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Optimal Consumption and Investment for Exponential Utility Function

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Abstract
We investigate an optimal consumption and investment problem for Black-Scholes type financial market on the whole investment interval $[0, T]$. We formulate various utility maximization problem, which can be solved explicitly. The method of solution uses the convex dual function (Legendre transform) of the utility function. Related to this concept, we introduce and study the convex dual of the value function for our problem.

Keywords: *Portfolio optimization; consumption; exponential utility; Convex duality.*

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1. Introduction

One of the principal questions in mathematical finance is the optimal investment consumption problem for continuous time market models. By applying results from stochastic control theory, explicit solutions have been obtained for some special cases (see e.g. Karatzas and Shreve [1], Korn [2] and references therein). Kluppelberg and Pergamenshchikov [3] considered the optimal investment/consumption problem with uniform risk limits throughout the investment horizon for power utility functions. In this paper, we investigate the optimal investment/consumption problem for exponential utility functions over the whole investment horizon $[0, T]$. Using a new approach, called martingale method of convex duality, we find all optimal solutions in explicit form. Our paper is organized as follows. Section 2 and 3 describe the market model and the set of consumption and portfolio processes from which the investor in this market is free to choose. Section 4 introduces the notion of utility function. We allow these functions to take the value $-\infty$ on a half-line extending to $-\infty$, which effectively places a lower constraint on consumption and/or wealth. Section 5 solves the problem of an agent who seeks to maximize expected utility from consumption and terminal wealth. The method of solution uses the convex dual function (Legendre transform) of the utility function. Related to this concept, we introduce and study the convex dual of the value function for the problem of Section 6. In Section 7, we present our main results.

2. The model

We consider a Black-Scholes type financial market consisting of one riskless bond and several risky stocks on the interval $[0, T]$. Their respective prices $(S_0(t))_{t\geq 0}$ and $(S_i(t))_{t\geq 0}$ for $i = 1, ..., d$ evolve according to the equation:

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, & S_0(0) = 1, \\ dS_i(t) = S_i(t)\mu_i(t)dt + S_i(t)\displaystyle\sum_{j=1}^{d} \sigma_{ij} dW_j(t), & S_i(0) = s_i > 0. \end{cases}$$

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Here $W(t) = (W_1(t), ..., W_d(t))'$ is a standard d-dimensional Brownian motion in $\mathbb{R}^d$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$; $r(t) \in \mathbb{R}$ is the risk-free rate process satisfying $\int_0^T |r(t)|dt < \infty$ almost surely, $\mu(t) = (\mu_1(t), ..., \mu_d(t))' \in \mathbb{R}^d$ is the vector of stock-appreciation rates and $\sigma(t) = (\sigma_{ij}(t))_{1\le i,j\le d}$ is the matrix of stock-volatilities satisfying $\sum_{n=1}^{N}\sum_{d=1}^{D}\int_0^T \sigma_{nd}^2(t)dt < \infty$ a.s. We also assume that the matrix $\sigma(t)$ is nonsingular for Lebesgue-almost all $t \ge 0$. We denote by $\mathcal{F}_t = \sigma\{W_s, s \le t\}, t \ge 0$, the filtration generated by the Brownian motion (augmented by the null sets). Furthermore, $|.|$ denotes the Euclidean norm for vectors and the corresponding matrix norm for matrices and prime denotes the transposed. We then introduce the martingale

$$\theta(t) := \sigma(t)^{-1}(\mu(t) - r(t)1) \quad \text{with } \ 1 = (1, ..., 1)' \in \mathbb{R}^d,$$

$$Z_0(t) := \exp\left\{-\int_0^t \theta'(s)dW_s - \frac{1}{2}\int_0^t \parallel \theta(s) \parallel^2 ds\right\}.$$

For a standard market, we define the standard martingale measure $\mathbf{P}_0$ on $\mathcal{F}_T$ by

$$\mathbf{P}_0(A) := \mathbf{E}\left[Z_0(T)\mathbb{1}_A\right], \ \forall A \in \mathcal{F}_T,$$

We say that $\mathbf{P}_0$ and $\mathbf{P}$ are equivalent on $\mathcal{F}_T$ and the drifted Brownian motion

$$W_0(t) := W(t) + \int_0^t \theta(s)ds, \ \ 0 \le t \le T.$$

According to Girsanov theorem, W_0 is a standard Brownian motion under $\mathbf{P}_0$ and

$$H_0(t) := \frac{Z_0(t)}{S_0(t)}.$$

For this model, the following condition will be imposed.

