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Bézier Curve with a Minimal Jerk Energy

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Abstract
We provide a method in order to determine a Bézier curve with a minimal jerk energy by means of associated matrices. By way of an application, we show that the unknown control points of the Bézier curve having a minimal jerk energy can be written as a linear combination of the known control points. Furthermore, for such a Bézier curve we obtain a general form of its matrix represention.

Keywords: *Bézier curve; minimal jerk energy; , control points.*

AMS Subject Classification (2010): *Primary: 53A04; 65D17.*

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1. Introduction

The notion of a Bézier curve was introduced by Pierre Bézier in 1960 in order to design a car body. Since then Bézier curves be come a common tool used in Computer Aided Geometric Design which were the subject of many works in the literature [5, 6, 8, 11, 12]. The most comprehensive information about Bézier curves and their geometry is available in the book by Farin [5], entitled "curves and surfaces for CAGD".

Geometric construction of curves with a minimal energy is one of the most important subject of Computer Aided Design, and it has been studied in many papers [1–3, 14, 17, 18]. In particular, Bézier curves with a minimal energy constitute a very interesting area of research. For instance, in Xu et al. [16] derive a necessary and sufficient condition on the control points for Bézier curves to have a minimal internal energy including stretch energy, strain energy and jerk energy. They propose geometric constructions of three kinds of Bézier curves with minimal internal energy, and then compare the corresponding three kinds of energy-minimizing Bézier curves. Eberly [4], obtains a relationship between the minimum bending energy and the degree elevation for Bézier curves, and finds unknowns of a cubic Bézier curve with the minimum bending energy by the aid of associated matrices. Following these works, we obtain that unknown control points of a Bézier curve with degree n having a minimal jerk energy can written as a linear combination of known control points.

Most fundamental descriptions and properties on Bézier curves needed throughout the sequel are stated in Section 2, where we also provide the definition of the jerk energy functional. The characterization of unknown control point of a Bézier curve with a minimal jerk energy in cubic case given in Section 3. By the help of associated matrices, we carry over such a characterization to the quartic case. Finally, we complete our characterization of unknown control points of any Bézier curve with degree n which has a minimal jerk energy following a similar methodology.

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2. Preliminaries

A Bézier curve of degree n is a parametric curve with control points $P_0, P_1, ..., P_n$, and it is expressed in terms of Bernstein polynomials given by

$$B_i^n(t) = \begin{pmatrix} n \\ i \end{pmatrix} (1-t)^{n-i} t^i,$$

where the binomial coefficients are

$$\begin{pmatrix} n \\ i \end{pmatrix} = \begin{cases} \frac{n!}{i!(n-i)!} & \text{if } 0 \leq i \leq n \\ 0 & \text{else.} \end{cases}$$

Therefore, a Bézier curve of degree n is explicitly defined by

$$\gamma(t) = \sum_{i=0}^{n} B_i^n(t) P_i, \qquad t \in [0,1]. \tag{2.1}$$

(see [5, 7]). In order to characterize any Bézier curve in terms of control points, it is necessary that the points P_0 and P_n must be initially determined; that is, the endpoints of the curve must be known in advance. By considering the energy as a function of the control points, we derive the necessary and sufficient condition on the control points for Bézier curves having a minimal jerk energy.

The curvature variation energy is given by

$$E_{cv}(P) = \int_0^{\ell} [\kappa'(s)]^2 \, dt,$$

where s is the arc parameter, ℓ is the arc length of $\gamma(t)$, and $\kappa(s)$ is the curvature of $\gamma(t)$. The previous nonlinear energy functional can be approximated by the jerk energy, where the jerk energy of a Bézier curve is defined as

$$E_{jerk}(\gamma) = \int_0^1 \left|\left|\gamma'''(t)\right|\right|^2 dt \tag{2.2}$$

(see [9, 10, 16]).

