

THE RANDOM OF LACUNARY STATISTICAL ON χ^2 OVER p -
METRIC SPACES DEFINED BY MUSIELAK

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ABSTRACT. Mursaleen introduced the concepts of statistical convergence in random 2-normed spaces. Recently Mohiuddine and Aiyup defined the notion of lacunary statistical convergence and lacunary statistical Cauchy in random 2-normed spaces. In this paper, we define and study the notion of lacunary statistical convergence and lacunary of statistical Cauchy sequences in random on χ^2 over p - metric spaces defined by Musielak and prove some theorems which generalizes Mohiuddine and Aiyup results.

1. INTRODUCTION

The concept of statistical convergence play a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modeling, and motion planning in robotics.

The notion of statistical convergence was introduced by Fast and Schoenberg independently. Over the years and under different names statistical convergence has been discussed in the theory of fourier analysis, ergodic theory and number theory. Later on it was further investigated by Fridy, Šalát, Çakalli, Maio and Kocinac, Miller, Maddox, Leindler, Mursaleen and Alotaibi, Mursaleen and Edely, and many others. In the recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-Čech compactification of the natural numbers. Moreover statistical convergence is closely related to the concept of convergence in probability.

The notion of statistical convergence depends on the density of subsets of \mathbb{N} . A subset of \mathbb{N} is said to have density $\delta(E)$ if

$$\delta(E) = \lim_{r,s \rightarrow \infty} \frac{1}{rs} \sum_{m=1}^r \sum_{n=1}^s \chi^2 E(mn) = 0.$$

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Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}) , where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [11], Moricz [19], Moricz and Rhoades [20], Basarir and Solankan [3], Tripathy [34-37], Turkmenoglu [30], and many others.

We procure the following sets of double sequences:

$$\begin{aligned} \mathcal{M}_u(t) &:= \left\{ (x_{mn}) \in w^2 : \sup_{m,n \in \mathbb{N}} |x_{mn}|^{t_{mn}} < \infty \right\}, \\ \mathcal{C}_p(t) &:= \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n \rightarrow \infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C} \right\}, \\ \mathcal{C}_{0p}(t) &:= \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n \rightarrow \infty} |x_{mn}|^{t_{mn}} = 1 \right\}, \\ \mathcal{L}_u(t) &:= \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\}, \\ \mathcal{C}_{bp}(t) &:= \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \cap \mathcal{M}_u(t), \end{aligned}$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n \rightarrow \infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets \mathcal{M}_u , \mathcal{C}_p , \mathcal{C}_{0p} , \mathcal{L}_u , \mathcal{C}_{bp} and \mathcal{C}_{0bp} , respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [7,8] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the α -, β -, γ - duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [42] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [21] have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Basar [1] have defined the spaces \mathcal{BS} , $\mathcal{BS}(t)$, \mathcal{CS}_p , \mathcal{CS}_{bp} , \mathcal{CS}_r and \mathcal{BV} of double sequences consisting of all double series whose sequence of partial sums are in the spaces \mathcal{M}_u , $\mathcal{M}_u(t)$, \mathcal{C}_p , \mathcal{C}_{bp} , \mathcal{C}_r and \mathcal{L}_u , respectively, and also examined some properties of those sequence spaces and determined the α - duals of the spaces \mathcal{BS} , \mathcal{BV} , \mathcal{CS}_{bp} and the $\beta(\vartheta)$ - duals of the spaces \mathcal{CS}_{bp} and \mathcal{CS}_r of double series. Basar and Sever [2] have introduced the Banach space \mathcal{L}_q of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space \mathcal{L}_q . Quite recently Subramanian and Misra [28] have studied the space $\chi_M^2(p, q, u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [18] as an extension of the definition of strongly Cesàro summable sequences. Connor [5] further extended this definition to a definition of strong A - summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong A - summability, strong A - summability with respect to a modulus, and A - statistical convergence. In [25] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [12]-[13], and [14] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{kl}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b \geq 0$ and $0 < p < 1$, we have

$$(1.1) \quad (a + b)^p \leq a^p + b^p$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij}$ ($m, n \in \mathbb{N}$).

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2 . A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \rightarrow 0$ as $m, n \rightarrow \infty$. The double gai sequences will be denoted by χ^2 . Let $\phi = \{\text{all finite sequences}\}$.

