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Dual-Quasi Elliptic Planar Motion

Murat BEKAR

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Abstract
Dual-quaternions are an elegant and useful mathematical tools for representing rigid-body (screw) motions in three-dimensional Euclidean space $\mathbb{R}^3$. The aim of this paper is to consider the algebra of dual semi-quaternions with their basic properties and generalize the results of the Euclidean-planar motion given by Blaschke and Grünwald to dual planar motion.

Keywords: *Dual-numbers, Dual Semi-Quaternions, Quasi-Elliptic Motion.*

AMS Subject Classification (2010): *11E88; 11R52; 35H30; 53A17; 53A20; 53A35.*

1. INTRODUCTION

Quaternions are discovered by Sir William R. Hamilton in the middle of 19^{th} century. He want to construct an algebraic system whose elements comprise of a real and two imaginary part. The predicament in the development of this algebraic system occured by defining the mulitiplication rule. He overcame with this problem by using the three imaginary part $\boldsymbol{i},\ \boldsymbol{j},\ \boldsymbol{k}$ with the multiplication rules

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1,$$

and called the real space spanned by the elements $1,\ \boldsymbol{i},\ \boldsymbol{j},\ \boldsymbol{k}$ as *quaternions*, see [1].

Quaternions have some applications in physics, kinematics, mechanism, computer simulations, etc. For example, rotations in three-dimensional Euclidean space $\mathbb{R}^3$ can be represented by real-quaternions, while rigid-body motions in $\mathbb{R}^3$ can be represent by dual-quaternions, see [2–4]. Also, quasi-elliptic motions obtained by the kinematic mapping of Blaschke and Grünwald can be represented by semi-quaternions, see [5, 6]. In this paper, firstly a brief summary of the concepts dual-projective plane, dual quasi-elliptic geometry and dual semi-quaternions are given. Afterwards, the results of the Euclidean-planar motion given by Blaschke and Grünwald are generalized to dual planar-motion.

2. PRELIMINARIES

In this section, an overview of each of the concepts dual-projective plane, dual-quasi elliptic geometry and dual semi-quaternions is given.

2.1 Dual-Projective Plane

The set of *dual-numbers* is defined to be

$$\mathbb{D} = \{\, A = a + \varepsilon a^* : (\, a, a^*) \in \mathbb{R}^2,\ \varepsilon \neq 0 \text{ and } \varepsilon^2 = 0 \,\}$$

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where ε is the *dual unit* and commutes with real-numbers that is $r\varepsilon = \varepsilon r$ for all $r \in \mathbb{R}$. Also, the real-numbers a and a^* are called the *non-dual* and the *dual parts* of A, respectively. If $a = 0$ (resp. $a \neq 0$), then A is said to be a *pure* (resp. *non-pure*). The set of all pure (resp. non-pure) dual-numbers is denoted by **D** (resp. $\breve{\mathbb{D}}$).

Let $A = a + \varepsilon a^*$ and $B = b + \varepsilon b^*$ be any two dual-numbers. Then,

- The *addition* of A and B is
$$A + B = (a + b) + \varepsilon(a^* + b^*).$$
- The *multiplication* (known as the *Study multiplication*) of A and B is
$$AB = BA = (ab) + \varepsilon(ab^* + ba^*).$$
- The *equality* of A and B is defined to be
$$A = B \quad \text{iff} \quad a = b \quad \text{and} \quad a^* = b^*.$$
- The *dual conjugate* of A is defined to be
$$A^\star = a - \varepsilon a^*.$$
- The *square root* of $A = a + \varepsilon a^*$ exists only for $a > 0$ and is defined to be
$$\sqrt{A} = \sqrt{a} + \varepsilon \frac{a^*}{2\sqrt{a}}.$$
- The *norm* of A is defined to be
$$N_A = AA^\star = A^\star A = a^2,$$
while its *modulus* is defined to be
$$\|A\| = \sqrt{N_A} = |a|$$
where $| \, , \, |$ denotes the absolute value of a real-number. If $N_A = \|A\| = 1$ (that is, $A = \pm 1 + \varepsilon a^*$), then A is said to be a *unit*.
- The *multiplicative inverse* of a dual-number is obtained by dividing its dual-conjugate by its norm. It is important to emphasize that the multiplicative inverse of a dual-number exists only if it is a non-pure one. For example, the multiplicative inverse of A exists only for $a \neq 0$ and is defined to be
$$A^{-1} = \frac{A^\star}{N_A} = \frac{1}{a} - \varepsilon \frac{a^*}{a^2}.$$

The *dual angle* $\Theta = \theta + \varepsilon\theta^* \in \mathbb{D}$ represents the relative displacement and orientation between any two lines $l_1, \ l_2$ in space $\mathbb{R}^3$, see Fg. 1, where

$\theta \in \mathbb{R}$ is the projected angle between the lines l_1 and l_2,

$\theta^* \in \mathbb{R}$ is the shortest distance between the lines l_1 and l_2.

