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A Note on the Relative Ritt Order of Entire Functions Represented by Vector Valued Dirichlet Series

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Abstract
In this paper we investigate some growth properties of entire functions represented by a vector valued Dirichlet series on the basis of relative Ritt order and relative Ritt lower order.

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1. Introduction, Definitions and Notations.

Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ (σ and t are real variables) defined by everywhere absolutely convergent *vector valued Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \tag{1.1}$$

where a_n's belong to a Banach space $(E, \|.\|)$ and λ_n's are non-negative real numbers such that $0 < \lambda_n < \lambda_{n+1}\,(n \geq 1)\,, \lambda_n \to \infty$ as $n \to \infty$ and satisfy the conditions

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty$$

and

$$\limsup_{n\to\infty} \frac{\log \|a_n\|}{\lambda_n} = -\infty\ .$$

If σ_a and σ_c denote respectively the abscissa of convergence and absolute convergence of (1.1), then in this case clearly $\sigma_a = \sigma_c = \infty$.

The function $M_f(\sigma)$ known as *maximum modulus* function corresponding to an entire function $f(s)$ defined by (1.1) is written as follows:

$$M_f(\sigma) = \underset{-\infty<t<\infty}{l.u.b.} |f(\sigma + it)|\ .$$

In the sequel the following two notations are used:

$$\begin{aligned} \log^{[k]} x &= \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots; \\ \log^{[0]} x &= x \end{aligned}$$

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and

$$\begin{aligned}\exp^{[k]} x &= \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots ;\\ \exp^{[0]} x &= x.\end{aligned}$$

Taking this into account, the *Ritt order* (See [1]), of $f(s)$, denoted by ρ_f, which is generally used in computational purpose, is defined in terms of the growth of $f(s)$ with respect to the $\exp\exp z$ function as follows:

$$\rho_f = \limsup_{\sigma\to\infty} \frac{\log\log M_f(\sigma)}{\log\log M_{\exp\exp z}(\sigma)} = \limsup_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma}.$$

Similarly, one can define the *Ritt lower order* of $f(s)$, denoted by λ_f in the following manner:

$$\lambda_f = \liminf_{\sigma\to\infty} \frac{\log\log M_f(\sigma)}{\log\log M_{\exp\exp z}(\sigma)} = \liminf_{\sigma\to\infty} \frac{\log^{[2]} M_f(\sigma)}{\sigma}.$$

Further an entire function $f(s)$ defined by (1.1) is said to be of *regular Ritt growth* if its *Ritt order* coincides with its *Ritt lower order.* Otherwise $f(s)$ is said to be of *irregular Ritt-growth*.

During the past decades, several authors {e.g., cf., [1], [2], [3], [5], [7]} have made intensive investigations on the properties of entire Dirichlet series related to *Ritt order*. Further, Srivastava [6] defined different growth parameters such as order and lower order of entire functions represented by *vector valued Dirichlet series*. He also obtained the results for coefficient characterization of order.

Srivastava [4] introduced the *relative Ritt order* between two entire functions represented by *vector valued Dirichlet series* to avoid comparing growth just with $\exp\exp z$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(\sigma) < M_g(\sigma\mu) \text{ for all } \sigma > \sigma_0(\mu)\}\\ &= \limsup_{\sigma\to\infty} \frac{M_g^{-1} M_f(\sigma)}{\sigma}.\end{aligned}$$

Similarly, one can define the *relative Ritt lower order* of $f(s)$ with respect to $g(s)$, denoted by $\lambda_g(f)$ in the following manner:

$$\lambda_g(f) = \liminf_{\sigma\to\infty} \frac{M_g^{-1} M_f(\sigma)}{\sigma}.$$

For entire functions, the notions of thier growth indicators such as *Ritt order* are classical in complex analysis and during the past decades, several researchers have already been exploring their studies in different directions using the classical growth indicators. But at that time, the concepts of *relative Ritt order* of entire functions and as well as their technical advantages of not comparing with the growths of $\exp\exp z$ are not at all known to the researchers of this area. Therefore the studies of the growths of composite entire functions in the light of their *relative Ritt order* are the prime concern of this paper. Actually in this paper we establish some newly developed results related to the growth rates of composite entire functions on the basis of their *relative Ritt order* (respectively *relative Ritt lower order*).