Assumption 2.1. The state price density process H_0 satisfies

$$\mathbf{E}\left[\int_0^T H_0(t)dt + H_0(T)\right] < +\infty.$$

A sufficient condition for these assumption is that $S_0(\cdot)$ be bounded away from zero on $[0, T]$, so that $H_0(\cdot)$ is bounded from above by a constant times the nonnegative supermartingale $Z_0(\cdot)$.

3. Portfolio and consumption processes

Definition 3.1. A portfolio process $\pi(\cdot) = (\pi_1(\cdot), ..., \pi_d(\cdot))^T$ is a measurable, $\{\mathcal{F}(t)\}$-adapted, $\mathbb{R}^d$-valued process satisfying $\int_0^T \|\pi(t)\|^2 dt < \infty$ a.s. A consumption process is an $\{\mathcal{F}(t)\}$ progressively measurable, nonnegative process $c(\cdot)$ satisfying $\int_0^T c(\cdot)dt < \infty$, almost surely.

The set of all consumption/portfolio process pairs which are admissible for x will be denoted by $\mathcal{A}$.

Definition 3.2. Given $x \ge 0$, we say that a consumption and portfolio process pair (c, π) is admissible at x, and write $(c, \pi) \in \mathcal{A}(x)$, if the wealth process $X^{x,c,\pi}(\cdot)$ corresponding to $x, \ c, \ \pi$ satisfies

$$X^{x,c,\pi}(t) \ge 0, \ 0 \le t \le T, \ \text{a.s.}$$

For $x < 0$, we set $\mathcal{A}(x) = \emptyset$.

Theorem 3.1. *[1] Let $x \ge 0$ be given, let $c(\cdot)$ be a consumption process, and let ξ be a nonnegative, $\mathcal{F}(T)$-measurable random variable such that*

$$\mathbf{E}\left[\int_0^T H_0(u)c(u)du + H_0(T)\xi\right] = x. \tag{3.1}$$

Then there exists a portfolio process $\pi(\cdot)$ such that the pair (c, π) is admissible at x and $\xi = X^{x,c,\pi}(T)$.

4. Utility functions

We desire to maximize our utility. In this section, we develop the properties of the utility functions that we consider. We also introduce the *convex dual* of an utility function.

Definition 4.1. An utility function is a concave, nondecreasing, upper semicontinuous function $U : \mathbb{R} \to [-\infty, \infty)$ satisfying:

i) the half-line $dom(U) := \{x \in \mathbb{R};\ U(x) > -\infty\}$ is a nonempty subset of $[0, \infty)$;

ii) U' is continuous, positive, and strictly decreasing on the interior of $dom(U)$, and

$$U'(\infty) := \lim_{x\to\infty} U'(x) = 0.$$

We set

$$\bar{x} := \inf\{x \in \mathbb{R}; U(x) > -\infty\}$$

so that $\bar{x} \in [0, \infty)$ and either $dom(U) = [\bar{x}, \infty)$ or $dom(U) = (\bar{x}, \infty)$.
We define

$$U'(\bar{x}+) := \lim_{x\to\bar{x}} U'(x),$$

so that $U'(\bar{x}+) \in (0, \infty]$.
In this work, we choose $U_1(t, x) = U_2(x) = 1 - \exp(-x)$ and set

$$U(x) := \begin{cases} 1 - \exp(-x) & x > 0 \\ 0 & x = 0 \\ -\infty & x < 0 \end{cases}$$

The Arrow-Pratt index of risk aversion, $-\dfrac{U''(x)}{U'(x)} = 1.$
We denote by $U'(t, x)$ the derivative of U with respect to its second variable, and we denote by $I(t, \cdot)$ the inverse of $U'(t, \cdot)$.

Formulation of the dual problem.