The trigonometric functions *sine*, *cosine* and *tangent* of a dual angle $\Theta = \theta + \varepsilon\theta^*$ are defined to be

$$\sin\Theta = \sin\theta + \varepsilon\theta^*\cos\theta,$$

$$\cos\Theta = \cos\theta - \varepsilon\theta^*\sin\theta,$$

$$\tan\Theta = \tan\theta + \varepsilon\theta^*\sec^2\theta.$$

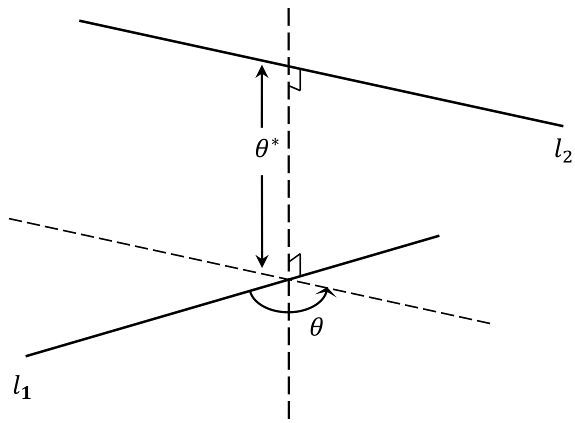

Figure 1. The dual angle $\Theta = \theta + \varepsilon\theta^* \in \mathbb{D}$ expresses the relationship between the l_1 and l_2 in $\mathbb{R}^3$.

For further information about dual-numbers see [7, 8].

The set

$$\mathbb{D}^2 = \{\, \widetilde{A} = (X, Y) : \; X, \; Y \in \mathbb{D} \,\}$$

is a two-dimensional module over the ring $\mathbb{D}$ and is called the *dual-plane*. The elements of $\mathbb{D}^2$ are also called the *dual-vectors* in $\mathbb{D}^2$. A dual-vector $\widetilde{A} = (X, Y)$ can be written in *dual form* as

$$\widetilde{A} = \vec{z} + \varepsilon\vec{z^*}$$

where $X = x + \varepsilon x^*$, $Y = y + \varepsilon y^* \in \mathbb{D}$ and $\vec{z} = (x, y)$, $\vec{z^*} = (x^*, y^*) \in \mathbb{R}^2$.

The set

$$\mathbb{D}^3 = \{\, \widehat{A} = (X, Y, Z) : \; X, \; Y, \; Z \in \mathbb{D} \,\}$$

is a three-dimensional module over the ring $\mathbb{D}$ and is called the *dual-space* (or $\mathbb{D} - module$). The elements of $\mathbb{D}^3$ are also called the *dual-vectors* in $\mathbb{D}^3$. A dual-vector $\widehat{A} = (X, Y, Z)$ can be written in *dual form* as

$$\widehat{A} = \vec{\boldsymbol{w}} + \varepsilon\vec{\boldsymbol{w}^*}$$

where

$$X = x + \varepsilon x^* \,, \;\; Y = y + \varepsilon y^* \,, \;\; Z = z + \varepsilon z^* \in \mathbb{D}$$

and

$$\vec{\boldsymbol{w}} = (x, y, z) \,, \;\; \vec{\boldsymbol{w}^*} = (x^*, y^*, z^*) \in \mathbb{R}^3.$$