2. Theorems.

In this section we present the main results.

Theorem 2.1. *Let f and g be any two entire functions represented by vector valued Dirichlet series such that $0 < \lambda_f \le \rho_f < \infty$ and $0 < \lambda_g \le \rho_g < \infty$. Then*

$$\frac{\lambda_f}{\rho_g} \le \lambda_g(f) \le \min\left\{\frac{\lambda_f}{\lambda_g}, \frac{\rho_f}{\rho_g}\right\} \le \max\left\{\frac{\lambda_f}{\lambda_g}, \frac{\rho_f}{\rho_g}\right\} \le \rho_g(f) \le \frac{\rho_f}{\lambda_g}.$$

Proof. From the definitions of ρ_f and λ_f, we have for all sufficiently large values of σ that

$$M_f(\sigma) \le \exp^{[2]}\{(\rho_f + \varepsilon)\sigma\}, \tag{2.1}$$

$$M_f(\sigma) \ge \exp^{[2]}\{(\lambda_f - \varepsilon)\sigma\} \tag{2.2}$$

and also for a sequence of values of σ tending to infinity we get

$$M_f(\sigma) \geq \exp^{[2]}\{(\rho_f - \varepsilon)\sigma\}, \tag{2.3}$$

$$M_f(\sigma) \leq \exp^{[2]}\{(\lambda_f + \varepsilon)\sigma\}. \tag{2.4}$$

Similarly from the definitions of ρ_g and λ_f, it follows for all sufficiently large values of σ that

$$\begin{aligned} M_g(\sigma) &\leq \exp^{[2]}\{(\rho_g + \varepsilon)\sigma\} \\ i.e.,\ \sigma &\leq M_g^{-1}\left[\exp^{[2]}\{(\rho_g + \varepsilon)\sigma\}\right] \\ i.e.,\ M_g^{-1}(\sigma) &\geq \left[\frac{\log^{[2]}\sigma}{(\rho_g + \varepsilon)}\right], \end{aligned} \tag{2.5}$$

$$\begin{aligned} M_g(\sigma) &\geq \exp^{[2]}\{(\lambda_g - \varepsilon)\sigma\} \\ i.e.,\ M_g^{-1}(\sigma) &\leq \left[\frac{\log^{[2]}\sigma}{(\lambda_g - \varepsilon)}\right] \end{aligned} \tag{2.6}$$

and for a sequence of values of σ tending to infinity we obtain

$$\begin{aligned} M_g(\sigma) &\geq \exp^{[2]}\{(\rho_g - \varepsilon)\sigma\} \\ i.e.\ M_g^{-1}(\sigma) &\leq \left[\frac{\log^{[2]}\sigma}{(\rho_g - \varepsilon)}\right], \end{aligned} \tag{2.7}$$

$$\begin{aligned} M_g(\sigma) &\leq \exp^{[2]}\{(\lambda_g + \varepsilon)\sigma\} \\ i.e.,\ M_g^{-1}(\sigma) &\geq \left[\frac{\log^{[2]}\sigma}{(\lambda_g + \varepsilon)}\right]. \end{aligned} \tag{2.8}$$

Now from (2.3) and in view of (2.5) , we get for a sequence of values of σ tending to infinity we get

$$M_g^{-1}M_f(\sigma) \geq M_g^{-1}\left[\exp^{[2]}\{(\rho_f - \varepsilon)\sigma\}\right]$$

$$\begin{aligned} i.e.,\ M_g^{-1}M_f(\sigma) &\geq \left[\frac{\log^{[2]}\exp^{[2]}\{(\rho_f - \varepsilon)\sigma\}}{(\rho_g + \varepsilon)}\right] \\ &= \frac{(\rho_f - \varepsilon)}{(\rho_g + \varepsilon)}\sigma \end{aligned}$$

$$i.e.,\ \frac{M_g^{-1}M_f(r)}{\sigma} \geq \frac{(\rho_f - \varepsilon)}{(\rho_g + \varepsilon)}.$$