Definition 4.2. Let U be an utility function. The convex dual of U is the convex function

$$\tilde{U}(y) := \sup_{x\in\mathbb{R}} \{U(x) - xy\}, \quad \forall y \in \mathbb{R}. \tag{4.1}$$

Except for the presence of some minus signs, $\tilde{U}(y)$ is the Legendre-Fenchel transform of U (Rockafellar [4], Ekeland and Temam [5]). Indeed, if we define the convex function

$$f(x) := -U(x), \ \ x \in \mathbb{R},$$

then the Legendre-Fenchel transform of f is

$$f^*(y) := \sup_{x\in\mathbb{R}} \{xy - f(x)\} = \tilde{U}(-y), \ \ y \in \mathbb{R}.$$

Lemma 4.1. *[1] Let $\tilde{U}$ be the convex dual of U. Then $\tilde{U} : \mathbb{R} \to (-\infty, \infty]$ is convex, nonincreasing, lower semicontinuous, and satisfies*

i)

$$\tilde{U}(y) = \begin{cases} U(I(y)) - yI(y), & y > 0, \\ U(\infty) := \lim_{x\to\infty} U(x), & y = 0, \\ \infty, & y < 0. \end{cases}$$

ii) The derivative $\tilde{U}'$ is defined, continuous, and nondecreasing on $(0, \infty)$, and

$$\tilde{U}'(y) = -I(y), \ \ 0 < y < \infty.$$

iii) For all $x \in \mathbb{R}$,

$$U(x) = \inf_{y \in \mathbb{R}} \left\{ \tilde{U}(y) + xy \right\}.$$

iv) For fixed $x \in (\bar{x}, \infty)$, *the function* $y \mapsto \tilde{U}(y) + xy$ *is uniquely minimized over* $\mathbb{R}$ *by* $y = U'(x)$; *i.e.,*

$$U(x) = \tilde{U}(U'(x)) + xU'(x).$$

5. The optimization problem

Definition 5.1. A (time-separable, von Neumann-Morgenstern) preference structure is a pair of functions $U_1 : [0,T] \times \mathbb{R} \to [-\infty, \infty)$ and $U_2 : \mathbb{R} \to [-\infty, \infty)$ as described below:

(i) For each $t \in [0,T]$, $U_1(t, \cdot)$ is a utility function, and the subsistence consumption

$$\bar{c}(t) := \inf \{c \in \mathbb{R}; U_1(t,c) > -\infty\}, \ \ 0 \leq t \leq T,$$

is a continuouse function of t, with values in $[0, \infty)$;

(ii) U_1 and U_1' (where the prime denotes differentiation with respect to the second argument) are continuous on the set

$$D_1 := \{(t,c) \in [0,T] \times (0,\infty);\ c > \bar{c}(t)\};$$

(iii) U_2 is an utility function, with subsistence terminal wealth defined by

$$\bar{x} := \inf \{x \in \mathbb{R};\ U_2(x) > -\infty\}.$$

Let an agent have an initial endowment $x \in \mathbb{R}$ and a preference structure (U_1, U_2). The agent can consider the problem whose elements of control are the admissible consumption and portfolio processes in $\mathcal{A}(x)$ of Definition 3.2.

Problem: Find an optimal pair $(c, \pi) \in \mathcal{A}(x)$ for the problem

$$V(x) := \sup_{(c,\pi) \in \mathcal{A}(x)} \mathbf{E} \left[\int_0^T U_1(t, c(t))dt + U_2(X^{x,c,\pi}(T)) \right] \tag{5.1}$$

of maximizing expected total utility from both consumption and terminal wealth.
In this work we choose $U_1(t,x) = U_2(x) = 1 - \exp(-x)$.
Because of the strict concavity of $U_1(t,\cdot)$ and U_2, if such a pair exists, the consumption process component $c(\cdot)$ and the corresponding terminal wealth $X^{x,c,\pi}(T)$ are uniquely determined (see Xu [6], theorem 2.4.5). Our goal is to compute the value function V of this problem and to characterize optimal pair (c, π) that attain the suprema in (5.1).

Remark 5.1. (i) Because $\bar{c}(\cdot)$ is continuous, there exists a finite number $\hat{c}$ such that $\hat{c} > (\bar{x} \vee \max_{0 \leq t \leq T} \bar{c}(t))$. From the continuity of U_1 on $D_1 \supset [0,T] \times [\hat{c}, \infty)$, we have

$$\int_0^T \mid 1 - \exp(-\hat{c}(t)) \mid dt + \mid 1 - \exp(-\hat{c}(T)) \mid < \infty$$

Furthermore, under the Assumption 2.1, the quantity

$$\mathcal{X}(\infty) := \mathbf{E} \left[\int_0^T H_0(t)\bar{c}(t)dt + H_0(T)\bar{x} \right], \tag{5.2}$$

is finite.

