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Integral Inequalities of Hermite-Hadamard Type for λ-MT-Convex Function

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Abstract
In this paper, we establish some Hermite-Hadamard type Integral inequalities for a new class of convex function called λ-MT-convex function. Our results generalize and extend some existing results in literature.

Keywords: *Hermite-Hadamard integral Inequalities; MT-convex function; λ-MT-convex function.*

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1. Introduction

In this section, some definitions and results used in this paper are presented.

Let $f : I \subseteq R \to R$ be a convex function defined on the interval $I = [a, b]$ of the real numbers and let $a, b \in [c, d]$ where $a < b$. Then, the following double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as the Hermite-Hadamard Integral Inequality [7]. The inequality, often referred to as the first fundamental result for convex functions with a natural geometrical interpretation and many applications, has attracted and continues to attract much interest in the field of Optimization, Mathematics and Engineering since its establishment in 1881 (see [2]). A good number of research papers and texts have been written on (1.1), providing new proofs, expositions, noteworthy extensions, generalizations and numerous applications (see [5]).

We give some definitions relating to convex functions below:

Definition 1.1. [2] A function $f : I \to R$ is said to be convex, if for every $x, y \in I$ and $t \in [0, 1]$, we have

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y). \tag{1.2}$$

Definition 1.2. [4] A function $f : [0, b] \to R$ is said to be m-convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$, and $t \in [0, 1]$, we have

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y). \tag{1.3}$$

In [11], Tunc and Yildirim defined the class of MT-convex functions as follows.

Definition 1.3. [11] A function $f : I \subseteq R \to R$ is said to belong to the class $MT(I)$ if f is nonnegative and $\forall x, y \in I$ and $t \in (0, 1)$ satisfies the inequality

$$f(tx + (1-t)y) \le \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y). \tag{1.4}$$

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Recently, Omotoyinbo and Mogbademu [9] introduced a new class of convex function as follows.

Definition 1.4. [9] A function $f : I \subseteq R \to R$ is said to belong to the class $m - MT(I)$ if f is nonnegative and $\forall x, y \in I$ and $t \in (0,1)$, with $m \in [0,1]$ satisfies the inequality

$$f(tx + m(1-t)y) \le \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.5}$$

Recently, Tunc and Yildirim [11] obtained the following two new inequalities of Hermite-Hadamard type for the class of MT-convex functions.

Theorem 1.1. *Let $f \in MT(I), a, b \in I$ with $a < b$ and $f \in L_1[a,b]$. Then, one has the inequality*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx$$

and

$$\frac{2}{b-a}\int_a^b \tau(x) f(x)dx \le \frac{f(a)+f(b)}{2}, \tag{1.6}$$

where $\tau(x) = \frac{\sqrt{(b-a)(x-a)}}{b-a}, x \in [a,b]$.

In [12], Tunc et al. established some Hermite-Hadamard inequalities for MT-convex functions. Indeed, they proved the following:

Theorem 1.2. *Let $f : [a,b] \subseteq R \to R$ be nonnegative MT-convex function and $f \in L_1[a,b]$. Then,*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{\pi}{4}(f(a)+f(b)). \tag{1.7}$$

Theorem 1.3. *Let $f, g \in [a,b] \to R$ be two nonnegative MT-convex functions and $f, g \in L_1[a,b]$. Then,*

$$\frac{8}{3} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \le M(a,b) + N(a,b), \tag{1.8}$$

where $M(a,b) = f(a)g(a) + f(b)g(b), N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 1.4. *Let $f, g : [a,b] \subseteq R \to R$ two nonnegative MT-convex functions and $f, g \in L_1([a,b])$. Then,*
$f\left(\frac{a+b}{2}\right)(g(a)+g(b)) + g\left(\frac{a+b}{2}\right)(f(a)+f(b))$

$$\le \frac{16}{3\pi} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + 2(f(a)+f(b))(g(a)+g(b)). \tag{1.9}$$

Motivated by the results of Dragomir et al. [3], Tunc and Yildirim [11], Omotoyinbo and Mogbademu [8], [9] and Tunc et al. [12], we introduce and define the following new class of convex functions.