As $\varepsilon > 0$ is arbitrary, it follows that

$$\rho_g(f) \geq \frac{\rho_f}{\rho_g}. \tag{2.9}$$

Analogously from (2.2) and in view of (2.8) , it follows for a sequence of values of σ tending to infinity that

$$M_g^{-1}M_f(\sigma) \geq M_g^{-1}\left[\exp^{[2]}\{(\lambda_f - \varepsilon)\sigma\}\right]$$

$$\begin{aligned} i.e.,\ M_g^{-1}M_f(\sigma) &\geq \left[\frac{\log^{[2]}\exp^{[2]}\{(\lambda_f - \varepsilon)\sigma\}}{(\lambda_g + \varepsilon)}\right] \\ &= \frac{(\lambda_f - \varepsilon)}{(\lambda_g + \varepsilon)}\sigma \end{aligned}$$

$$i.e., \ \frac{M_g^{-1} M_f(r)}{\sigma} \geq \frac{(\lambda_f - \varepsilon)}{(\lambda_g + \varepsilon)} \ .$$

Since $\varepsilon\,(> 0)$ is arbitrary, we get from above that

$$\rho_g(f) \geq \frac{\lambda_f}{\lambda_g} \ . \tag{2.10}$$

Again in view of (2.6) , we have from (2.1) for all sufficiently large values of σ that

$$M_g^{-1} M_f(\sigma) \leq M_g^{-1}\left[\exp^{[2]}\{(\rho_f + \varepsilon)\sigma\}\right]$$

$$\begin{aligned} i.e., \ M_g^{-1} M_f(\sigma) &\leq \left[\frac{\log^{[2]}\exp^{[2]}\{(\rho_f + \varepsilon)\sigma\}}{(\lambda_g - \varepsilon)}\right] \\ &= \frac{(\rho_f + \varepsilon)}{(\lambda_g - \varepsilon)}\sigma \end{aligned}$$

$$i.e., \ \frac{M_g^{-1} M_f(\sigma)}{\sigma} \leq \frac{(\rho_f + \varepsilon)}{(\lambda_g - \varepsilon)} \ .$$

Since $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\rho_g(f) \leq \frac{\rho_f}{\lambda_g} \ . \tag{2.11}$$

Again from (2.2) and in view of (2.5) with the same reasoning we get that

$$\lambda_g(f) \geq \frac{\lambda_f}{\rho_g} \ . \tag{2.12}$$

Also in view of (2.7) , we get from (2.1) for a sequence of values of σ tending to infinity that

$$M_g^{-1} M_f(\sigma) \leq M_g^{-1}\left[\exp^{[2]}\{(\rho_f + \varepsilon)\sigma\}\right]$$

$$\begin{aligned} i.e., \ M_g^{-1} M_f(\sigma) &\leq \left[\frac{\log^{[2]}\exp^{[2]}\{(\rho_f + \varepsilon)\sigma\}}{(\rho_g - \varepsilon)}\right] \\ &= \frac{(\rho_f + \varepsilon)}{(\rho_g - \varepsilon)}\sigma \end{aligned}$$

$$i.e., \ \frac{M_g^{-1} M_f(\sigma)}{\sigma} \leq \frac{(\rho_f + \varepsilon)}{(\rho_g - \varepsilon)} \ .$$