(ii) For our Problem, we must have initial wealth at least $\mathcal{X}(\infty)$ in order to avoid expected utility of $-\infty$. Indeed, for this problem, the preference structure forces the constraints

$$c(t) \geq \bar{c}(t), \ \ \text{a.e.} t \in [0,T], \tag{5.3}$$

$$X^{x,c,\pi}(T) \geq \bar{x}, \text{ a.e.} \tag{5.4}$$

For otherwise $\mathbf{E}\left[\int_0^T 1-\exp(-c(t))dt + 1 - \exp(-X^{x,c,\pi}(T))\right]$ would be $-\infty$. But (5.3), (5.4), and (5.2) imply

$$\mathbf{E}\left[\int_0^T H_0(t)c(t)dt + H_0(T)X^{x,c,\pi}(T)\right] \geq \mathcal{X}(\infty) \tag{5.5}$$

For $x = \mathcal{X}(\infty)$, any $(c,\pi) \in \mathcal{A}(x)$ satisfying (5.3), (5.4) must actually satisfy $c(t) = \bar{c}(t)$, $X^{x,c,\pi}(T) = \bar{x}$. According to Theorem 3.1 there is in fact a portfolio process $\bar{\pi}$ for which $X^{\mathcal{X}(\infty),\bar{c},\bar{\pi}}(T) = \bar{x}$, and we conclude that

$$V(x) = \begin{cases} \int_0^T 1-\exp(-\bar{c}(t))dt + 1 - \exp(-\bar{x}), & x = \mathcal{X}(\infty) \\ -\infty, & x < \mathcal{X}(\infty). \end{cases} \tag{5.6}$$

6. Utility from consumption and terminal wealth

We define the function

$$\mathcal{X}(y) := \mathbf{E}\left[\int_0^T H_0(t)I(t, yH_0(t))dt + H_0(T)I(yH_0(T))\right], \; 0 < y < \infty. \tag{6.1}$$

Assumption 6.1. $\mathcal{X}(y) < \infty, \; \forall y \in (0,\infty)$.

Lemma 6.1. *[1] Under Assumption 6.1, the function $\mathcal{X}$ is nonincreasing and continuous on $(0,\infty)$, and strictly decreasing on $(0,r)$, where $\mathcal{X}(0+) := \lim_{y\to 0} \mathcal{X}(y) = \infty$ and $\mathcal{X}(\infty) := \lim_{y\to\infty} \mathcal{X}(y)$ is given by (5.2), and*

$$r := \sup\{y > 0;\; \mathcal{X}(y) > \mathcal{X}(\infty)\} > 0. \tag{6.2}$$

In particular, the function $\mathcal{X}$ restricted to $(0,r)$ has a strictly decreasing inverse function $\mathcal{Y} : (\mathcal{X}(\infty),\infty) \to (0,r)$, so that

$$\mathcal{X}(\mathcal{Y}(x)) = x, \;\; \forall x \in (\mathcal{X}(\infty),\infty) \tag{6.3}$$

We only need to consider initial wealth x in the domain $(\mathcal{X}(\infty),\infty)$ of $\mathcal{Y}(\cdot)$. For such an x, we know from budget constraint and Theorem 3.1 that our problem amounts to maximizing $\mathbf{E}\left[\int_0^T (1-\exp(-c(t)))dt + (1-\exp(-\xi))\right]$ over pairs (c,ξ), consisting of a consumption process $c(.)$ and a nonnegative $\mathcal{F}(T)$-measurable random variable ξ, that satisfy the budget constraint, namely, $\mathbf{E}\left[\int_0^T H_0(t)c(t)dt + H_0(T)\xi\right] \leq x$. Now, if $y > 0$ is a "Lagrange multiplier" that enforces this constraint, the problem reduces to the unconstrained maximization of
$\mathbf{E}\left[\int_0^T (1-\exp(-c(t)))dt + (1-\exp(-\xi))\right] + y\left(x - \mathbf{E}\left[\int_0^T H_0(t)c(t)dt + H_0(T)\xi\right]\right)$.
But this expression is

$$xy + \mathbf{E}\left[\int_0^T (1-\exp(-c(t))) - yH_0(t)c(t)\right]dt + \mathbf{E}\left[(1-\exp(-\xi)) - yH_0(T)\xi\right]$$
$$\leq xy + \mathbf{E}\left[\int_0^T \tilde{U}_1(t, yH_0(t))dt + \tilde{U}_2(yH_0(T))\right],$$