Definition 1.5. A nonnegative function $f : I \subseteq R \to R$ is said to be a λ-MT-convex function or said to belong to the class λ-$MT(I)$ if $\forall x, y \in I$, $\lambda \in (0, \frac{1}{2}]$ and $t \in (0,1)$ the following inequality is satisfied

$$f(tx + (1-t)y) \le \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} f(y). \tag{1.10}$$

Remark 1.1. An example to illustrate this type of function is:
$f : [1,2] \to R, f(x) = x^k, k \in \left(0, \frac{1}{1000}\right)$.
By choosing $I = [1,2], \lambda \in [\frac{1}{3}, \frac{2}{5}], t = 0.5, x = 1, y = 2$, inequality (1.10) is satisfied. Thus, f is λ-MT-convex.

The purpose of this paper is to establish some new Hermite-Hadamard type integral inequalities for λ-MT-convex function, thereby extending known results in literature, using simple analytical techniques.

2. Main Results

Theorem 2.1. *Let $f, g \in [a,b] \to R$ be two nonnegative $\lambda - MT$-convex functions and $f, g \in L_1([a,b])$ with $a, b \in I$ and $a < b$. Then*

$$8\lambda^2 f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \leq (\lambda^2 - \lambda + 1)(M(a,b) + N(a,b)), \tag{2.1}$$

where $M(a,b) = f(a)g(a) + f(b)g(b), N(a,b) = f(a)g(b) + f(b)g(a)$.

Proof. Since f and g are $\lambda - MT$-convex, then from Definition 1.5,

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(a) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} f(b) \\ &\leq \frac{1}{2}\left(\frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\right)(f(a) + f(b)) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} g\left(\frac{a+b}{2}\right) &\leq \frac{\sqrt{t}}{2\sqrt{1-t}} g(a) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} g(b) \\ &\leq \frac{1}{2}\left(\frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\right)(g(a) + g(b)). \end{aligned} \tag{2.3}$$

Multiplying (2.2) and (2.3), we obtain

$$\begin{aligned} f\left(\tfrac{a+b}{2}\right) g\left(\tfrac{a+b}{2}\right) &\leq \left[\tfrac{1}{4}\left(\tfrac{\sqrt{t}}{\sqrt{1-t}} + \tfrac{(1-\lambda)\sqrt{1-t}}{\lambda\sqrt{t}}\right)\right]^2 (f(a) + f(b))(g(a) + g(b)) \\ &= \tfrac{1}{16}\left(\tfrac{\lambda^2 t^2 + 2\lambda(1-\lambda)t(1-t) + (1-\lambda)^2(1-t)^2}{\lambda^2 t(1-t)}\right)(f(a) + f(b))(g(a) + g(b)). \end{aligned} \tag{2.4}$$

It is easy to see that (2.4) gives

$$\begin{aligned} 16\lambda^2 t(1-t) f(\tfrac{a+b}{2}) g(\tfrac{a+b}{2}) &\leq (\lambda^2 t^2 + 2\lambda(1-\lambda)t(1-t) \\ &\quad + (1-\lambda)^2(1-t)^2)(f(a) + f(b))(g(a) + g(b)). \end{aligned} \tag{2.5}$$

Integrating both sides of (2.5) wrt t over $[0,1]$ to get

$$\begin{aligned} 16\lambda^2 f(\tfrac{a+b}{2}) g(\tfrac{a+b}{2}) \int_0^1 t(1-t)dt &\leq (f(a) + f(b))(g(a) + g(b))(\lambda^2 \int_0^1 t^2 dt \\ &\quad + 2\lambda(1-\lambda) \int_0^1 t(1-t)dt \\ &\quad + (1-\lambda)^2 \int_0^1 (1-t)^2 dt). \end{aligned} \tag{2.6}$$

Substituting $\int_0^1 t(1-t)dt = \frac{1}{6}, \int_0^1 t^2 dt = \int_0^1 (1-t)^2 dt = \frac{1}{3}$ in (2.5) and simplifying further gives the inequality (2.1). ☐

Remark 2.1. Setting $\lambda = \frac{\sqrt{3}}{2}$ in Theorem 2.1 gives

$$\begin{aligned} \tfrac{8}{3} f\left(\tfrac{a+b}{2}\right) g\left(\tfrac{a+b}{2}\right) &\leq \left(\tfrac{4-\sqrt{3}}{3}\right)(M(a,b) + N(a,b)) \\ &\leq M(a,b) + N(a,b) \end{aligned}$$

which is Theorem 2.5 of Tunc et al. [12].