3. Bézier Curve with a Minimal Jerk Energy

In this section, we give the necessary condition satisfied by control points of cubic Bézier curve with a minimal jerk energy. From the equation (2.1) we obtain a cubic Bézier curve as follows

$$\gamma(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t)t^2 P_2 + t^3 P_3,$$

where P_0, P_2, P_3 are known control points, and P_1 is the unknown control point which will be determined in order that the corresponding cubic Bézier curve to have a minimal jerk energy. For that purpose we next state some derivatives of the Bézier curve;

$$\gamma'(t) = -3(1-t)^2 P_0 + 3(1-t)(1-3t)P_1 + 3t(2-3t)P_2 + 3t^2 P_3,$$

$$\gamma''(t) = 6((1-t)P_0 + (3t-2)P_1 + (1-3t)P_2 + tP_3)$$

and

$$\gamma'''(t) = 6(-P_0 + 3P_1 - 3P_2 + P_3).$$

By using the third derivative of the Bézier curve we obtain the jerk energy functional (2.2) in the following form

$$E_{jerk}(\gamma) = 36 \left|\left|-P_0 + 3P_1 - 3P_2 + P_3\right|\right|^2.$$

It follows that such a Bézier curve to have a minimal jerk energy, the control points should satisfy the equation

$$-P_0 + 3P_1 - 3P_2 + P_3 = 0,$$

from which we conclude the following proposition.

Proposition 3.1. *Let P_0, P_2 and P_3 be the known control points of a cubic Bézier curve. The unknown control point P_1 can be constructed in the form*

$$P_1 = \frac{3P_2 + P_0 - P_3}{3}.$$

Therefore the corresponding cubic Bézier curve has a minimal jerk energy, and the curve reduces to a quadratic Bézier curve.

Example 3.1. If the known control points of a Bézier curve are $P_0 = (0,0)$, $P_2 = (2,3)$ and $P_3 = (4,5)$, then the unknown control point P_1 of the Bézier curve with a minimal jerk energy is obtained as

$$P_1 = \left(\frac{2}{3}, \frac{4}{3}\right).$$

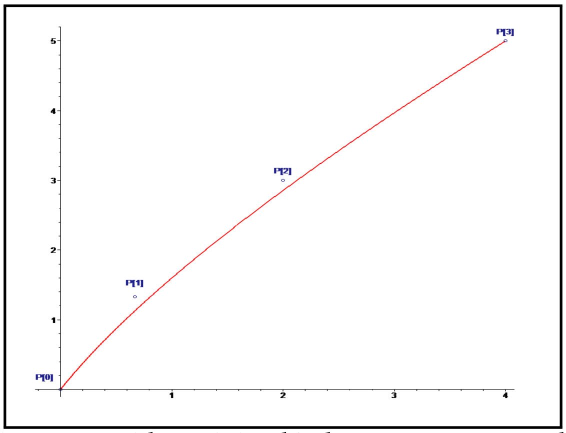

Figure 1. The Bézier curve with a minimal jerk energy corresponding Example 3.1.

Now we consider a quartic Bézier curve given by

$$\gamma(t) = (1-t)^4 P_0 + 4(1-t)^3 t P_1 + 6(1-t)^2 t^2 P_2 + 4(1-t)t^3 P_3 + t^4 P_4,$$

where P_0, P_3 and P_4 are known control points, P_1 and P_2 are the unknown control points which will be determined in order that the corresponding quartic Bézier curve to have a minimal jerk energy. The first, second and third derivatives of the Bézier curve are obtained as follows

$$\gamma'(t) = -4(1-t)^3 P_0 + 4(1-t)^2(1-4t)P_1 + 12(1-t)(1-2t)tP_2 + 4t^2(-4t+3)P_3 + 4t^3 P_4,$$

$$\gamma''(t) = 12((1-t)^2 P_0 + 2(1-t)(2t-1)P_1 + (6t^2 - 6t + 1)P_2 + 2t(1-2t)P_3 + t^2 P_4)$$

