frac{\kappa^2}{\nu\left(\kappa^2+\tau^2\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)^{'} \right)(s)$$

$$= \frac{k_g^{'}(s)}{\left(k_g^{\,2}(s)+1\right)^{3/2}}$$

If we solve this differential equation, we have

$$\frac{k_g(s)}{\sqrt{k_g^{\,2}(s)+1}} = ms+n$$

where $n \in R$. Then,

$$\frac{k_g{}^2(s)}{k_g{}^2(s)+1} = (ms+n)^2$$

$$\frac{k_g{}^2(s)+1-1}{k_g{}^2(s)+1} = (ms+n)^2$$

$$1 - \frac{1}{k_g{}^2(s)+1} = (ms+n)^2$$

$$k_g{}^2(s) = \frac{1}{1-(ms+n)^2} - 1$$

$$k_g{}^2(s) = \frac{(ms+n)^2}{1-(ms+n)^2}.$$

$\square$

Furthermore, we can give a similar theorem for (2.7),

Theorem 3.3. *Let $\gamma(s)$ be a unit speed spherical curve on S^2; b, m, n, θ be constant numbers; and a be a constant vector. The geodesic curvature of $\gamma(s)$ satisfies*

$$k_g{}^2(s) = \frac{(ms+n)^2}{1-(ms+n)^2}$$

if and only if

$$c(s) = b\int_{s_0}^{s} \gamma(\varphi)\,d\varphi + b\cot\theta \int_{s_0}^{s} p(\varphi)\,d\varphi + a$$

is a slant helix.

Proof. Let, for γ

$$(3.6) \qquad\qquad k_g{}^2(s) = \frac{(ms+n)^2}{1-(ms+n)^2}.$$

For the curve $c(s)$ we have

$$c'(s) = b\left(\gamma(s) + \cot\theta p(s)\right)$$
$$c''(s) = b\left(1 - \cot\theta k_g(s)\right)p(s)$$
$$c'''(s) = -b\cot\theta k_g{}'(s)p(s) + b\left(1 - \cot\theta k_g(s)\right)\left(-\gamma(s) + k_g(s)p(s)\right)$$
$$\kappa(s) = \varepsilon\frac{\sin^2\theta(1-\cot\theta k_g(s))}{b}$$
$$\tau(s) = \frac{\sin^2\theta(k_g(s)+\cot\theta)}{b}$$
$$\nu(s) = \varepsilon b\csc\theta.$$

where $\varepsilon = \pm 1$.

The geodesic curvature of the spherical image of the principal normal indicatrix of c is as follows

$$\sigma(s) = \left(\frac{\kappa^2}{\nu \left(\kappa^2 + \tau^2\right)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \right)(s)$$

$$= \left(\frac{\varepsilon \sin^3 \theta k_g{}'}{b^3 \left(\frac{(\varepsilon^2 + \cot^2 \theta - 2(-1+\varepsilon^2)\cot\theta k_g + (1+\varepsilon^2 \cot^2 \theta)k_g{}^2)\sin^4 \theta}{a^2} \right)^{3/2}} \right)(s)$$

$$= \left(\frac{\varepsilon k_g{}'}{\sin^3 \theta \left(\left(1 + \cot^2 \theta\right)\left(1 + k_g{}^2\right) \right)^{3/2}} \right)(s)$$

$$= \left(\frac{\varepsilon k_g{}'}{\sin^3 \theta \left(\frac{1}{\sin^2 \theta} \left(1 + k_g{}^2\right) \right)^{3/2}} \right)(s)$$

$$= \left(\frac{\varepsilon k_g{}'}{\left(1 + k_g{}^2\right)^{3/2}} \right)(s)$$

So we have

$$(3.7) \qquad \sigma(s) = \frac{\varepsilon k_g{}'(s)}{\left(k_g{}^2(s) + 1\right)^{3/2}}$$

Now, let's take $u(s) = ms + n$ then we have (3.6) as follows

$$(3.8) \qquad k_g{}^2(s) = \frac{u^2(s)}{1 - u^2(s)}.$$

If we take the derivates of the both sides of (3.8) for s we have

$$2k_g(s)\, k_g{}'(s) = \left(\frac{2uu'\left(1 - u^2\right) - \left(-2uu'\right)u^2}{\left(1 - u^2\right)^2} \right)(s)$$

$$k_g(s)\, k_g{}'(s) = \left(\frac{uu'}{\left(1 - u^2\right)^2} \right)(s)$$

$$(3.9) \qquad k_g{}'(s) = \left(\left(\frac{uu'}{\left(1 - u^2\right)^2} \right) \left(\varepsilon \sqrt{\frac{1 - u^2}{u^2}} \right) \right)(s).$$

Putting (3.9) and (3.8) in (3.7), we have

$$\sigma(s) = \frac{\varepsilon k_g^{'}(s)}{\left(k_g^{\ 2}(s) + 1\right)^{3/2}}$$

$$= \left(\varepsilon^2 \frac{\frac{\sqrt{1-u^2}uu^{'}}{|u|(1-u^2)^2}}{\left(\frac{u^2}{1-u^2} + 1\right)^{3/2}}\right)(s)$$

$$= \left(\frac{\sqrt{1-u^2}uu^{'}}{|u|(1-u^2)^2}(1-u^2)^{3/2}\right)(s)$$

$$= \left(\frac{(1-u^2)^2}{(1-u^2)^2}\frac{u}{|u|}u^{'}\right)(s)$$

$$= \frac{ms+n}{|ms+n|}m$$

$$= \varepsilon m$$

which is constant.

Conversely, let $c(s)$ be a *slant helix*, then the geodesic curvature of the spherical image of the principal normal indicatrix of c is a constant function. So we can take

$$\sigma(s) = \left(\frac{\kappa^2}{\nu(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)^{'}\right)(s) = m$$

where $m \in R$. Therefore, from (3.7)

$$m = \left(\frac{\kappa^2}{\nu(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)^{'}\right)(s)$$

$$= \frac{\varepsilon k_g^{'}(s)}{\left(k_g^{\ 2}(s)+1\right)^{3/2}}$$

If we solve this differential equation, we have

$$\frac{\varepsilon k_g(s)}{\sqrt{k_g^{\ 2}(s)+1}} = ms+n$$

where $n \in R$. Then,

$$\frac{k_g^{\ 2}(s)}{k_g^{\ 2}(s)+1} = (ms+n)^2$$

$$\frac{k_g^{\ 2}(s)+1-1}{k_g^{\ 2}(s)+1} = (ms+n)^2$$

$$1 - \frac{1}{k_g^{\ 2}(s)+1} = (ms+n)^2$$

$$k_g^{\ 2}(s) = \frac{(ms+n)^2}{1-(ms+n)^2}.$$

$\square$

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DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION, UNIVERSITY OF AHI EVRAN, KIRŞEHIR, TURKEY
    E-mail address: bulent.altunkaya@ahievran.edu.tr

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, UNIVERSITY OF AHI EVRAN, KIRŞEHIR, TURKEY
    E-mail address: lkula@ahievran.edu.tr