Theorem 2.2. *Let $f, g : [a,b] \subseteq R \to R$ be two nonnegative λ-MT-convex functions and $f, g \in L_1([a,b])$ with $a, b \in I$ where $a < b$. Then*

$$\begin{aligned} f\left(\tfrac{a+b}{2}\right)(g(a) + g(b)) + g\left(\tfrac{a+b}{2}\right)(f(a) + f(b)) &\leq \tfrac{16\lambda}{3\pi} f\left(\tfrac{a+b}{2}\right) g\left(\tfrac{a+b}{2}\right) \\ &+ \tfrac{4}{3\lambda\pi}(\lambda^4 - 2\lambda^3 + 2\lambda^2 + \lambda + 1) \\ &\times (M(a,b) + N(a,b)) \end{aligned}$$

where M(a, b)= f(a)g(a) + f(b)g(b), N(a,b)= f(a)g(b)+f(b)g(a).

Proof. Since $f, g \in \lambda\text{-}MT(I)$, then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(a) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} f(b) \\ &\leq \frac{1}{2}\left(\frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\right)(f(a)+f(b)), \qquad (2.7) \\ g\left(\frac{a+b}{2}\right) &\leq \frac{\sqrt{t}}{2\sqrt{1-t}} g(a) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}} g(b) \\ &\leq \frac{1}{2}\left(\frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}\right)(g(a)+g(b)). \qquad (2.8) \end{aligned}$$

Recall: For $p, q, r, s, t \in R^{+}$, if $p \leq s$, and $r \leq q$ then $pq + rs \leq ps + qr$.

$$\begin{aligned} &\frac{1}{4} f\left(\frac{a+b}{2}\right)\left(\frac{\sqrt{t}}{\sqrt{1-t}} + \frac{(1-\lambda)\sqrt{1-t}}{\lambda\sqrt{t}}\right)(g(a)+g(b)) \\ &+\frac{1}{4} g\left(\frac{a+b}{2}\right)\left(\frac{\sqrt{t}}{\sqrt{1-t}} + \frac{(1-\lambda)\sqrt{1-t}}{\lambda\sqrt{t}}\right)(f(a)+f(b)) \\ &\leq f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \left[\frac{1}{4}\left(\frac{\sqrt{t}}{\sqrt{1-t}} + \frac{(1-\lambda)\sqrt{1-t}}{\lambda\sqrt{t}}\right)\right]^{2}(f(a)+f(b))(g(a)+g(b)). \qquad (2.9) \end{aligned}$$

Simplifying (2.9), we get

$$\begin{aligned} &\tfrac{1}{4} f\left(\tfrac{a+b}{2}\right)\left(\tfrac{\sqrt{t}}{\sqrt{1-t}} + \tfrac{(1-\lambda)\sqrt{1-t}}{\lambda\sqrt{t}}\right)(g(a)+g(b)) + \tfrac{1}{4} g(\tfrac{a+b}{2})(\tfrac{\sqrt{t}}{\sqrt{1-t}} + \tfrac{(1-\lambda)\sqrt{1-t}}{\lambda\sqrt{t}})(f(a)+f(b)) \\ &\leq f(\tfrac{a+b}{2}) g(\tfrac{a+b}{2}) + \left(\tfrac{\lambda^2 t^2 + 2\lambda(1-\lambda)t(1-t) + \lambda^2(1-\lambda)^2(1-t)^2}{16\lambda^2 t(1-t)}\right)(f(a)+f(b))(g(a)+g(b)). \end{aligned} \qquad (2.10)$$

Multiplying both sides of the inequality (2.10) by $16\lambda^2 t(1-t)$, gives

$$\begin{aligned} &4f\left(\tfrac{a+b}{2}\right)(\lambda^2 t^{\frac{3}{2}}(1-t)^{\frac{1}{2}} + \lambda(1-\lambda)t^{\frac{1}{2}}(1-t)^{\frac{3}{2}})(g(a)+g(b)) \\ &+4g(\tfrac{a+b}{2})(\lambda^2 t^{\frac{3}{2}}(1-t)^{\frac{1}{2}} + \lambda(1-\lambda)t^{\frac{1}{2}}(1-t)^{\frac{3}{2}})(f(a)+f(b)) \\ &\leq 16\lambda^2 t(1-t) f(\tfrac{a+b}{2}) g(\tfrac{a+b}{2}) + (\lambda^2 t^2 + 2(1-\lambda)t(1-t) \\ &+\lambda^2(1-\lambda)^2(1-t)^2)(f(a)+f(b))(g(a)+g(b)). \end{aligned} \qquad (2.11)$$

