MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES
4 (1) 63-68 (2016) ©MSAEN

Some Results of Best Proximity Point in Regular Cone Metric Spaces

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(Communicated by Nihal YILMAZ ÖZGÜR)

Abstract
The purpose of this paper is to provide sufficient conditions for the existence of a best proximity point for various types of cyclic contraction maps. Our results extend and improve certain recent results in the literature.

Keywords: *Best proximity point; cyclic contraction map; cone L- function; regular cone metric.*

AMS Subject Classification (2010): *Primary: 41A65; Secondary: 41A52; 46N10.*

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1. Introduction and preliminaries

Let A and B nonempty subsets of a metric space X. If there is a pair $(x_0, y_0) \in A \times B$ for which $d(x_0, y_0) = d(A, B)$, that

$$d(A, B) = inf\{d(x, y) : x \in A, y \in B\},$$

then the pair (x_0, y_0) is called a best proximity pair for A and B. We can find the best proximity pair of the sets A and B, by considering a map $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. The point $x \in A \cup B$ is a best proximity point for T if, $d(x, Tx) = d(A, B)$. A map $T : A \cup B \to A \cup B,\ T(A) \subseteq B,\ T(B) \subseteq A$ is called cyclic contraction [3] if, for some $k \in (0, 1)$ the condition

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B),$$

holds for all $x \in A,\ y \in B$.
In 2003, Kirk et al. proved fixed point results for cyclic contraction maps [8]. In 2006, Eldered and Veeramani obtained best proximity point results for cyclic contraction maps [3].

Theorem 1.1. *[3] Let A and B be nonempty closed subsets of a complete metric space X. Let $T : A \cup B \to A \cup B$ be a cyclic contraction map. Let $x_0 \in A$ and define $x_{n+1} = Tx_n$. Suppose $\{x_{2n}\}$ has a convergent subsequence in A. Then there exists $x \in A$ such that $d(x, Tx) = d(A, B)$.*

Best proximity point theory of cyclic contraction maps has been studied by many authors see [1, 3, 9] and references therein. In 2007, Huang and Zhang [6] introduced cone metric spaces as a generalization of metric spaces. Then in [10] some results about characterization of best approximations in the cone metric spaces are studied. In 2011, Haghi et al [4] obtained best proximity points for cyclic contraction maps on regular cone metric spaces. In 2012, Karapınar [7], obtained best proximity point for certain cyclic contraction maps in metric spaces. In 2013, Amini and et al [2], introduce a new class of cyclic generalized contraction maps and it is shown that the best proximity point property for closed and convex subsets of a uniformly convex Banach space holds.

Received : *10–January–2015,* ***Accepted :*** *15–January–2016*

In this paper, we obtain some existence of best proximity point theorems for various types of cyclic contraction maps, which are the generalization of some results in the literature. To prove our results in the next section we recall some definitions and facts. In the present paper E stands for a real Banach space. A subset P of E is called a cone if and only if

(i) P is closed, nonempty and $P \neq \{0\}$;
(ii) $a, b \in \mathbb{R}^+$ and $x, y \in P$ implies $ax + by \in P$;
(iii) $x \in P$ and $-x \in P$ implies $x = 0$.

We define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

A map $f : P \to P$ is said to be increasing (strictly increasing) whenever $x \leq y$ implies that $f(x) \leq f(y)$ ($x < y$ implies that $f(x) < f(y)$).

A cone P is said to be normal if there is a number $M > 0$ such that for all $x, y \in E$

$$0 \leq x \leq y \quad implies \quad \|x\| \leq M\|y\|.$$

The least positive number M satisfying the above inequality is called the normal constant of cone P. [6]

Definition 1.1. [12] A nonempty subset A of (X, d), is said to be bounded above if there exists $c \in intP$ such that $c - d(x, y) \in P$ for all $x, y \in A$.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \ldots \leq y$ for some $y \in E$, then there is $x \in E$ such that $lim_{n\to\infty} \| x_n - x \|= 0$. Equivalently, the cone P is regular if and only if, every decreasing sequence which is bounded from below is convergent.

Lemma 1.1. *[11] Every regular cone is normal.*

Definition 1.2. [6] Let X be a nonempty set. Suppose that a mapping $d : X \times X \to E$ satisfies:
(d1) $0 \leq d(x, y)$ for every $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for every $x, y \in X$;
(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for every $x, y, z \in X$.
Then d is called a cone metric and (X, d) is called a cone metric space.