(where $\tilde{U}_1 = \tilde{U}_2 := \sup_{x\in\mathbb{R}}\{(1-\exp(-x)) - xy\}, \;\; \forall y \in \mathbb{R}$.)
with equality if and only if

$$c(t) = I(t, yH_0(t)), \;\; 0 \leq t \leq T \;\text{ and }\; \xi = I(yH_0(T)).$$

(recall (4.1) and Lemma (4.1i)). Quite clearly, $y = \mathcal{Y}(x)$ is the only value of $y > 0$ for which the above pair (c,ξ) satisfies the budget constraint with equality. Thus, for every $x \in (\mathcal{X}(\infty),\infty)$, we are led to the **candidate optimal terminal wealth**

$$\xi := I(\mathcal{Y}(x)H_0(T)) \tag{6.4}$$

and the **candidate optimal consumption process**

$$c(t) := I(t, \mathcal{Y}(x)H_0(t)), \;\; 0 \le t \le T. \tag{6.5}$$

From (6.1), (6.3), we have

$$\mathbf{E}\left[\int_0^T H_0(u)c(u)du + H_0(T)\xi\right] = \mathcal{X}(\mathcal{Y}(x)) = x, \tag{6.6}$$

Theorem 6.2. *[1] Suppose that both Assumptions 2.1 and 6.1 hold, let $x \in (\mathcal{X}(\infty), \infty)$ be given, let ξ and $c(\cdot)$ be given by (6.4), (6.5), and let $\pi(\cdot)$ be such that $(c,\pi) \in \mathcal{A}(x), \;\; \xi = X^{x,c,\pi}(T)$. Then $(c,\pi) \in \mathcal{A}(x)$, and (c,π) is optimal for our problem*

$$V(x) = \mathbf{E}\left[\int_0^T (1 - \exp(-c(t))dt + (1 - \exp(-X^{x,c,\pi}(T)))\right]. \tag{6.7}$$

Corollary 6.1. *[1] Under the assumptions of Theorem 6.2, the optimal wealth process $X(t) = X^{x,c,\pi}(t)$ is*

$$X(t) = \frac{1}{H_0(t)}\mathbf{E}\left[\int_t^T H_0(u)c(u)du + H_0(T)\xi \mid \mathcal{F}(t)\right], \;\; 0 \le t \le T. \tag{6.8}$$

Furthermore, the optimal portfolio π is given by

$$\sigma'(t)\pi(t) = \frac{\psi(t)}{H_0(t)} + X(t)\theta(t), \tag{6.9}$$

in terms of integrand $\psi(\cdot)$ in the stochastic integral representation $M(t) = x + \int_0^t \psi'(u)dW(u)$ of the martingale

$$M(t) := \mathbf{E}\left[\int_0^T H_0(u)c(u)du + H_0(T)\xi \mid \mathcal{F}(t)\right], \;\; 0 \le t \le T. \tag{6.10}$$

The value function V is then given as

$$V(x) = G(\mathcal{Y}(x)), \;\; \mathcal{X}(\infty) < x < \infty, \tag{6.11}$$

where

$$G(y) := \mathbf{E}\left[\int_0^T (1 - (yH_0(t)))dt + (1 - (yH_0(T)))\right], \;\; 0 < y < \infty \tag{6.12}$$

7. Main result

Theorem 7.1. *Suppose that both Assumptions 2.1 and 6.1 hold, the optimal value of $V(x)$ for Problem 5.1 is given by*

$$V(x) = (T+1) - \mathbf{E}\left[\int_0^T H_0(t)dt + H_0(T)\right]\exp\left(\frac{\mathcal{X}(1) - x}{\mathbf{E}\left[\int_0^T H_0(t)dt + H_0(T)\right]}\right), \; 0 < x < \infty$$