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \mathfrak{S}_{ij}$ for all $m, n \in \mathbb{N}$; where \mathfrak{S}_{ij} denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i, j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if (\mathfrak{S}_{mn}) is a Schauder basis for X . Or equivalently $x^{[m,n]} \rightarrow x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \rightarrow (x_{mn})$ ($m, n \in \mathbb{N}$) are also continuous.

Let M and Φ are mutually complementary modulus functions. Then, we have

(i) For all $u, y \geq 0$,

$$(1.2) \quad uy \leq M(u) + \Phi(y), \text{ (Young's inequality)[16].}$$

(ii) For all $u \geq 0$,

$$(1.3) \quad u\eta(u) = M(u) + \Phi(\eta(u)).$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$(1.4) \quad M(\lambda u) \leq \lambda M(u).$$

Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p$ ($1 \leq p < \infty$), the spaces ℓ_M coincide with the classical sequence space ℓ_p .

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = \sup \{ |v| u - (f_{mn})(u) : u \geq 0 \}, m, n = 1, 2, \dots$$

is called the complementary function of a Musielak-modulus function f . For a given Musielak modulus function f , the Musielak-modulus sequence space t_f and

its subspace h_f are defined as follows:

$$t_f = \left\{ x \in w^2 : I_f(|x_{mn}|)^{1/m+n} \rightarrow 0 \text{ as } m, n \rightarrow \infty \right\},$$

$$h_f = \left\{ x \in w^2 : I_f(|x_{mn}|)^{1/m+n} \rightarrow 0 \text{ as } m, n \rightarrow \infty \right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = \sup_{mn} \left\{ \inf \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} \left(\frac{|x_{mn}|^{1/m+n}}{mn} \right) \right) \leq 1 \right\}.$$

If X is a sequence space, we give the following definitions:

- (i) X' = the continuous dual of X ;
- (ii) $X^\alpha = \{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X\}$;
- (iii) $X^\beta = \{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convergent, for each } x \in X\}$;
- (iv) $X^\gamma = \{a = (a_{mn}) : \sup_{mn} \geq 1 \left| \sum_{m,n=1}^{M,N} a_{mn}x_{mn} \right| < \infty, \text{ for each } x \in X\}$;
- (v) Let X be an FK -space $\supset \phi$; then $X^f = \{f(\mathfrak{S}_{mn}) : f \in X'\}$;
- (vi) $X^\delta = \{a = (a_{mn}) : \sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X\}$;

$X^\alpha, X^\beta, X^\gamma$ are called α - (or Köthe-Toeplitz) dual of X, β - (or generalized-Köthe-Toeplitz) dual of X, γ -dual of X, δ - dual of X , respectively. X^α is defined by Gupta and Kamptan [16]. It is clear that $X^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\beta \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c, c_0$ and ℓ_∞ , where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here c, c_0 and ℓ_∞ denote the classes of convergent, null and bounded scalar valued single sequences, respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar in [1]. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\},$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$. The generalized difference double notion has the following representation:

$$\Delta^m x_{mn} = \Delta^{m-1} x_{mn} - \Delta^{m-1} x_{mn+1} - \Delta^{m-1} x_{m+1n} + \Delta^{m-1} x_{m+1n+1},$$

and also this generalized difference double notion has the following binomial representation:

$$\Delta^m x_{mn} = \sum_{i=0}^m \sum_{j=0}^m (-1)^{i+j} \binom{m}{i} \binom{m}{j} x_{m+i, n+j}.$$

2. DEFINITION AND PRELIMINARIES

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w where $n \leq w$. A real valued function $d_p(x_1, \dots, x_n) = \|(d_1(x_1), \dots, d_n(x_n))\|_p$ on X satisfying the following four conditions:

- (i) $\|(d_1(x_1), \dots, d_n(x_n))\|_p = 0$ if and only if $d_1(x_1), \dots, d_n(x_n)$ are linearly dependent,
- (ii) $\|(d_1(x_1), \dots, d_n(x_n))\|_p$ is invariant under permutation,
- (iii) $\|(\alpha d_1(x_1), \dots, d_n(x_n))\|_p = |\alpha| \|(d_1(x_1), \dots, d_n(x_n))\|_p, \alpha \in \mathbb{R}$
- (iv) $d_p((x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)) = (d_X(x_1, x_2, \dots, x_n)^p + d_Y(y_1, y_2, \dots, y_n)^p)^{1/p}$ for $1 \leq p < \infty$; (or)
- (v) $d((x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)) := \sup \{d_X(x_1, x_2, \dots, x_n), d_Y(y_1, y_2, \dots, y_n)\}$, for $x_1, x_2, \dots, x_n \in X, y_1, y_2, \dots, y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n -vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\begin{aligned} \|(d_1(x_1), \dots, d_n(x_n))\|_E &= \sup(|\det(d_{mn}(x_{mn}))|) \\ &= \sup \left(\begin{vmatrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \dots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \dots & d_{2n}(x_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \dots & d_{nn}(x_{nn}) \end{vmatrix} \right), \end{aligned}$$

where $x_i = (x_{i1}, \dots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \dots, n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the p - metric. Any complete p - metric space is said to be p - Banach metric space.

Let X be a linear metric space. A function $w : X \rightarrow \mathbb{R}$ is called paranorm, if

- (1) $w(x) \geq 0$, for all $x \in X$;
- (2) $w(-x) = w(x)$, for all $x \in X$;
- (3) $w(x+y) \leq w(x) + w(y)$, for all $x, y \in X$;
- (4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \rightarrow \sigma$ as $m, n \rightarrow \infty$ and (x_{mn}) is a sequence of vectors with $w(x_{mn} - x) \rightarrow 0$ as $m, n \rightarrow \infty$, then $w(\sigma_{mn}x_{mn} - \sigma x) \rightarrow 0$ as $m, n \rightarrow \infty$.

A paranorm w for which $w(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [41], Theorem 10.4.2, p.183).

By the convergence of a double sequence we mean the convergence on the Pringsheim sense that is, a double sequence $x = (x_{mn})$ has Pringsheim limit L (denoted by

P - $\lim x = L$) provided that given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{mn} - L| < \epsilon$ whenever $m, n > n$. We shall write more briefly as P -convergent.

The double sequence $\theta_{rs} = \{(m_r, n_s)\}$ is called double lacunary sequence if there exist two increasing of integers such that

$$\begin{aligned} m_0 = 0, \varphi_r = m_r - m_{r-1} \rightarrow \infty \text{ as } r \rightarrow \infty \text{ and} \\ n_0 = 0, \varphi_s = n_s - n_{s-1} \rightarrow \infty \text{ as } s \rightarrow \infty. \end{aligned}$$

Notations: $m_{rs} = m_r n_s, h_{rs} = \varphi_r \varphi_s, \theta_{rs}$ is determined by

$$\begin{aligned} I_{rs} = \{(m, n) : m_{r-1} < m \leq m_r \text{ and } n_{s-1} < n \leq n_s\}, \\ q_r = \frac{k_r}{k_{r-1}}, \bar{q}_s = \frac{n_s}{n_{s-1}} \text{ and } q_{rs} = q_r \bar{q}_s. \end{aligned}$$

The notion of λ -double gai and double analytic sequences as follows: Let $\lambda = (\lambda_{mn})_{m,n=0}^{\infty}$ be a strictly increasing sequences of positive real numbers tending to infinity, that is

$$0 < \lambda_{00} < \lambda_{11} < \dots \text{ and } \lambda_{mn} \rightarrow \infty \text{ as } m, n \rightarrow \infty$$

and said that a sequence $x = (x_{mn}) \in w^2$ is λ -convergent to 0, called the λ -limit of x , if $\mu_{mn}(x) \rightarrow 0$ as $m, n \rightarrow \infty$, where $\mu_{mn}(x)$ is defined by

$$\frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} (\Delta^{m-1} \lambda_{m,n} - \Delta^{m-1} \lambda_{m,n+1} - \Delta^{m-1} \lambda_{m+1,n} + \Delta^{m-1} \lambda_{m+1,n+1}) |x_{mn}|^{1/m+n}.$$