Integrating both sides of (2.11) wrt t over $[0,1]$,

$$\begin{aligned} &4f\left(\frac{a+b}{2}\right)\left(\lambda^2 \int_0^1 t^{\frac{3}{2}}(1-t)^{\frac{1}{2}} dt + \lambda(1-\lambda)\int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{3}{2}} dt\right)(g(a)+g(b) \\ &+4g\left(\frac{a+b}{2}\right)\left(\lambda^2 \int_0^1 t^{\frac{3}{2}}(1-t)^{\frac{1}{2}} dt + \lambda(1-\lambda)\int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{3}{2}}\right)(f(a)+f(b)) \\ &\leq 16\lambda^2 f\left(\tfrac{a+b}{2}\right) g\left(\tfrac{a+b}{2}\right)\int_0^1 t(1-t)dt \\ &+\left(\lambda^2 \int_0^1 t^2 dt + 2(1-\lambda)\int_0^1 t(1-t)dt + \lambda^2(1-\lambda)^2 \int_0^1 (1-t)^2 dt\right)(f(a)+f(b))(g(a)+g(b)). \qquad (2.12) \end{aligned}$$

Substitute the following equalities in (2.12),

$$\int_0^1 t^{\frac{3}{2}}(1-t)^{\frac{1}{2}} dt = \int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{3}{2}} dt = \frac{\pi}{16}$$

$$\int_0^1 t(1-t)dt = \frac{1}{6}, \int_0^1 t^2 dt = \int_0^1 (1-t)^2 dt = \frac{1}{3}.$$

This completes the proof. □

Theorem 2.3. *Let $f : [a,b] \subseteq R \to R$ be a nonnegative $\lambda - MT$-convex functions and $f \in L_1([a,b])$ with $a, b \in I$ where $a < b$. Then*

$$\frac{1}{((\frac{\lambda}{1-\lambda})b - a)}\int_a^{(\frac{\lambda}{1-\lambda})b} f(x)dx + \frac{1}{(b-(\frac{\lambda}{1-\lambda})a)}\int_{(\frac{\lambda}{1-\lambda})a}^{b} f(y)dy \le \frac{\pi}{4}\left(\frac{1}{\lambda}\right)(f(a)+f(b))$$

Proof. Since $f \in \lambda - MT(I)$. Then, we can write

$$f\left(ta + \frac{\lambda}{1-\lambda}(1-t)b\right) \le \frac{\sqrt{t}}{2\sqrt{1-t}}f(a) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}f(b), \tag{2.13}$$

$$f\left(tb + \frac{\lambda}{1-\lambda}(1-t)a\right) \le \frac{\sqrt{t}}{2\sqrt{1-t}}f(b) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}f(a). \tag{2.14}$$

Adding (2.13) and (2.14) gives

$$f\left(ta + \frac{\lambda}{1-\lambda}(1-t)b\right) + f\left(tb + \frac{\lambda}{1-\lambda}(1-t)a\right) \le \frac{1}{2}\left(\frac{\sqrt{t}}{\sqrt{1-t}} + \frac{(1-\lambda)\sqrt{1-t}}{\lambda\sqrt{t}}\right)(f(a)+f(b)). \tag{2.15}$$

Integrating both sides of (2.15) wrt to t,

$$\int_0^1 f\left(ta + \frac{\lambda}{1-\lambda}(1-t)b\right)dt + \int_0^1 f\left(tb + \frac{\lambda}{1-\lambda}(1-t)a\right)dt$$
$$\le \frac{1}{2}\left(\int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}dt + \frac{1-\lambda}{\lambda}\int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}dt\right)(f(a)+f(b)). \tag{2.16}$$