and

$$\gamma'''(t) = 24((t-1)P_0 + (3-4t)P_1 + (6t-3)P_2 + (1-4t)P_3 + tP_4),$$

respectively. Then the integrand of the jerk energy functional can be expressed with the help of a matrix;

$$\|\gamma'''(t)\|^2 = 576 P^T \begin{bmatrix} t-1 \\ 3-4t \\ 6t-3 \\ 1-4t \\ t \end{bmatrix} \begin{bmatrix} t-1 & 3-4t & 6t-3 & 1-4t & t \end{bmatrix} P,$$

where P is the matrix with 5 columns and 1 row given by

$$P = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}.$$

By integrating of the matrix terms, we obtain the jerk energy

$$E_{jerk}(\gamma) = 96P^T MP,$$

where M is the following symmetric matrix

$$M = \begin{bmatrix} 2 & -5 & 3 & 1 & -1 \\ -5 & 14 & -12 & 2 & 1 \\ 3 & -12 & 18 & -12 & 3 \\ 1 & 2 & -12 & 14 & -5 \\ -1 & 1 & 3 & -5 & 2 \end{bmatrix}.$$

Obviously the eigenvalues of M with multiplicity 3 are $35, 15$ and 0. For the eigenvalue 35, the corresponding eigenspace is $1-$dimensional, and it is spanned by the unit-length eigenvector $V_0 = (1, -4, 6, -4, 1)/\sqrt{70}$. On the other hand, for the eigenvalue 15, the eigenspace is again $1-$dimensional, and it is spanned by the unit-length eigenvector $V_1 = (-1, 2, 0, -2, 1)/\sqrt{10}$. Finally for the eigenvalue 0, the resulting eigenspace is $3-$dimensional, and it is spanned by the unit-length eigenvectors $V_2 = (3, 1, 0, 0, 1)/\sqrt{11}$, $V_3 = (-2, 6, 9, 7, 0)/\sqrt{170}$ and $V_4 = (24, 30, 11, -33, -102)/\sqrt{13090}$. Therefore $\{V_0, V_1, V_2, V_3, V_4\}$ is the orthonormal set and we can write the eigenvectors as a rotation matrix

$$R = \begin{bmatrix} V_0 & V_1 & V_2 & V_3 & V_4 \end{bmatrix}. \tag{3.1}$$

By using the diagonal matrix $D = diag(35, 15, 0, 0, 0)$, the symmetric matrix M can be written as $M = RDR^T$. On the other hand, we may define

$$\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = Q = R^T P = \frac{1}{130900} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -1 & 2 & 0 & -2 & 1 \\ 3 & 1 & 0 & 0 & 1 \\ 6 & 3 & 1 & 0 & 0 \\ -8 & -3 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}$$

by means of (3.1). Thus we compute the jerk energy of the curve as

$$\begin{aligned} E_{jerk}(\gamma) &= 96P^T MP = 96P^T RDR^T P = 96Q^T DQ \\ &= 3350 \|Q_0\|^2 + 1440 \|Q_1\|^2 . \end{aligned} \tag{3.2}$$

When Q_0 and Q_1 are equal to zero, the jerk energy (3.2) will be minimal which in turn forces

$$\begin{aligned} P_0 - 4P_1 + 6P_2 - 4P_3 + P_4 &= 0 \\ -P_0 + 2P_1 - 2P_3 + P_4 &= 0, \end{aligned}$$

from which we deduced the following proposition.

Proposition 3.2. *Let P_0, P_3 and P_4 are the known control points of a Bézier curve. The unknown control points P_1 and P_2 can be constructed as*

$$P_1 = \frac{P_0 + 2P_3 - P_4}{2} \text{ and } P_2 = \frac{P_0 + 8P_3 - 3P_4}{6}.$$

Therefore the corresponding quartic Bézier curve has a minimal jerk energy, and the curve reduces to quadratic Bézier curve.