Let

$$\widehat{A} = (X, Y, Z) = \vec{\boldsymbol{w}} + \varepsilon\vec{\boldsymbol{w}^*} \,, \;\; \widehat{B} = (P, Q, R) = \vec{\boldsymbol{v}} + \varepsilon\vec{\boldsymbol{v}^*}$$

be any two dual-vectors where

$$X = x + \varepsilon x^* \,, \;\; Y = y + \varepsilon y^* \,, \;\; Z = z + \varepsilon z^* \in \mathbb{D}$$

$$P = p + \varepsilon p^* \,, \;\; Q = q + \varepsilon q^* \,, \;\; R = r + \varepsilon r^* \in \mathbb{D}$$

and

$$\vec{\boldsymbol{w}} = (x, y, z) \,, \;\; \vec{\boldsymbol{w}^*} = (x^*, y^*, z^*) \in \mathbb{R}^3$$
$$\vec{\boldsymbol{v}} = (p, q, r) \,, \;\; \vec{\boldsymbol{v}^*} = (p^*, q^*, r^*) \in \mathbb{R}^3.$$

Then,

1. The *addition* of $\widehat{A} = (X, Y, Z) = \vec{\boldsymbol{w}} + \varepsilon \vec{\boldsymbol{w}}^{*}$ and $\widehat{B} = (P, Q, R) = \vec{\boldsymbol{v}} + \varepsilon \vec{\boldsymbol{v}}^{*}$ is

$$\begin{aligned} \widehat{A} + \widehat{B} &= (X + P,\ Y + Q,\ Z + R) \\ &= (\vec{\boldsymbol{w}} + \vec{\boldsymbol{v}}) + \varepsilon(\vec{\boldsymbol{w}}^{*} + \vec{\boldsymbol{v}}^{*}) \end{aligned}$$

2. The *inner product* of $\widehat{A} = (X, Y, Z) = \vec{\boldsymbol{w}} + \varepsilon \vec{\boldsymbol{w}}^{*}$ and $\widehat{B} = (P, Q, R) = \vec{\boldsymbol{v}} + \varepsilon \vec{\boldsymbol{v}}^{*}$ is

$$\begin{aligned} < \widehat{A}, \widehat{B} >_d &= XP + YQ + ZR \\ &= \ < \vec{\boldsymbol{w}}, \vec{\boldsymbol{v}} > + \varepsilon\,(< \vec{\boldsymbol{w}}, \vec{\boldsymbol{v}}^{*} > + < \vec{\boldsymbol{w}}^{*}, \vec{\boldsymbol{v}} >) \end{aligned}$$

   where "$<,>$" denotes the usual inner-product in $\mathbb{R}^3$. If $< \widehat{A}, \widehat{B} >_d = 0$, then $\widehat{A}$ and $\widehat{B}$ are said to be *perpendicular* in the sense of *dual*.

3. The *vector product* of $\widehat{A} = (X, Y, Z) = \vec{\boldsymbol{w}} + \varepsilon \vec{\boldsymbol{w}}^{*}$ and $\widehat{B} = (P, Q, R) = \vec{\boldsymbol{v}} + \varepsilon \vec{\boldsymbol{v}}^{*}$ is

$$\begin{aligned} \widehat{A} \times_d \widehat{B} &= (YR - QZ, ZP - RX, XQ - PY) \\ &= \vec{\boldsymbol{w}} \times \vec{\boldsymbol{v}} \ + \ \varepsilon\,(\vec{\boldsymbol{w}} \times \vec{\boldsymbol{v}}^{*} + \vec{\boldsymbol{w}}^{*} \times \vec{\boldsymbol{v}}) \end{aligned}$$

   where "$\times$" denotes the usual vector-product in $\mathbb{R}^3$.

For further information about dual-plane and dual-space see [7, 9, 10].

Let $\widehat{P_0} = (X_0, Y_0, Z_0)$ be a point and $\widehat{\boldsymbol{N}} = (A, B, C)$ be a non-zero vector in $\mathbb{D}^3$. The plane passes through $\widehat{P_0}$ and perpendicular to $\widehat{\boldsymbol{N}}$ consists of all the points $\widehat{P} = (X, Y, Z) \in \mathbb{D}^3$ such that the vector from $\widehat{P_0}$ to $\widehat{P}$ (that is, $\overrightarrow{\widehat{P_0}\widehat{P}}$) is perpendicular to the vector $\widehat{\boldsymbol{N}}$, see Fig. 2. In other words,

$$< \widehat{\boldsymbol{N}}, \overrightarrow{\widehat{P_0}\widehat{P}} >_d = \ A(X - X_0) + B(Y - Y_0) + C(Z - Z_0) = 0.$$

Thus, the plane through $\widehat{P_0}$ and perpendicular to $\widehat{\boldsymbol{N}}$ has the equation *(in general form)*

$$AX + BY + CZ = D$$

where $D = AX_0 + BY_0 + CZ_0$. For the special case $D = 0$, the plane passes through the origin.