Substituting $x = ta + \frac{\lambda}{1-\lambda}(1-t)b,\ dx = (a - (\frac{\lambda}{1-\lambda})b)\,dt$,

$$y = tb + \frac{\lambda}{1-\lambda}(1-t)a,\ dy = (b - (\frac{\lambda}{1-\lambda})a)\,dt$$

where $\int_0^1 t^{-\frac{1}{2}}(1-t)^{\frac{1}{2}}dt = \int_0^1 t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt = \frac{\pi}{2}$ into (2.16) and simplifying to get:

$$\frac{1}{((\frac{\lambda}{1-\lambda})b - a)}\int_a^{(\frac{\lambda}{1-\lambda})b} f(x)dx + \frac{1}{(b-(\frac{\lambda}{1-\lambda})a)}\int_{(\frac{\lambda}{1-\lambda})a}^{b} f(y)dy \le \frac{\pi}{4}\left(\frac{1}{\lambda}\right)(f(a)+f(b)).$$

Hence, the proof is completed.

Remark 2.2. Let $f : [a,b] \subseteq R \to R$ be a nonnegative MT-convex function and $f \in L_1[a,b]$. By choosing $\lambda = \frac{1}{2}$ and $x = y$ we obtain

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{\pi}{4}(f(a)+f(b)),$$

which is Theorem 2.3 of Tunc et al. [12].

Remark 2.3. Alternatively, we can integrate directly either of the $\lambda - MT$-convex functions

$$f\left(ta + \frac{\lambda}{1-\lambda}(1-t)b\right) \le \frac{\sqrt{t}}{2\sqrt{1-t}}f(a) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}f(b)$$

or

$$f\left(tb + \frac{\lambda}{1-\lambda}(1-t)a\right) \le \frac{\sqrt{t}}{2\sqrt{1-t}}f(b) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}f(a)$$

to obtain

$$\frac{1}{((\frac{\lambda}{1-\lambda})b - a)}\int_a^{(\frac{\lambda}{1-\lambda})b} f(x)dx \le \frac{\pi}{4}\left(f(a) + \left(\frac{1-\lambda}{\lambda}\right)f(b)\right)$$

or

$$\frac{1}{((\frac{\lambda}{1-\lambda})a - b)}\int_b^{(\frac{\lambda}{1-\lambda})a} f(x)dx \le \frac{\pi}{4}\left(f(b) + \left(\frac{1-\lambda}{\lambda}\right)f(a)\right).$$

Theorem 2.4. *Let $f : [a, b] \subseteq R \to R$ be a nonnegative $\lambda - MT$-convex functions and $f \in L_1([a, b])$ with $a, b \in I$ where $a < b$. Then*

$$f\left(\frac{a+(\frac{\lambda}{1-\lambda})b}{2}\right) \leq \frac{1}{2}\left(\frac{1}{((\frac{\lambda}{1-\lambda})b-a)}\int_a^{(\frac{\lambda}{1-\lambda})b} f(x)dx + \frac{1}{(b-(\frac{\lambda}{1-\lambda})a)}\int_{(\frac{\lambda}{1-\lambda})a}^{b} f(y)dy\right). \tag{2.17}$$

Proof. Since $f \in \lambda - MT(I)$. Then, for all $x, y \in I$,

$$f\left(ta + \frac{\lambda}{1-\lambda}(1-t)b\right) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(a) + \frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}f(b). \tag{2.18}$$

Substituting $t = \frac{1}{2}$ in inequality (2.18), we have,

$$f\left(\frac{a+(\frac{\lambda}{1-\lambda})b}{2}\right) \leq \frac{f(a)+(\frac{\lambda}{1-\lambda})f(b)}{2}$$

That is, with $x = ta + \frac{\lambda}{1-\lambda}(1-t)b, dx = (a-(\frac{\lambda}{1-\lambda})b)dt$,

$$y = \frac{\lambda}{1-\lambda}(1-t)a + tb, dy = (b-(\frac{\lambda}{1-\lambda})a)dt,$$

where $\int_0^1 t^{-\frac{1}{2}}(1-t)^{\frac{1}{2}}dt = \int_0^1 t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt = \frac{\pi}{2}$.

$$f\left(\frac{a+(\frac{\lambda}{1-\lambda})b}{2}\right) \leq \frac{1}{2}\left(\int_0^1 f\left(ta + \frac{\lambda}{1-\lambda}(1-t)b\right)dt + \int_0^1 f\left(\frac{\lambda}{1-\lambda}(1-t)a + tb\right)dt\right). \tag{2.19}$$

Further simlification of inequality (2.19) completely gives (2.17). Hence, the proof is completed. □

Remark 2.4. Setting $\lambda = \frac{1}{2}$, in (2.17) we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx,$$

which is the first part of Hadamard's inequality and Theorem 2a of Tunc and Yildirim[11].