Example 3.2. Assume that $P_0 = (2, 1)$, $P_3 = (3, 7)$ and $P_4 = (9, 5)$ are the known control points of a Bézier curve. Then the unknown control points P_1 and P_2 can be computed as $P_1 = \left(-\frac{1}{2}, 5\right)$ and $P_2 = \left(-\frac{1}{6}, 7\right)$ respectively. Thus, the Bézier curve has a minimal jerk energy.

4. Determination of unknown control points in degree n

In this final section, we generalize our results in cubic and quartic cases to Bézier curves of degree n. In order to achieve that we employ permutation matrices into our calculations that allows us to rearrange the positions of the unknown control points.

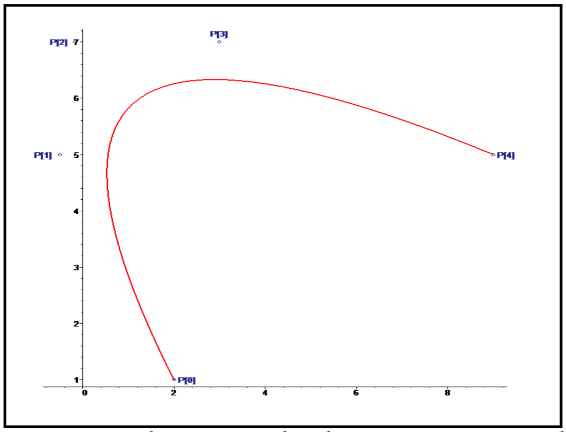

Figure 2. The Bézier curve with a minimal jerk energy corresponding Example 3.2.

Now, we assume that P_0, P_2, P_3 and P_4 are the known control points of a quartic Bézier curve. Then, we will find unknown control point P_1 such that the Bézier curve has a minimal jerk energy. Using the matrix P and the permutation matrix J given by

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

we define the matrix $\widehat{P}$ for which its first row is the unknown control point P_1. So,

$$\widehat{P} = JP = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_0 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}.$$

Moreover, if we set $\widehat{M} = JMJ^T$, then the jerk energy functional can be expressed by

$$\frac{1}{4}E_{jerk}(\gamma) = \widehat{P}^T \widehat{M} \widehat{P} = \widehat{P}^T \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \widehat{P}, \tag{4.1}$$

where A is the $1 \times 1-$matrix, B is the $1 \times 4-$matrix and C is the $4 \times 4-$matrix. Let U be the first row of the matrix $\widehat{P}$ and K be the remaining last three rows of $\widehat{P}$. As a result of algebraic operations on these matrices, we may reduce the equation (4.1) to

$$\frac{1}{4}E_{jerk}(\gamma) = U^T AU + 2K^T B^T U + K^T CK.$$

When the gradient of the jerk energy $E_{jerk}(\gamma)$, which is a function of U, is zero, i.e,

$$AU + BK = 0, \tag{4.2}$$

then the jerk energy $E_{jerk}(\gamma)$ becomes minimal. Since A is invertible, we can solve the linear system (4.2) to obtain the unknown control point P_1. Indeed, from M and J, we get

$$\widehat{M} = \begin{bmatrix} 14 & -5 & -12 & 2 & 1 \\ -5 & 2 & 3 & 1 & -1 \\ -12 & 3 & 18 & -12 & 3 \\ 2 & 1 & -12 & 14 & -5 \\ 1 & -1 & 3 & -5 & 2 \end{bmatrix}.$$

By using the equation (4.2), we find the unknown control point in the following from

$$P_1 = \frac{1}{14}\left(5P_0 + 12P_2 - 2P_3 - P_4\right). \tag{4.3}$$

Example 4.1. Let $P_0 = (1,4)$, $P_2 = (2,-2)$, $P_3 = (4,3)$ and $P_4 = (7,1)$ be the known control points of a Bézier curve. From the equation (4.3), we can find

$$P_1 = \left(1, -\frac{11}{14}\right)$$

as the unknown control point of the Bézier curve with a minimal jerk energy.