Example 1.1. [5] Let $E = (L^1[0, 1], \| \cdot \|_1)$, $P = \{f \in E : f \geq 0 \quad a.e.\}$, (X, ρ) be a metric space and $d : X \times X \to E$ defined by $d(x, y) = f_{x,y}$, where $f_{x,y}(t) = \rho(x, y)t^2$. Then (X, d) is a regular cone metric space. In fact, if $\{f_n\}_{n\geq 1}$ is an increasing sequence and there is $g \in L^1$ such that $f_1 \leq f_2 \leq \ldots \leq f_n \leq \ldots \leq g$ for almost every where x, then $\{f_n\}_{n\geq 1}$ converges to a function f a.e. on X. Then, $f_n \leq f \leq g$ (a.e.) for all $n \geq 1$. Thus $g - f_1 \in L^1,\ g - f_n \leq g - f_1$ for all $n \geq 1$ and $lim_{n\to\infty} g - f_n = g - f$ (a.e.)
Hence by the Lebesgue dominated convergence theorem, $f \in L^1$ and $lim_{n\to\infty} \| f_n - f \|_1= 0$. So, the cone P is regular.

Definition 1.3. [4] Let A and B nonempty subsets of cone metric space (X, d). An element $p \in P$ is said to be a lower bound for $A \times B$ whenever

$$p \leq d(a, b),$$

for all $(a, b) \in A \times B$. If $p \geq q$ for all lower bound q for $A \times B$, then p is called the greatest lower bound for $A \times B$. We denote it by $d(A, B)$.

Clearly, $d(A, B)$ is a unique vector in P.

Definition 1.4. [4] A map $\psi : P \to P$ is called cone L- function whenever $\psi(0) = 0,\ \psi(s) > 0$ for all $s \in P$ with $s \neq 0$ and there exists $\delta_s \gg 0$ such that $\psi(t) \leq s$ for all $s \leq t \leq s + \delta_s$.

Lemma 1.2. *[4] Let $\psi : P \to P$ be a cone L-function and $\{s_n\}$ a decreacing sequence in P such that $s_{n+1} < \psi(s_n)$ for all $n \geq 1$. Then $lim_{n\to\infty} s_n = 0$.*

2. Main results

Throughout this section, E is a normed space, (X, d) is regular cone metric space, $\leq$ is the partial ordering with respect to P and A, B are nonempty subsets of X.

Theorem 2.1. *Let $T : A \cup B \to A \cup B$ be a map such that $T(A) \subseteq B$, $T(B) \subseteq A$ and*

$$d(Tx, Ty) \leq k \max\{d(x, y), (1/2)\{d(Tx, x) + d(Ty, y)\}\} + (1-k)d(a, b), \tag{2.1}$$

for all $(a, b), (x, y) \in A \times B$, for some $k \in (0, 1)$. Then, $d(A, B)$ exists.

Proof. The $\max\{d(x, y), (1/2)\{d(Tx, x) + d(Ty, y)\}\} = d(x, y)$ is known (see [4]). Let

$$\max\{d(x, y), (1/2)\{d(Tx, x) + d(Ty, y)\}\} = (1/2)\{d(Tx, x) + d(Ty, y)\}.$$

Take $x_0 \in A \cup B$, set $x_{n+1} = Tx_n$ and $d_{n+1} = d(x_{n+1}, x_n)$ for all $n \geq 1$. Then

$$d_{n+1} \leq (k/2)\{d_{n+1} + d_n\} + (1-k)d(a, b),$$

for all $(a, b) \in A \times B$, which is equivalent to

$$d_{n+1} \leq \frac{k/2}{1-k/2} d_n + \frac{1-k}{1-k/2} d(a, b),$$

for each (a, b) in $A \times B$. It follows that $d_{n+1} \leq d_n$ for all $n \geq 1$. By the regularity of the cone P, there exists $p \in P$ such that $d_n \to p$ as $n \to \infty$. Thus $p \leq d(a, b)$ holds for any (a, b) in $A \times B$. Now if q is a lower bound for $A \times B$, then $q \leq d_n$ for all $n \geq 1$, and so, $q \leq p$. Therefore, $d(A, B) = p$. □

Note that, inequality (2.1) is equivalent to

$$d(Tx, Ty) \leq k \max\{d(x, y), (1/2)\{d(Tx, x) + d(Ty, y)\}\} + (1-k)d(A, B),$$

in metric spaces.