The sequence $x = (x_{mn}) \in w^2$ is λ -double analytic if $\sup_{uv} |\mu_{mn}(x)| < \infty$. If $\lim_{mn} x_{mn} = 0$ in the ordinary sense of convergence, then

$$\lim_{mn} \frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} (\Delta^{m-1} \lambda_{m,n} - \Delta^{m-1} \lambda_{m,n+1} - \Delta^{m-1} \lambda_{m+1,n} + \Delta^{m-1} \lambda_{m+1,n+1}) ((m+n)! |x_{mn} - 0|)^{1/m+n} = 0.$$

This implies that

$$\begin{aligned} & \lim_{mn} \frac{1}{\varphi_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} \frac{\lim_{mn} |\mu_{mn}(x) - 0|}{((m+n)! |x_{mn} - 0|)^{1/m+n}} \\ & (\Delta^{m-1} \lambda_{m,n} - \Delta^{m-1} \lambda_{m,n+1} - \Delta^{m-1} \lambda_{m+1,n} + \Delta^{m-1} \lambda_{m+1,n+1}) = 0, \end{aligned}$$

which yields that $\lim_{uv} \mu_{mn}(x) = 0$ and hence $x = (x_{mn}) \in w^2$ is λ -convergent to 0.

Let I^2 - be an admissible ideal of $2^{\mathbb{N} \times \mathbb{N}}$, θ_{rs} be a double lacunary sequence, $f = (f_{mn})$ be a Musielak-modulus function and $(X, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p)$ be a p -metric space, $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. By $w^2(p-X)$ we denote the space of all sequences defined over $(X, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p)$. The following inequality will be used throughout the paper. If $0 \leq q_{mn} \leq \sup q_{mn} = H, K = \max(1, 2^{H-1})$ then

$$(2.1) \quad |a_{mn} + b_{mn}|^{q_{mn}} \leq K \{|a_{mn}|^{q_{mn}} + |b_{mn}|^{q_{mn}}\}$$

for all m, n and $a_{mn}, b_{mn} \in \mathbb{C}$. Also $|a|^{q_{mn}} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In the present paper we define the following sequence spaces:

$$\begin{aligned} & \left[\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^2} = \\ & \left\{ r, s \in I_{rs} : \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \geq \epsilon \right\} \in I^2 \end{aligned}$$

and

$$\left[\Lambda_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} = \left\{ r, s \in I_{rs} : \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \geq K \right\} \in I^2.$$

If we take $f_{mn}(x) = x$, we get

$$\left[\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} = \left\{ r, s \in I_{rs} : \left[\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \geq \epsilon \right\} \in I^2$$

and

$$\left[\Lambda_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} = \left\{ r, s \in I_{rs} : \left[\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \geq K \right\} \in I^2.$$

If we take $q = (q_{mn}) = 1$, we get

$$\left[\chi_{f\mu}^2, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^I = \left\{ r, s \in I_{rs} : \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right] \geq \epsilon \right\} \in I^2$$

and

$$\left[\Lambda_{f\mu}^2, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} = \left\{ r, s \in I_{rs} : \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right] \geq K \right\} \in I^2.$$

In the present paper we plan to study some topological properties and inclusion relation between the above defined sequence spaces.

$$\left[\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$$

and

$$\left[\Lambda_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2},$$

which we shall discuss in this paper.

3. MAIN RESULTS

3.1. Theorem. Let $f = (f_{mn})$ be a Musielak-modulus function, $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers, the sequence spaces

$$\left[\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$$

and

$$\left[\Lambda_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$$

are linear spaces.

Proof: It is routine verification. Therefore the proof is omitted.

3.2. Theorem. Let $f = (f_{mn})$ be a Musielak-modulus function, $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers, the sequence space $\left[\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^2}$ is a paranormed space with respect to the paranorm defined by

$$g(x) = \inf \left\{ \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \leq 1 \right\}.$$

Proof: Clearly $g(x) \geq 0$ for $x = (x_{mn}) \in \left[\chi_{f\mu}^{2q}, \|(d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{I^2}$.