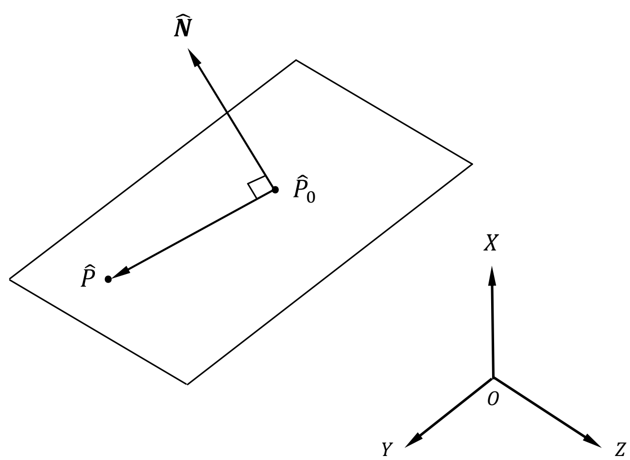

Figure 2. The dual-plane passing through the point $\widehat{P_0}$ and perpendicular to the vector $\widehat{\boldsymbol{N}}$ in $\mathbb{D}^3$

Let the linear space $\mathbb{D}^3$ be equipped with the coordinates X, Y, Z. The set of the one-dimensional subspaces of $\mathbb{D}^3$ is called the *dual-projective plane* and is denoted by $\mathbb{P}^2$. A two-dimensional dual-plane $\mathbb{D}^2$ can be embedded into $\mathbb{P}^2$ by

$$\widetilde{A} = (Y, Z) \longmapsto \widehat{A}\mathbb{D} = (1, Y, Z)\mathbb{D} = (\lambda, \lambda Y, \lambda Z), \ \ \lambda \in \mathbb{D}.$$

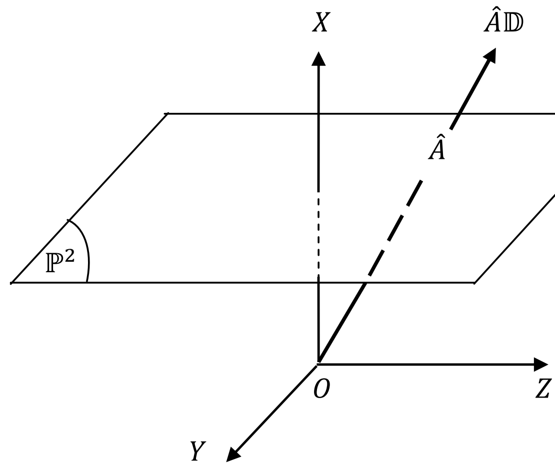

Figure 3. Dual-projective plane $\mathbb{P}^2$ where $\widehat{A} = (1, Y, Z) \in \mathbb{P}^2$.

For the special case $X = 0$, the point $\widehat{A}\mathbb{D} = (X, Y, Z)\mathbb{D}$ is called the *ideal point* , see Fig. 3.

If $\widehat{A}\mathbb{D} = (X, Y, Z)\mathbb{D}$ is a proper point (that is $X \in \breve{\mathbb{D}}$), its coordinates are recovered in $\mathbb{D}^2$ by

$$(X, Y, Z)\mathbb{D} \in \mathbb{P}^2 \iff (Y/X,\ Z/X) \in \mathbb{D}^2.$$

If a line L of $\mathbb{D}^2$ is parallel to the vector (L_1, L_2), then it has the ideal point $(0, L_1, L_2)\mathbb{D}$. The plane $X = 0$ corresponds to the *ideal* (or *absolute*) line W which contains all the ideal points. The *homogeneous coordinate vector* of the point $\widehat{A}\mathbb{D}$ is the coordinate vector $\widehat{A}$ which is represented as column vector in matrix notation.