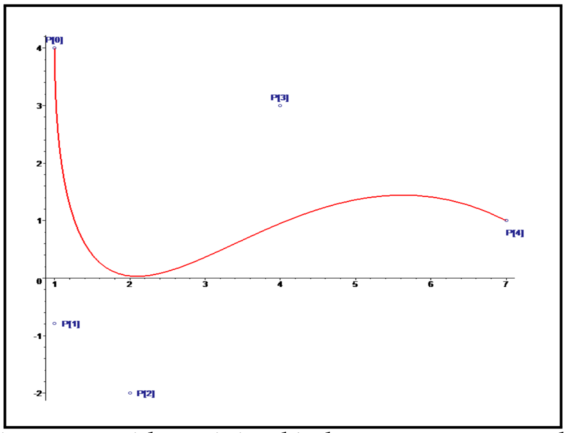

Figure 3. The Bézier curve with a minimal jerk energy corresponding Example 4.1.

In the case where there are two unknown control points, we consider a similar approach. So suppose that P_0, P_3, P_4 are the known control points of a quartic Bézier curve in which P_1 and P_2 are unknown control points. Then the corresponding Bézier curve has the jerk energy functional given by

$$\frac{1}{4}E_{jerk}(\gamma) = \widehat{P}^T \widehat{M} \widehat{P} = \widehat{P}^T \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \widehat{P}, \tag{4.4}$$

where A is the $2\times 2-$matrix, B is the $2\times 3-$matrix and C is the $3\times 3-$matrix. Once again, let U be the first two rows of $\widehat{P}$ and K be the remaining last three rows of $\widehat{P}$. From the equation (4.4), we have that

$$\frac{1}{4}E_{jerk}(\gamma) = U^T A U + 2K^T B^T U + K^T C K.$$

When the gradient of the jerk energy $E_{jerk}(\gamma)$ is zero, that is,

$$AU + BK = 0, \tag{4.5}$$

the Bézier curve has a minimal jerk energy. The solution of the equation (4.5) is obvious, since A is invertible. When we employ the permutation matrix

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

we deduce that

$$\widehat{P} = \begin{bmatrix} P_1 \\ P_2 \\ P_0 \\ P_3 \\ P_4 \end{bmatrix}, \; U = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \; K = \begin{bmatrix} P_0 \\ P_3 \\ P_4 \end{bmatrix}$$

and

$$\widehat{M} = \begin{bmatrix} 14 & -12 & -5 & 2 & 1 \\ -12 & 18 & 3 & -12 & 3 \\ -5 & 3 & 2 & 1 & -1 \\ 2 & -12 & 1 & 14 & -5 \\ 1 & 3 & -1 & -5 & 2 \end{bmatrix}.$$

By using the equation (4.5), we have

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 14 & -12 \\ -12 & 18 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} -5 & 2 & 1 \\ 3 & -12 & 3 \end{bmatrix} \begin{bmatrix} P_0 \\ P_3 \\ P_4 \end{bmatrix}.$$

It then follows that the equations

$$\begin{aligned} P_1 - 12P_2 - 5P_0 + 2P_3 + P_4 &= 0 \\ -12P_1 + 18P_2 + 3P_0 - 12P_3 + 3P_4 &= 0 \end{aligned}$$

must hold. Therefore, we conclude that

$$P_1 = \frac{P_0 + 2P_3 - P_4}{2} \text{ and } P_2 = \frac{P_0 + 8P_3 - 3P_4}{6},$$

as in the case of Proposition 3.2.