The optimal terminal wealth is given by

$$X^{x,c,\pi}(T) = \xi = \frac{x - \mathcal{X}(1)}{\mathbf{E}\left[\int_0^T H_0(t)dt + H_0(T)\right]} - \ln(H_0(T)).$$

The optimal consumption is given by

$$c(t) = \frac{x - \mathcal{X}(1)}{\mathbf{E}\left[\int_0^T H_0(t)dt + H_0(T)\right]} - \ln(H_0(t)).$$

where

$$\mathcal{X}(1) = -\left(\mathbf{E}\int_0^T H_0(t)\ln H_0(t)dt + H_0(T)\ln H_0(T)\right).$$

Proof. $U_1(t,x) = U_2(x) = 1 - \exp(-x),\ \forall (t,x) \in [0,T]\times(0,\infty)$.
We have $I_1(t,y) = I_2(y) = -\ln y$ for $0 < y < \infty$, and

$$\begin{aligned}
\mathcal{X}(y) &= \mathbf{E}\left[\int_0^T H_0(t)I_1(t,yH_0(t))dt + H_0(T)I_2(yH_0(T))\right] \\
&= \mathbf{E}\left[\int_0^T H_0(t)(-\ln(yH_0(t))dt + H_0(T)(-\ln(yH_0(t))\right] \\
&= -\ln y\mathbf{E}\left[\int_0^T H_0(t)dt + H_0(T)\right] + \mathcal{X}(1),\ 0 < y < \infty
\end{aligned}$$

$$\mathcal{Y}(x) = \exp\left(\frac{\mathcal{X}(1) - x}{\mathbf{E}\left[\int_0^T H_0(t)dt + H_0(T)\right]}\right),\ \ 0 < x < \infty.$$

The optimal terminal wealth and the optimal consumption process are given as

$$\begin{aligned}
\xi &= I(\mathcal{Y}(x)H_0(T)) \\
&= -\ln(\mathcal{Y}(x)H_0(T)) \\
&= -\ln\left(H_0(T)\exp\left(\frac{x - \mathcal{X}(1)}{\mathbf{E}\left[\int_0^T H_0(t)dt + H_0(T)\right]}\right)\right) \\
&= \frac{x - \mathcal{X}(1)}{\mathbf{E}\left[\int_0^T H_0(t)dt + H_0(T)\right]} - \ln(H_0(T))
\end{aligned}$$

$$\begin{aligned}
c(t) &= I(t,\mathcal{Y}(x)H_0(t)) \\
&= \frac{x - \mathcal{X}(1)}{\mathbf{E}\left[\int_0^T H_0(t)dt + H_0(T)\right]} - \ln(H_0(t))
\end{aligned}$$

and

$$\begin{aligned}
X(t) &= \frac{1}{H_0(t)}\mathbf{E}\left[\int_t^T H_0(u)c(u)du + H_0(T)\xi \mid \mathcal{F}(t)\right],\ \ 0 \le t \le T. \\
&= \frac{x + 2\mathcal{X}(1)}{H_0(t)}\mathbf{E}\left[\int_t^T \frac{H_0(u)}{\int_0^T H_0(u)du + H_0(T)}du + \frac{H_0(T)}{\int_0^T H_0(t)dt + H_0(T)} \mid \mathcal{F}(t)\right].
\end{aligned}$$

Finally

$$\begin{aligned}
G(y) &= \mathbf{E}\left[\int_0^T (1 - (yH_0(t)))dt + (1 - (yH_0(T)))\right],\ \ 0 < y < \infty \\
&= (T+1) - y\mathbf{E}\left[\int_0^T H_0(t)dt + H_0(T)\right].
\end{aligned}$$

$$G(\mathcal{Y}(x)) = (T+1) - \mathbf{E}\left[\int_0^T H_0(t)dt + H_0(T)\right] \exp\left(\frac{\mathcal{X}(1) - x}{\mathbf{E}\left[\int_0^T H_0(t)dt + H_0(T)\right]}\right).$$

□

8. Conclusion

In this paper we study an optimal consumption and investment problem for Black-Scholes type financial market on the whole investment interval $[0, T]$. By choosing a particular utility function and using the method of convex dual function, we formulate various utility maximization problem, which can be solved explicitly. We also study the convex dual of the value function for our problem.

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