Since $\varepsilon\,(> 0)$ is arbitrary, we get from above that

$$\lambda_g(f) \leq \frac{\rho_f}{\rho_g} \ . \tag{2.13}$$

Similarly from (2.4) and in view of (2.6) , it follows for a sequence of values of σ tending to infinity that

$$M_g^{-1} M_f(\sigma) \leq M_g^{-1}\left[\exp^{[2]}\{(\lambda_f + \varepsilon)\sigma\}\right]$$

$$\begin{aligned} i.e., \ M_g^{-1} M_f(\sigma) &\leq \left[\frac{\log^{[2]}\exp^{[2]}\{(\lambda_f + \varepsilon)\sigma\}}{(\lambda_g - \varepsilon)}\right] \\ &= \frac{(\lambda_f + \varepsilon)}{(\lambda_g - \varepsilon)}\sigma \end{aligned}$$

$$i.e., \frac{M_g^{-1} M_f(\sigma)}{\sigma} \leq \frac{(\lambda_f + \varepsilon)}{(\lambda_g - \varepsilon)} .$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from above that

$$\lambda_g(f) \leq \frac{\lambda_f}{\lambda_g} . \tag{2.14}$$

The theorem follows from (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14). □

Corollary 2.1. *Let f be an entire function represented by vector valued Dirichlet series with Ritt order ρ_f and Ritt lower order λ_f. Also let g be an entire function of regular Ritt growth. Then*

$$\lambda_g(f) = \frac{\lambda_f}{\rho_g} \quad \text{and} \quad \rho_g(f) = \frac{\rho_f}{\rho_g} .$$

In addition, if $\rho_f = \rho_g$, then

$$\rho_g(f) = \lambda_f(g) = 1 .$$

Corollary 2.2. *Let f and g be any two entire functions represented by vector valued Dirichlet series with regular Ritt growth. Then*

$$\lambda_g(f) = \rho_g(f) = \frac{\rho_f}{\rho_g} .$$

Corollary 2.3. *Let f and g be any two entire functions represented by vector valued Dirichlet series with regular Ritt growth. Also suppose that $\rho_f = \rho_g$. Then*

$$\lambda_g(f) = \rho_g(f) = \lambda_f(g) = \rho_f(g) = 1 .$$

Corollary 2.4. *Let f and g be any two entire functions represented by vector valued Dirichlet series with regular Ritt growth and $0 < \lambda_g < \rho_g < \infty$ respectively. Then*

$$\rho_g(f) \cdot \rho_f(g) = \lambda_g(f) \cdot \lambda_f(g) = 1 .$$

Corollary 2.5. *Let f and g be any two entire functions represented by vector valued Dirichlet series and either f is not of regular Ritt growth or g is not of regular Ritt growth, then*

$$\lambda_g(f) \cdot \lambda_f(g) < 1 < \rho_g(f) \cdot \rho_f(g) .$$

Corollary 2.6. *Let f be an entire function represented by vector valued Dirichlet series with $0 < \lambda_f < \rho_f < \infty$. Then for any entire function g represented by vector valued Dirichlet series,*

$$\begin{aligned} (i)\ \lambda_g(f) &= \infty \text{ when } \rho_g = 0 , \\ (ii)\ \rho_g(f) &= \infty \text{ when } \lambda_g = 0 , \\ (iii)\ \lambda_g(f) &= 0 \text{ when } \rho_g = \infty \end{aligned}$$

and

$$(iv)\ \rho_g(f) = 0 \text{ when } \lambda_g = \infty .$$

Corollary 2.7. *Let g be an entire function represented by vector valued Dirichlet series with $0 < \lambda_g < \rho_g < \infty$. Then for any entire function f represented by vector valued Dirichlet series,*

$$\begin{aligned} (i)\ \rho_g(f) &= 0 \text{ when } \rho_f = 0 , \\ (ii)\ \lambda_g(f) &= 0 \text{ when } \lambda_f = 0 , \\ (iii)\ \rho_g(f) &= \infty \text{ when } \rho_f = \infty \end{aligned}$$

and

$$(iv)\ \lambda_g(f) = \infty \text{ when } \lambda_f = \infty .$$

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