Since $f_{mn}(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then

$$\inf \left\{ \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \leq 1 \right\}.$$

Suppose that $\mu_{mn}(x) \neq 0$ for each $m, n \in \mathbb{N}$. Then

$$\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \rightarrow \infty.$$

It follows that

$$\left(\left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \right)^{1/H} \rightarrow \infty,$$

which is a contradiction. Therefore $\mu_{mn}(x) = 0$. Let

$$\left(\left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \right)^{1/H} \leq 1$$

and

$$\left(\left[f_{mn} \left(\|\mu_{mn}(y), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \right)^{1/H} \leq 1.$$

Then by using Minkowski's inequality, we have

$$\begin{aligned} & \left(\left[f_{mn} \left(\|\mu_{mn}(x+y), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \right)^{1/H} \leq \\ & \left(\left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \right)^{1/H} + \\ & \left(\left[f_{mn} \left(\|\mu_{mn}(y), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \right)^{1/H}. \end{aligned}$$

So we have

$$\begin{aligned} g(x+y) &= \inf \left\{ \left[f_{mn} \left(\|\mu_{mn}(x+y), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \leq 1 \right\} \leq \\ & \inf \left\{ \left[f_{mn} \left(\|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \leq 1 \right\} + \\ & \inf \left\{ \left[f_{mn} \left(\|\mu_{mn}(y), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \leq 1 \right\}. \end{aligned}$$

Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, to prove that the scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf \left\{ \left[f_{mn} \left(\|\mu_{mn}(\lambda x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \leq 1 \right\}.$$

Then

$$g(\lambda x) = \inf \left\{ (|\lambda| t)^{q_{mn}/H} : \left[f_{mn} \left(\|\mu_{mn}(\lambda x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{q_{mn}} \leq 1 \right\},$$

where $t = \frac{1}{|\lambda|}$. Since $|\lambda|^{qmn} \leq \max(1, |\lambda|^{supp_{mn}})$, we have

$$g(\lambda x) \leq \max(1, |\lambda|^{supp_{mn}}) \inf \left\{ t^{qmn/H} : \left[f_{mn} \left(\|\mu_{mn}(\lambda x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right) \right]^{qmn} \leq 1 \right\}.$$

This completes the proof.

3.3. Theorem. (i) If the sequence (f_{mn}) satisfies uniform Δ_2 -condition, then

$$\begin{aligned} & \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{\varphi} I^{2\alpha} = \\ & \left[\chi_g^{2q\mu}, \|\mu_{uv}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{\varphi} I^{2\alpha}. \end{aligned}$$

(ii) If the sequence (g_{mn}) satisfies uniform Δ_2 -condition, then

$$\begin{aligned} & \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{\varphi} I^{2\alpha} = \\ & \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{\varphi} I^{2\alpha}. \end{aligned}$$

Proof: (i) Let the sequence (f_{mn}) satisfies uniform Δ_2 -condition, we get

$$\begin{aligned} & \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{\varphi} I^{2\alpha} \subset \\ & \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{\varphi} I^{2\alpha}. \end{aligned}$$

To prove the inclusion

$$\begin{aligned} & \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{\varphi} I^{2\alpha} \subset \\ & \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{\varphi} I^{2\alpha}, \end{aligned}$$

let $a \in \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{\varphi} I^{2\alpha}$. Then for all $\{x_{mn}\}$

with $(x_{mn}) \in \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{\varphi} I^{2\alpha}$ we have

$$(3.1) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn} a_{mn}| < \infty.$$

Since the sequence (f_{mn}) satisfies uniform Δ_2 -condition, then

$$(y_{mn}) \in \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{\varphi} I^{2\alpha},$$

we get $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| \frac{\varphi_{rs} y_{mn} a_{mn}}{\Delta^m \lambda_{mn} (m+n)!} \right| < \infty$ by (3.2). Thus

$$\begin{aligned} (\varphi_{rs} a_{mn}) & \in \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{\varphi} I^{2\alpha} \\ & = \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{\varphi} I^{2\alpha} \end{aligned}$$

and hence

$$(a_{mn}) \in \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p \right]_{\theta_{rs}}^{\varphi} I^{2\alpha}.$$

This gives that

$$\begin{aligned} & \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^{2\alpha}} \subset \\ & \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}. \end{aligned}$$

We are granted with (3.1) and (3.3)

$$\begin{aligned} & \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs_2}}^{I^{2\alpha}} = \\ & \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}. \end{aligned}$$

(ii) Similarly, one can prove that

$$\begin{aligned} & \left[\chi_g^{2q\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs_2}}^{I^{2\alpha}} \subset \\ & \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \end{aligned}$$

if the sequence (g_{mn}) satisfies uniform Δ_2 - condition.