Theorem 2.5. *Let $f, g : [a, b] \subseteq R \to R$ be two nonnegative $\lambda - MT$-convex functions and $f \in L_1([a, b])$ with $a, b \in I$ where $a < b$. Then*

$$g(a)\frac{\lambda^2}{((\frac{\lambda}{1-\lambda})b-a)^3}\int_a^{(\frac{\lambda}{1-\lambda})b}((\frac{\lambda}{1-\lambda})b-x)^{\frac{3}{2}}(x-a)^{\frac{1}{2}}f(x)dx$$

$$+g(b)\frac{\lambda(1-\lambda)}{((\frac{\lambda}{1-\lambda})b-a)^3}\int_a^{(\frac{\lambda}{1-\lambda})b}((\frac{\lambda}{1-\lambda})b-x)^{\frac{1}{2}}(x-a)^{\frac{3}{2}}f(x)dx$$

$$f(a)\frac{\lambda^2}{((\frac{\lambda}{1-\lambda})b-a)^3}\int_a^{(\frac{\lambda}{1-\lambda})b}((\frac{\lambda}{1-\lambda})b-x)^{\frac{3}{2}}(x-a)^{\frac{1}{2}}g(x)dx$$

$$+f(b)\frac{\lambda(1-\lambda)}{((\frac{\lambda}{1-\lambda})b-a)^3}\int_a^{(\frac{\lambda}{1-\lambda})b}((\frac{\lambda}{1-\lambda})b-x)^{\frac{1}{2}}(x-a)^{\frac{3}{2}}g(x)dx$$

$$\leq \frac{1}{2}\left(\frac{\lambda^3}{3}f(a)g(a) + \frac{(1-\lambda)^2}{3}f(b)g(b) + \frac{\lambda(1-\lambda)}{6}(f(a)g(b)+f(b)g(a))\right)$$

$$+\frac{2\lambda^2}{((\frac{\lambda}{1-\lambda})b-a)^3}\int_a^{(\frac{\lambda}{1-\lambda})b}((\frac{\lambda}{1-\lambda})b-x)(x-a)f(x)g(x)dx$$

Proof. Since f and g are $\lambda - MT$-convex, we can write

$$f\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)\le\frac{\sqrt{t}}{2\sqrt{1-t}}f(a)+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}f(b),$$

$$g\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)\le\frac{\sqrt{t}}{2\sqrt{1-t}}g(a)+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}g(b).$$

Using the basic inequality, $pq+rs\le ps+qr$, whenever $p\le s$ and $r\le q$ for any $p,q,r,s\in R^{+}$ we have

$$\begin{aligned}&f\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)\left(\frac{\sqrt{t}}{2\sqrt{1-t}}g(a)+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}g(b)\right)\\&+g\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)\left(\frac{\sqrt{t}}{2\sqrt{1-t}}f(a)+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}f(b)\right)\\&\le\left(\frac{\sqrt{t}}{2\sqrt{1-t}}f(a)+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}f(b)\right)\left(\frac{\sqrt{t}}{2\sqrt{1-t}}g(a)+\frac{(1-\lambda)\sqrt{1-t}}{2\lambda\sqrt{t}}g(b)\right)\\&+f\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)g\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right).\end{aligned}\tag{2.20}$$

This gives

$$\begin{aligned}&g(a)\frac{\sqrt{t}\sqrt{1-t}}{1-t}f\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)+g(b)\frac{(1-\lambda)\sqrt{t}\sqrt{1-t}}{\lambda t}f\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)\\&+f(a)\frac{\sqrt{t}\sqrt{1-t}}{1-t}g\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)+f(b)\frac{(1-\lambda)\sqrt{t}\sqrt{1-t}}{\lambda t}g\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)\\&\le\frac{1}{2}\left(\frac{t}{1-t}f(a)g(a)+\frac{(1-\lambda)}{\lambda}f(a)g(b)+\frac{(1-\lambda)}{\lambda}f(b)g(a)+\frac{(1-\lambda)^2(1-t)}{\lambda^2 t}f(b)g(b)\right)\\&+2f\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)g\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right).\end{aligned}\tag{2.21}$$