Example 4.2. Let $P_0 = (3, 5)$, $P_3 = (4, 11)$ and $P_4 = (9, 3)$ be the known control points of a Bézier curve. Then, the unknowns control point of the Bézier curve with a minimal jerk energy are

$$P_1 = (1, 12) \text{ and } P_2 = \left(\frac{4}{3}, \frac{42}{3}\right).$$

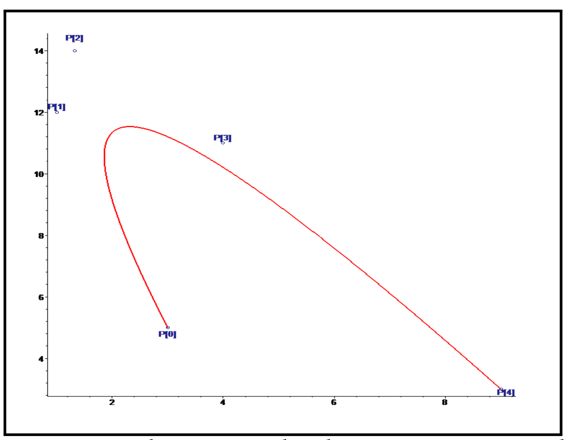

Figure 4. The Bézier curve with a minimal jerk energy corresponding Example 4.2.

Now, we reach the general characterization of unknown control points of a Bézier curve with degree n which has a minimal jerk energy.

Theorem 4.1. *Let $\gamma(t)$ be a Bézier curve with degree n. If p and q are the number of the unknown control points and the known control points respectively, then the unknown control points of the Bézier curve with a minimal jerk energy are the solutions of the equation system*

$$AU + BK = 0,$$

such that

$$J = \begin{pmatrix} L & O \\ O^T & I \end{pmatrix}$$

and

$$\widehat{M}=\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}=JMJ^T,$$

where

$$L=(\ell_{ij})=\begin{cases} 1, & i+1=j \\ 1, & i=p+1 \text{ and } j=1 \\ 0, & \text{the other case} \end{cases}, \quad 1\le i,j\le p+1,$$

is the $(p+1)\times(p+1)-$matrix, O is a zero matrix with $(p+1)-$columns and $(q-1)-$rows, I is the unit matrix $(q-1)\times(q-1)$, A is the $p\times p-$matrix, B is the $p\times q-$matrix, C is the $q\times q-$matrix, U is the $p\times 1-$matrix of the unknown control points and K is the $q\times 1-$matrix of the known control points. In addition, for $n\ge 3$ and $0\le i,j\le n$, the entry $m_{(i+1)(j+1)}$ of the matrix M is

$$m_{(i+1)(j+1)}=\frac{(n!)^2}{(2n-5)!}\sum_{k,l=0}^{3}\frac{(-1)^{k+l}\binom{3}{k}\binom{3}{l}(2n-i-j-k-l)!(i+j+k+l-6)!}{(i+k-3)!(j+l-3)!(n-i-k)!(n-j-l)!}.$$

Proof. Let

$$P=\begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_n \end{pmatrix}$$

be the matrix of control points of a Bézier curve with degree n. In order to locate the unknown control points in to initial rows of the matrix P, we define the matrix J in a way that

$$\widehat{P}=\begin{pmatrix} P_1 \\ \vdots \\ P_p \\ P_0 \\ \vdots \\ P_n \end{pmatrix}=\begin{pmatrix} L & O \\ O^T & I \end{pmatrix}\begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_n \end{pmatrix}=JP.$$