3.4. Proposition. If $0 < q_{mn} < p_{mn} < \infty$ for each m and n , then

$$\begin{aligned} & \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \subseteq \\ & \left[\Lambda_{f\mu}^{2p}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}. \end{aligned}$$

Proof: The proof is standard, so we omit it.

3.5. Proposition. (i) If $0 < \inf q_{mn} \leq q_{mn} < 1$ then

$$\begin{aligned} & \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \subset \\ & \left[\Lambda_{f\mu}^2, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}. \end{aligned}$$

(ii) If $1 \leq q_{mn} \leq \sup q_{mn} < \infty$, then

$$\begin{aligned} & \left[\Lambda_{f\mu}^2, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \subset \\ & \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}. \end{aligned}$$

Proof: The proof is standard, so we omit it.

3.6. Proposition. Let $f' = (f'_{mn})$ and $f'' = (f''_{mn})$ are sequences of Musielak functions, we have

$$\begin{aligned} & \left[\Lambda_{f'\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \cap \\ & \left[\Lambda_{f''\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \subseteq \\ & \left[\Lambda_{f'+f''\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}. \end{aligned}$$

Proof: The proof is easy so we omit it.

3.7. Proposition. For any sequence of Musielak functions $f = (f_{mn})$ and $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. Then

$$\left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \subset \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}.$$

Proof: The proof is easy so we omit it.

3.8. Proposition. The sequence space

$$\left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$$

is solid.

Proof: Let $x = (x_{mn}) \in \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$, (i.e)

$$\sup_{mn} \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} < \infty.$$

Let (α_{mn}) be double sequence of scalars such that $|\alpha_{mn}| \leq 1$ for all $m, n \in \mathbb{N} \times \mathbb{N}$. Then we get

$$\sup_{mn} \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(\alpha x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2} \leq \sup_{mn} \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}.$$

This completes the proof.

3.9. Proposition. The sequence space

$$\left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$$

is monotone

Proof: The proof follows from Proposition 3.8.

3.10. Proposition. If $f = (f_{mn})$ be any Musielak function. Then

$$\left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]_{\theta_{rs}}^{I^2} \subset \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^{**}} \right]_{\theta_{rs}}^{I^2}$$

if and only if $\sup_{r,s \geq 1} \frac{\varphi_{rs}^*}{\varphi_{rs}^{**}} < \infty$.

Proof: Let

$$x \in \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]_{\theta_{rs}}^{I^2}$$

and

$$N = \sup_{r,s \geq 1} \frac{\varphi_{rs}^*}{\varphi_{rs}^{**}} < \infty.$$

Then we get

$$\begin{aligned} & \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi_{rs}^{**}} \right]_{\theta_{rs}^s}^{I^2} = \\ N & \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi_{rs}^*} \right]_{\theta_{rs}}^{I^2} = 0. \end{aligned}$$

Thus $x \in \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi_{rs}^{**}} \right]_{\theta_{rs}}^{I^2}$.

Conversely, suppose that

$$\begin{aligned} & \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]_{N_\theta}^I \subset \\ & \left[\Lambda_{f\mu}^{2qu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi_{rs}^{**}} \right]_{\theta_{rs}}^{I^2} \end{aligned}$$

and

$$x \in \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]_{\theta_{rs}}^{I^2}.$$

Then

$$\left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]_{\theta_{rs}}^{I^2} < \epsilon,$$

for every $\epsilon > 0$. Suppose that $\sup_{r,s \geq 1} \frac{\varphi_{rs}^*}{\varphi_{rs}^{**}} = \infty$, then there exists a sequence of members (rs_{jk}) such that $\lim_{j,k \rightarrow \infty} \frac{\varphi_{jk}^*}{\varphi_{jk}^{**}} = \infty$. Hence, we have

$$\left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi_{rs}^*} \right]_{\theta_{rs}}^{I^2} = \infty.$$

Therefore

$$x \notin \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi_{rs}^{**}} \right]_{\theta_{rs}}^{I^2},$$

which is a contradiction. This completes the proof.