It can be eaily seen that (2.21) gives

$$\begin{aligned}&g(a)\lambda^2t\sqrt{t}\sqrt{1-t}f\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)+g(b)\lambda(1-\lambda)(1-t)\sqrt{t}\sqrt{1-t}f\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)\\&+f(a)\lambda^2t\sqrt{t}\sqrt{1-t}g\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)+f(b)\lambda(1-\lambda)(1-t)\sqrt{t}\sqrt{1-t}g\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)\\&\le\frac{1}{2}\left(\lambda^2t^2f(a)g(a)+\lambda(1-\lambda)t(1-t)(f(a)g(b)+f(b)g(a))+(1-\lambda)^2(1-t)^2f(b)g(b)\right)\\&+2\lambda^2t(1-t)\left(f\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)g\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)\right)\end{aligned}\tag{2.22}$$

Integrating both sides of (2.22) over $[0,1]$ wrt to t,

$$\begin{aligned}&g(a)\lambda^2\int_0^1 t\sqrt{t}\sqrt{1-t}f\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)dt\\&+g(b)\lambda(1-\lambda)\int_0^1(1-t)\sqrt{t}\sqrt{1-t}f\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)dt\\&+f(a)\lambda^2\int_0^1 t\sqrt{t}\sqrt{1-t}g\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)dt\\&+f(b)\lambda(1-\lambda)\int_0^1(1-t)\sqrt{t}\sqrt{1-t}g\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)dt\end{aligned}$$

$$\leq \frac{1}{2}\left(\lambda^2\int_0^1 t^2 dt f(a)g(a)+\lambda(1-\lambda)(f(a)g(b)+f(b)g(a))\right)$$
$$+\frac{1}{2}\left(\int_0^1 t(1-t)dt+(1-\lambda)^2 f(b)g(b)\int_0^1(1-t)^2 dt\right)$$
$$+2\lambda^2\int_0^1 t(1-t)f\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)g\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)dt \tag{2.23}$$

By using $x = ta+\frac{\lambda}{1-\lambda}(1-t)b,\ dx = \left(a-(\frac{\lambda}{1-\lambda})b\right)dt$ in (2.23), we obtain

$$g(a)\lambda^2\int_0^1 t\sqrt{t}\sqrt{1-t}f\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)dt$$
$$= g(a)\frac{\lambda^2}{((\frac{\lambda}{1-\lambda})b-a)^3}\int_a^{(\frac{\lambda}{1-\lambda})b}((\frac{\lambda}{1-\lambda})b-x)^{\frac{3}{2}}(x-a)^{\frac{1}{2}}f(x)dx,$$
$$g(b)\lambda(1-\lambda)\int_0^1(1-t)\sqrt{t}\sqrt{1-t}f\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)dt$$
$$= g(b)\frac{\lambda(1-\lambda)}{((\frac{\lambda}{1-\lambda})b-a)^3}\int_a^{(\frac{\lambda}{1-\lambda})b}((\frac{\lambda}{1-\lambda})b-x)^{\frac{1}{2}}(x-a)^{\frac{3}{2}}f(x)dx,$$
$$f(a)\lambda^2\int_0^1 t\sqrt{t}\sqrt{1-t}g\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)dt$$
$$= f(a)\frac{\lambda^2}{((\frac{\lambda}{1-\lambda})b-a)^3}\int_a^{(\frac{\lambda}{1-\lambda})b}((\frac{\lambda}{1-\lambda})b-x)^{\frac{3}{2}}(x-a)^{\frac{1}{2}}g(x)dx,$$
$$f(b)\lambda(1-\lambda)\int_0^1(1-t)\sqrt{t}\sqrt{1-t}g\left(ta+\frac{\lambda}{1-\lambda}(1-t)b\right)dt$$
$$= f(b)\frac{\lambda(1-\lambda)}{((\frac{\lambda}{1-\lambda})b-a)^3}\int_a^{(\frac{\lambda}{1-\lambda})b}((\frac{\lambda}{1-\lambda})b-x)^{\frac{1}{2}}(x-a)^{\frac{3}{2}}g(x)dx,$$
$$\int_0^1 t^2 dt = \int_0^1(1-t)^2 dt = \frac{1}{3}, \int_0^1 t(1-t)dt = \frac{1}{6}.$$

Hence, this completes the proof. □

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