If we define the matrix $\widehat{M}=JMJ^T$, the jerk energy functional is given by

$$\frac{1}{4}E_{jerk}(\gamma)=\widehat{P}^T\widehat{M}\widehat{P}=\widehat{P}^T\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}\widehat{P},$$

where A is the $p\times p-$matrix, B is the $p\times q-$matrix, C is the $q\times q-$matrix, and the matrix $\widehat{P}$ consists of the $p\times 1-$matrix U of the unknown control points and the $q\times 1-$matrix K of the known control points. By using the some algebraic calculus, we obtain that

$$\frac{1}{4}E_{jerk}(\gamma)=U^TAU+2K^TB^TU+K^TCK. \tag{4.6}$$

When the gradient of the jerk energy functional (4.6) is zero, i.e.,

$$AU+BK=0,$$

then the jerk energy is minimal. Thus, there is only one solution of the linear equation system depending on U, if A is an invertible matrix. If A is not an invertible matrix, then there are infinite solutions of this linear equation system. So, we have the unknown control points of a Bézier curve with degree n which has a minimal jerk energy. □

For instance, if P_0, P_4, P_5 are the known control points and P_1, P_2, P_3 are the unknown control points of a Bézier curve with degree 5, we may obtain unknown control points by using the above construction so that the Bézier curve has a minimal jerk energy. Since

$$M=\begin{pmatrix} 6 & -15 & 0 & 0 & 0 & -1 \\ -15 & 40 & -30 & 0 & 5 & 0 \\ 0 & -30 & 30 & -10 & 0 & 0 \\ 0 & 0 & -10 & 30 & 30 & 10 \\ 0 & 5 & 0 & 30 & 40 & -15 \\ -1 & 0 & 0 & 10 & -15 & 6 \end{pmatrix}$$

and

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

for the Bézier curve with degree 5, we get

$$\widehat{M} = \begin{pmatrix} 40 & -30 & 0 & -15 & 5 & 0 \\ -30 & 30 & -10 & 0 & 0 & 0 \\ 0 & -10 & 30 & 0 & 30 & 10 \\ -15 & 0 & 0 & 6 & 0 & -1 \\ 5 & 0 & 30 & 0 & 40 & -15 \\ 0 & 0 & 10 & -1 & -15 & 6 \end{pmatrix}.$$

So,

$$A = \begin{pmatrix} 40 & -30 & 0 \\ -30 & 30 & -10 \\ 0 & -10 & 30 \end{pmatrix}, \quad B = \begin{pmatrix} -15 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 30 & 10 \end{pmatrix},$$

from which we obtain the unknown control points as

$$P_1 = \frac{96P_0 - 40P_4 + 24P_5}{40}, \; P_2 = \frac{-27P_0 + 33P_4 + 8P_5}{10} \text{ and } P_3 = \frac{-9P_0 + 21P_4 + 6P_5}{10}.$$

Example 4.3. If the known control points of a Bézier curve are $P_0 = (-1, 2)$, $P_4 = (8, 2)$ and $P_5 = (11, 17)$, then the unknowns control point of the Bézier curve with a minimal jerk energy can be given

$$P_1 = \left(-\frac{19}{5}, 13\right), P_2 = \left(\frac{379}{10}, \frac{74}{5}\right) \text{ and } P_3 = \left(\frac{243}{10}, \frac{63}{5}\right).$$

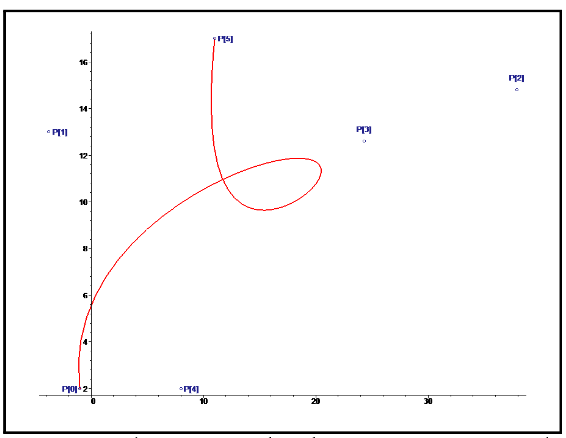

Figure 5. The Bézier curve with a minimal jerk energy corresponding to Example 4.3.

5. Acknowledgment

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