Theorem 2.2. *Suppose that the conditions of Theorem 2.1 hold, $x_0 \in A$ and $x_{n+1} = Tx_n$ for all $n \geq 1$. If $\{x_{2n}\}$ has a convergent subsequence in A, then there exists $x \in A$ such that $d(x, Tx) = d(A, B)$.*

Proof. Let $\{x_{2n_k}\}$ be the convergent subsequence of $\{x_{2n}\}$ in A with $x_{2n_k} \to x \in A$. Since

$$p = d(A, B) \leq d(x, x_{2n_k-1}) \leq d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}),$$

for each $k \geq 1$, $\{d(x_{2n_k}, x_{2n_k-1})\}$ is a subsequence of $\{d_n\}$, hence $d(x, x_{2n_k-1}) \to p$ as $n \to \infty$. As

$$p \leq d(Tx, x_{2n_k}) \leq d(x, x_{2n_k-1}),$$

for all $k \geq 1$. It follows that $d(x, Tx) = p = d(A, B)$. □

Theorem 2.3. *Let $T : A \cup B \to A \cup B$ be a map such that $T(A) \subseteq B$, $T(B) \subseteq A$ and*

$$d(Tx, Ty) \leq ad(x, y) + b\{d(Tx, x) + d(Ty, y)\} + cd(a, b), \tag{2.2}$$

for all $(a, b), (x, y) \in A \times B$, where a, b, c are constant such that $a, b, c \geq 0$ and $a + 2b + c < 1$. Then $d(A, B)$ exists.

Proof. Take $x_0 \in A \cup B$. Set $x_{n+1} = Tx_n$ and $d_{n+1} = d(x_{n+1}, x_n)$ for all $n \geq 1$. Then

$$d_{n+1} \leq ad_n + b\{d_{n+1} + d_n\} + cd(a, b),$$

for all $(a, b) \in A \times B$. So

$$d_{n+1} \leq \frac{a+b}{1-b} d_n + \frac{c}{1-b} d(a, b),$$

for each $(a, b) \in A \times B$. So

$$d_{n+1} \leq kd_n + (1-k)d(a, b),$$

for all (a, b) in $A \times B$, where $k = (a + b)/(1 - b)$. Hence, $d_{n+1} \leq d_n$ for all $n \geq 1$. Similar to the prove of Theorem 2.1 we obtain the result. □

Note that, inequality (2.2) is equivalent to

$$d(Tx,Ty) \le ad(x,y) + b\{d(Tx,x) + d(Ty,y)\} + cd(A,B),$$

in metric spaces.

Theorem 2.4. *Suppose that the conditions of Theorem 2.3 hold, $x_0 \in A$ and $x_{n+1} = Tx_n$ for all $n \ge 1$. If $\{x_{2n}\}$ has a convergent subsequence in A. Then there exists $x \in A$ such that $d(x,Tx) = d(A,B)$.*

Proof. The proof is similar to the proof of Theorem 2.2. □

Now, we will consider the best proximity points for a pair of mapping (S,T), such that $S,T : A \cup B \to A \cup B,\ S(A) \subseteq B$ and $T(B) \subseteq A$.

Theorem 2.5. *Let $S,T : A \cup B \to A \cup B$ such that $S(A) \subseteq B,\ T(B) \subseteq A$ and*

$$d(Sx,Ty) \le kd(x,y) + (1-k)d(a,b), \tag{2.3}$$

for all $(a,b),(x,y) \in A \times B$, for some $k \in (0,1)$. Then, $d(A,B)$ exists.

Proof. Take $x_0 \in A$, then $Sx_0 \in B$, so there exists $y_0 \in B$ such that $y_0 = Sx_0$. Now $Ty_0 \in A$, so there exists $x_1 \in A$ such that $x_1 = Ty_0$. Inductively, we define sequence $\{x_n\}$ and $\{y_n\}$ in A and B, respectively by

$$x_{n+1} = Ty_n, \quad y_n = Sx_n. \tag{2.4}$$

Set $d_n = d(x_n, Sx_n)$. Since

$$\begin{aligned} d_{n+1} &\le kd(y_n, x_{n+1}) + (1-k)d(a,b) \\ &\le k^2 d_n + (1-k^2)d(a,b), \end{aligned}$$

for all (a,b) in $A \times B$. It follows that $d_{n+1} \le d_n$. Similar to the prove of Theorem 2.1 we obtain the result. □

Note that, inequality (2.3) is equivalent to

$$d(Sx,Ty) \le kd(x,y) + (1-k)d(A,B),$$

in metric spaces. Also, in case $S = T$, Theorem 2.5 reduce to the Theorem 2.1 in [4].