3.11. Proposition. If $f = (f_{mn})$ be any Musielak function. Then

$$\begin{aligned} & \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi^*} \right]_{\theta_{rs}^s}^{I^2} = \\ & \left[\Lambda_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi_{rs}^{**}} \right]_{\theta_{rs}}^{I^2} \end{aligned}$$

if and only if $\sup_{r,s \geq 1} \frac{\varphi_{rs}^*}{\varphi_{rs}^{**}} < \infty$, $\sup_{r,s \geq 1} \frac{\varphi_{rs}^{**}}{\varphi_{rs}^*} > \infty$.

Proof: It is easy to prove so we omit.

3.12. Proposition. The sequence space

$$\left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^{\varphi} \right]_{\theta_{rs}}^{I^2}$$

is not solid.

Proof: The result follows from the following example.

Example: Consider

$$x = \begin{pmatrix} x_{mn} \\ 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \cdot & & & \\ \cdot & & & \\ 1 & 1 & \dots & 1 \end{pmatrix} \in \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}.$$

Let

$$\alpha_{mn} = \begin{pmatrix} -1^{m+n} & -1^{m+n} & \dots & -1^{m+n} \\ -1^{m+n} & -1^{m+n} & \dots & -1^{m+n} \\ \cdot & & & \\ \cdot & & & \\ -1^{m+n} & -1^{m+n} & \dots & -1^{m+n} \end{pmatrix},$$

for all $m, n \in \mathbb{N}$. Then

$$\alpha_{mn} x_{mn} \notin \left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}.$$

Hence

$$\left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$$

is not solid.

3.13. Proposition. The sequence space

$$\left[\chi_{f\mu}^{2q}, \|\mu_{mn}(x), (d(x_1), d(x_2), \dots, d(x_{n-1}))\|_p^\varphi \right]_{\theta_{rs}}^{I^2}$$

is not monotone

Proof: The proof follows from Proposition 3.12.

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REFERENCES

- [1] B. Altay and F. Başar, Some new spaces of double sequences, *J. Math. Anal. Appl.*, **309**(1) (2005), 70-90.
- [2] F. Başar and Y. Sever, The space \mathcal{L}_p of double sequences, *Math. J. Okayama Univ.*, **51** (2009), 149-157.
- [3] M. Başarır and O. Solançan, On some double sequence spaces, *J. Indian Acad. Math.*, **21**(2) (1999), 193-200.
- [4] T. J. I'A. Bromwich, An introduction to the theory of infinite series *Macmillan and Co.Ltd.*, New York, (1965).
- [5] J. Connor, On strong matrix summability with respect to a modulus and statistical convergence, *Canad. Math. Bull.*, **32**(2) (1989), 194-198.
- [6] P. Chandra and B. C. Tripathy, On generalized Kothe-Toeplitz duals of some sequence spaces, *Indian Journal of Pure and Applied Mathematics*, **33**(8) (2002), 1301-1306.
- [7] A. Gökhan and R. Çolak, The double sequence spaces $c_2^P(p)$ and $c_2^{PB}(p)$, *Appl. Math. Comput.*, **157**(2) (2004), 491-501.
- [8] A. Gökhan and R. Çolak, Double sequence spaces ℓ_2^∞ , *ibid.*, **160**(1) (2005), 147-153.
- [9] M. Gupta and S. Pradhan, On Certain Type of Modular Sequence space, *Turk J. Math.*, **32** (2008), 293-303.