2.2 Dual-Quasi Elliptic Geometry

Suppose that L is a non-horizontal line in $\mathbb{D}^3$ intersecting the dual-planes $Z = 0,\ Z = +1,\ Z = -1$, respectively, at the points $M,\ L_1,\ L_2$. And suppose that $L_1^{'},\ L_2^{'}$ are the normal piercing points, respectively, of the points $L_1,\ L_2$ on the plane $Z = 0$. Moreover, let $\beta^-,\ \beta^+$ be two mappings that rotates, respectively, the points $L_2^{'},\ L_1^{'}$ in the plane $Z = 0$ around the point M (which is the midpoint of the points $L_1^{'},\ L_2^{'}$) with a positive oriented hyperbolic right angle as

$$L_2^{'}\beta^- = L_2^{''},\quad L_1^{'}\beta^+ = L_1^{''},$$

see Fig. 4. In this case, β^- and β^+ are linear mappings from the dual-space of lines onto the horizontal dual-plane $Z = 0$, and there exists an ordered point pair $L_1^{''},\ L_2^{''}$ associated with every ideal line W. That means, if we choose an ordered pair of dual in the plane $Z = 0$, then there exists a unique line L associated with them. Also, there exists an invertible relationship $L \leftrightarrow (L_1^{''},\ L_2^{''})$ between the non-horizontal lines L of $\mathbb{D}^3$ and the ordered pairs $(L_1^{''},\ L_2^{''})$ of dual.

2.3 Dual Semi-Quaternions

A *dual semi-quaternion* is defined by

$$Q = Q_0 + Q_1\boldsymbol{i} + Q_2\boldsymbol{j} + Q_3\boldsymbol{k}$$

where $Q_i = q_i + \varepsilon q_i^*$ are dual-numbers for $i = 0, 1, 2, 3$. Also, $1,\ \boldsymbol{i},\ \boldsymbol{j},\ \boldsymbol{k}$ may be interpreted as the four basic vectors of Cartesian set of coordinates satisfying the following non-commutative multiplication rules

$$\boldsymbol{i}^2 = -1,\ \ \boldsymbol{j}^2 = \boldsymbol{k}^2 = 0,$$
$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k},\ \ \boldsymbol{jk} = -\boldsymbol{kj} = 0,\ \ \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}.$$

The set of all dual semi-quaternions is denoted by $\mathbb{H}_{\mathbb{D}S}$. A dual semi-quaternion $Q = Q_0 + Q_1\boldsymbol{i} + Q_2\boldsymbol{j} + Q_3\boldsymbol{k}$ can be written as a sum of a *scalar part* $S_Q = Q_0 \in \mathbb{D}$ and *vector part* $\boldsymbol{V}_Q = Q_1\boldsymbol{i} + Q_2\boldsymbol{j} + Q_3\boldsymbol{k} \in \mathbb{D}^3$ that is $Q = S_Q + \boldsymbol{V}_Q$. If $S_Q = 0$, then Q is said to be a *pure* and is denoted by boldface letter $\boldsymbol{Q}$. The set of all pure dual-quaternions is

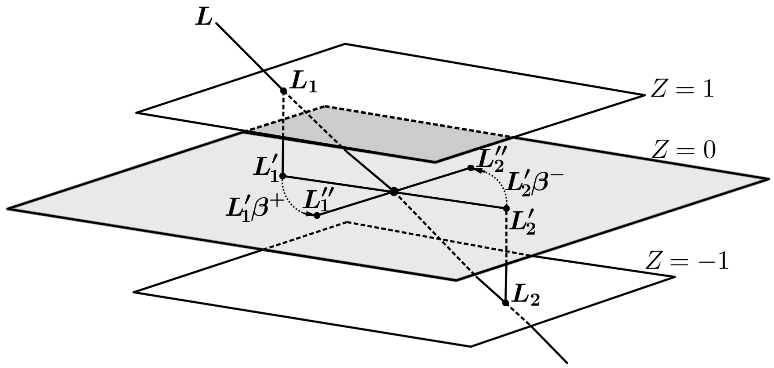

Figure 4. Dual-quasi elliptic motion.

denoted by $\mathbf{H}_{\mathbf{DS}}$.