Theorem 2.6. *Suppose that the conditions of Theorem 2.5 hold and the sequence $\{x_n\}$ and $\{y_n\}$ are generated by (2.4) for some $x_0 \in A \cup B$. If both $\{x_n\}$ and $\{y_n\}$ have a convergent subsequence in A and B respectively, then there exist $x \in A$ and $y \in B$ such that*

$$d(x,Sx) = d(A,B) = d(y,Ty).$$

Proof. Set $d_n = d(x_n, Sx_n)$. Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ such that $y_{n_k} \to y$. The relation

$$p = d(A,B) \le d(Ty_{n_k}, y) \le d(y_{n_k}, y) + d(y_{n_k}, Ty_{n_k}),$$

holds for each $k \ge 1$. Since

$$d(y_{n_k}, Ty_{n_k}) \le kd_{n_k} + (1-k)d(a,b),$$

for all $(a,b) \in A \times B$. It follows that $d(y_{n_k}, Ty_{n_k}) \le d_{n_k}$. Since $\{d(Sx_{n_k}, x_{n_k})\}$ is a subsequence of $\{d_n\}$, hence $\lim_{k\to\infty} d(Sx_{n_k}, x_{n_k}) = p$. Thus

$$\lim_{k\to\infty} d(y_{n_k}, Ty_{n_k}) = p.$$

So $d(Ty_{n_k}, y) \to p$ as $k \to \infty$. Now, for each $k \ge 1$

$$\begin{aligned} d(Ty, y_{n_k}) &\le kd(y, x_{n_k}) + (1-k)d(a,b) \\ &\le k\{d(y, y_{n_k}) + d(y_{n_k}, x_{n_k})\} + (1-k)d(a,b). \end{aligned}$$

i.e.

$$p = d(A, B) \leq d(Ty, y_{n_k}) \leq k\{d(y, y_{n_k}) + d_{n_k}\} + (1 - k)d(a, b),$$

for all $(a, b) \in A \times B$. Letting $k \to \infty$, we have $d(Ty, y) = p = d(A, B)$.
Similarly, it can be proved that $d(x, Sx) = d(A, B)$. □

In the following Theorem, the distance of A and B is obtained by considering the pair mapping (S, T) in a regular cone metric space.

Theorem 2.7. *Let $\psi : P \to P$ be a cone L- function. $S, T : A \cup B \to A \cup B$ such that $T(B) \subseteq A,\ S(A) \subseteq B$ and*

$$d(Sx, Ty) - p < \psi(d(x, y) - p),$$

for all $(x, y) \in A \times B$ with $p < d(x, y)$, where p is lower bounded for $A \times B$. Then $d(A, B) = p$.

Proof. Let $\{x_n\}$ and $\{y_n\}$ be as follows $x_{n+1} = Ty_n,\ Sx_n = y_{n+1}$ for some $(x_0, y_0) \in A \times B,\ n \in \mathbb{N}$. Also let $d_{n+1} = d(x_{n+1}, y_{n+1})$, we have

$$d_{n+1} - p < \psi(d_n - p) \leq d_n - p.$$

By the regularity of the cone P, we have $d_{n+1} \leq d_n$. Hence, there exists $q \in P$ such that $lim_{n\to\infty} d_n = q$. Then $p \leq q$. Put $s_n = d_n - p$. Since, $s_n > 0$, we have $s_{n+1} < \psi(s_n) \leq s_n$. By Lemma 1.8, $lim_{n\to\infty} s_n = 0$. Thus, $lim_{n\to\infty} d_n = p$ and so $d(A, B) = p = q$. □

Note that, in case $S = T$. Theorem 2.7 reduce to the Theorem 2.4 in [4].

Acknowledgment

This research was partially supported by Center of Excellence for Mathematics (University of Isfahan).

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