- [10] G. Goes and S. Goes. Sequences of bounded variation and sequences of Fourier coefficients, *Math. Z.*, **118** (1970), 93-102.
- [11] G. H. Hardy, On the convergence of certain multiple series, *Proc. Camb. Phil. Soc.*, **19** (1917), 86-95.
- [12] H. J. Hamilton, Transformations of multiple sequences, *Duke Math. J.*, **2** (1936), 29-60.
- [13] H. J. Hamilton, A Generalization of multiple sequences transformation, *Duke Math. J.*, **4** (1938), 343-358.
- [14] H. J. Hamilton, Preservation of partial Limits in Multiple sequence transformations, *Duke Math. J.*, **4** (1939), 293-297.
- [15] M. A. Krasnoselskii and Y. B. Rutickii, Convex functions and Orlicz spaces, *Gorningen, Netherlands*, (1961).
- [16] P. K. Kamthan and M. Gupta, Sequence spaces and series, Lecture notes, Pure and Applied Mathematics, *65 Marcel Dekker, In c., New York*, (1981).
- [17] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, **10** (1971), 379-390.
- [18] I. J. Maddox, Sequence spaces defined by a modulus, *Math. Proc. Cambridge Philos. Soc.*, **100(1)** (1986), 161-166.
- [19] F. Moricz, Extentions of the spaces c and c_0 from single to double sequences, *Acta. Math. Hung.*, **57(1-2)** (1991), 129-136.
- [20] F. Moricz and B. E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Camb. Phil. Soc.*, **104** (1988), 283-294.
- [21] M. Mursaleen and O. H. H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.*, **288(1)** (2003), 223-231.
- [22] M. Mursaleen, Almost strongly regular matrices and a core theorem for double sequences, *J. Math. Anal. Appl.*, **293(2)** (2004), 523-531.
- [23] M. Mursaleen and O. H. H. Edely, Almost convergence and a core theorem for double sequences, *J. Math. Anal. Appl.*, **293(2)** (2004), 532-540.
- [24] H. Nakano, Concave modulars, *J. Math. Soc. Japan*, **5** (1953), 29-49.
- [25] A. Pringsheim, Zurtheorie der zweifach unendlichen zahlenfolgen, *Math. Ann.*, **53** (1900), 289-321.
- [26] W. H. Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, *Canad. J. Math.*, **25** (1973), 973-978.
- [27] G. M. Robison, Divergent double sequences and series, *Amer. Math. Soc. Trans.*, **28** (1926), 50-73.
- [28] N. Subramanian and U. K. Misra, The semi normed space defined by a double gai sequence of modulus function, *Fasciculi Math.*, **45** (2010), 111-120.
- [29] B. C. Tripathy, On statistically convergent double sequences, *Tamkang J. Math.*, **34(3)** (2003), 231-237.
- [30] A. Turkmenoglu, Matrix transformation between some classes of double sequences, *J. Inst. Math. Comp. Sci. Math. Ser.*, **12(1)** (1999), 23-31.
- [31] B. C. Tripathy and S. Mahanta, On a class of vector valued sequences associated with multiplier sequences, *Acta Math. Applicata Sinica (Eng. Ser.)*, **20(3)** (2004), 487-494.
- [32] B. C. Tripathy and M. Sen, Characterization of some matrix classes involving paranormed sequence spaces, *Tamkang Journal of Mathematics*, **37(2)** (2006), 155-162.
- [33] B. C. Tripathy and A. J. Dutta, On fuzzy real-valued double sequence spaces ${}_2\mathcal{L}_F^p$, *Mathematical and Computer Modelling*, **46 (9-10)** (2007), 1294-1299.
- [34] B. C. Tripathy and B. Sarma, Statistically convergent difference double sequence spaces, *Acta Mathematica Sinica*, **24(5)** (2008), 737-742.
- [35] B. C. Tripathy and B. Sarma, Vector valued double sequence spaces defined by Orlicz function, *Mathematica Slovaca*, **59(6)** (2009), 767-776.
- [36] B. C. Tripathy and A. J. Dutta, Bounded variation double sequence space of fuzzy real numbers, *Computers and Mathematics with Applications*, **59(2)** (2010), 1031-1037.
- [37] B. C. Tripathy and B. Sarma, Double sequence spaces of fuzzy numbers defined by Orlicz function, *Acta Mathematica Scientia*, **31 B(1)** (2011), 134-140.
- [38] B. C. Tripathy and P. Chandra, On some generalized difference paranormed sequence spaces associated with multiplier sequences defined by modulus function, *Anal. Theory Appl.*, **27(1)** (2011), 21-27.

- [39] B. C. Tripathy and A. J. Dutta, Lacunary bounded variation sequence of fuzzy real numbers, *Journal in Intelligent and Fuzzy Systems*, **24(1)** (2013), 185-189.
- [40] J. Y. T. Woo, On Modular Sequence spaces, *Studia Math.*, **48** (1973), 271-289.
- [41] A. Wilansky, Summability through Functional Analysis, *North-Holland Mathematical Studies*, North-Holland Publishing, Amsterdam, **Vol.85** (1984).
- [42] M. Zeltser, Investigation of Double Sequence Spaces by Soft and Hard Analytical Methods, *Dissertationes Mathematicae Universitatis Tartuensis* 25, *Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu*, **2001**.

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