Let $Q = S_Q + \boldsymbol{V}_Q = Q_0 + Q_1\boldsymbol{i} + Q_2\boldsymbol{j} + Q_3\boldsymbol{k}$ and $P = S_P + \boldsymbol{V}_P = P_0 + P_1\boldsymbol{i} + P_2\boldsymbol{j} + P_3\boldsymbol{k}$ be any two dual semi-quaternions. Then,

- The *addition* of Q and P is

$$\begin{aligned} Q + P &= (S_Q + S_P) + (\boldsymbol{V}_Q + \boldsymbol{V}_P) \\ &= (Q_0 + P_0) + (Q_1 + P_1)\boldsymbol{i} + (Q_2 + P_2)\boldsymbol{j} + (Q_3 + P_3)\boldsymbol{k} \end{aligned}$$

- The *multiplication* of Q and P is

$$\begin{aligned} QP &= S_Q S_P - < \boldsymbol{V}_Q, \boldsymbol{V}_P >_d^{'} + S_Q\boldsymbol{V}_P + S_P\boldsymbol{V}_Q + \boldsymbol{V}_Q \times_d^{'} \boldsymbol{V}_P \\ &= (Q_0P_0 - Q_1P_1) + (Q_1P_0 + Q_0P_1)\,\boldsymbol{i} + \\ &\quad (Q_2P_0 + Q_3P_1 + Q_0P_2 - Q_1P_3)\,\boldsymbol{j} + \\ &\quad (Q_0P_3 + Q_1P_2 + Q_3P_0 - Q_2P_1)\,\boldsymbol{k} \end{aligned}$$

  where $< \boldsymbol{V}_Q, \boldsymbol{V}_P >_d^{'} = Q_1P_1$ and $\boldsymbol{V}_Q \times_d^{'} \boldsymbol{V}_P = 0\,\boldsymbol{i} + (Q_3P_1 - Q_1P_3)\,\boldsymbol{j} + (Q_1P_2 - Q_2P_1)\,\boldsymbol{k}$.

- The *equality* of Q and P is defined to be

$$Q = P \quad \text{iff} \quad S_Q = S_P \quad \text{and} \quad \boldsymbol{V}_Q = \boldsymbol{V}_P.$$

- The *quaternionic conjugate* of Q is defined to be

$$\overline{Q} = S_Q - \boldsymbol{V}_Q = Q_0 - Q_1\boldsymbol{i} - Q_2\boldsymbol{j} - Q_3\boldsymbol{k},$$

  while its *dual conjugate* is defined to be

$$Q^\star = S_Q^\star + \boldsymbol{V}_Q^\star = Q_0^\star + Q_1^\star\boldsymbol{i} + Q_2^\star\boldsymbol{j} + Q_3^\star\boldsymbol{k}.$$

- The *norm* of Q is defined to be

$$N_Q = Q\overline{Q} = \overline{Q}Q = Q_0^2 + Q_1^2 = (q_0^2 + q_1^2) \; + \; 2\varepsilon(q_0q_0^* + q_1q_1^*),$$

  while its *modulus* is defined to be

$$\|Q\| = \sqrt{N_Q} = \sqrt{q_0^2 + q_1^2} \; + \; \varepsilon\frac{q_0q_0^* + q_1q_1^*}{\sqrt{q_0^2 + q_1^2}}$$

  for $q_0^2 + q_1^2 \neq 0$. If $N_Q = \|Q\| = 1$, then Q is said to be a *unit*.

- The *multiplicative inverse* of a dual semi-quaternion is obtained by dividing its quaternionic-conjugate by its norm. It is important to emphasize that the multiplicative inverse of a dual semi-quaternion exists only if its norm is non-zero. For example, the multiplicative inverse of Q exists only for $q_0 \neq 0 \neq q_1$ and is defined to be

$$Q^{-1} = \frac{\overline{Q}}{N_Q}.$$

The *matrix representation* of a dual semi-quaternion $Q = Q_0 + Q_1\boldsymbol{i} + Q_2\boldsymbol{j} + Q_3\boldsymbol{k}$ can be given by

$$Q := \mathrm{Q} = \begin{pmatrix} Q_0 & -Q_1 & 0 & 0 \\ Q_1 & Q_0 & 0 & 0 \\ Q_2 & Q_3 & Q_0 & -Q_1 \\ Q_3 & -Q_2 & Q_1 & Q_0 \end{pmatrix}.$$

For the special case, if Q is unit then Q can be given in general form as

$$\begin{pmatrix} \cos\Theta & -\sin\Theta & 0 & 0 \\ \sin\Theta & \cos\Theta & 0 & 0 \\ Q_2 & Q_3 & \cos\Theta & -\sin\Theta \\ Q_3 & -Q_2 & \sin\Theta & \cos\Theta \end{pmatrix}$$

where $\Theta = \theta + \varepsilon\theta^* \in \mathbb{D}$. It can be easily checked that Q is orthogonal because $\mathrm{I} = \mathrm{Q}^{\mathrm{T}}\,\mathrm{I}\,\mathrm{Q}$ and $\det \mathrm{Q} = 1$ for the metric tensor

$$\mathrm{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where Q^{T} denotes the transpose of Q. Hence, unit dual semi-quaternions can be used to represent rotations.

The algebra $\mathbb{H}_{\mathbb{D}\mathbb{S}}$ is isomorphic to the Clifford algebra $Cl_{0,1,3}$ (i.e., $\mathbb{H}_{\mathbb{D}\mathbb{S}} \cong Cl_{0,1,3}$) in dimension 4 when we identify the quaternionic units $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, respectively, with e_1, e_2, $e_{12}(= e_1e_2)$, and the dual unit ε with $e_{34}(= e_3e_4)$ which commutes with a subalgebra of $Cl_{0,1,3}$ generated by e_1 and e_2. Here, the standard anti-commuting generators e_i, $i = 1, 2, 3, 4$, satisfy

$$e_1^2 = -1,\ e_2^2 = e_3^2 = e_4^2 = 0 \text{ and } e_ie_j = -e_je_i \text{ for } i \neq j.$$

3. DUAL-QUASI ELLIPTIC MOTION IN $\mathbb{D}^3$

Let $Q = Q_0 + Q_1\boldsymbol{i} + Q_2\boldsymbol{j} + Q_3\boldsymbol{k}$ be a unit dual semi-quaternion and Q_1 be a non-pure dual-number, i.e., $Q_1 \in \breve{\mathbb{D}}$. Then the map

$$\beta_Q : Q\mathbb{D} \in \mathbb{P}^2 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 2(Q_1Q_2 + Q_0Q_3) & Q_0^2 - Q_1^2 & -2Q_0Q_1 \\ 2(Q_1Q_3 - Q_0Q_2) & 2Q_0Q_1 & Q_0^2 - Q_1^2 \end{pmatrix} = \mathrm{A}$$

corresponds to each point a negative oriented rotation in the dual-projective plane $\mathbb{P}^2$. That is because the matrix A is orthogonal (i.e., $\mathrm{I} = \mathrm{A}^{\mathrm{T}}\,\mathrm{I}\,\mathrm{A}$) with $\det \mathrm{A} = 1$.

Proposition 3.1. *The unit dual semi-quaternion*

$$Q = -\cos\frac{\Theta}{2} + \sin\frac{\Theta}{2}\boldsymbol{i} + Q_2\boldsymbol{j} + Q_3\boldsymbol{k}$$

represents a negative oriented rotation in dual-plane $\mathbb{D}^2$, *where* $\sin(\Theta/2)$ *is a non-pure dual-number, i.e.,* $\sin(\Theta/2) \in \breve{\mathbb{D}}$.

Proof. If we take $Q_0 = -\cos(\Theta/2)$, $Q_1 = \sin(\Theta/2)$ then the map

$$f_Q : \begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} Q_0^2 - Q_1^2 & -2Q_0Q_1 \\ 2Q_0Q_1 & Q_0^2 - Q_1^2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} 2(Q_1Q_2 + Q_0Q_3) \\ 2(Q_1Q_3 - Q_0Q_2) \end{pmatrix}$$

represents a dual-planar motion. Also, it is straightforward to show that

$$f_Q(\frac{Q_2}{\sin\frac{\Theta}{2}}, \frac{Q_3}{\sin\frac{\Theta}{2}}) = (\frac{Q_2}{\sin\frac{\Theta}{2}}, \frac{Q_3}{\sin\frac{\Theta}{2}}).$$

Thus, the linear map f_Q represents a negative oriented rotation through an angle $\Theta \in \mathbb{D}$ about the center

$$M = (\frac{Q_2}{\sin\frac{\Theta}{2}}, \frac{Q_3}{\sin\frac{\Theta}{2}}) \in \mathbb{D}^2$$

in dual-plane $\mathbb{D}^2$. □

Corollary 3.1. *If a non-horizontal line L is incident with a point $P \in \mathbb{P}^2$, then the rotation corresponding to P maps $L_2^{''}$ to $L_1^{''}$ as in Fig. 4.*

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Affiliations

MURAT BEKAR
ADDRESS: Necmettin Erbakan University, Department of Mathematics and Computer Sciences, 42090, Konya-TURKEY.
E-MAIL: mbekar@konya.edu.tr