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Local T_0 Approach Spaces

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Abstract
In this paper, we characterize local T_0 distance-approach spaces and gauge-approach spaces and compare them with usual T_0 approach spaces.

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1. Introduction

In 1989, Robert Lowen [10] introduced theory of approach spaces which are generalization of metric spaces and topological spaces, based upon point-to-set distances rather than point-to-point distances. The most fundamental motivation was to solve the problem of infinite product of metric spaces. The another motivation for introducing approach spaces is to unify metric, uniformity, topological concepts and theories of convergence.

There are various ways to generalize the usual T_0 axiom of topology to set-based topological category and the relationship between different forms of generalized T_0 axiom in topological category have been studied in [9] and [14] and [15]. In 1991, Baran [4] introduced local T_0 axiom of topology to set-based topological category to define the notion of closedness in set-based topological category that have been used in the notion of regular, completely regular and normal objects ([6], [7]). Another use of the local T_0 axiom is to define local Hausdorff objects in a topological category [5].

In this paper, we characterize local T_0 distance-approach spaces and gauge- approach spaces and examine how these are related to each other. Finally, we investigate the relationship between these local T_0 approach spaces and usual T_0 approach space defined in [13].

2. Preliminaries

Recall, [1], [2] or [3], that a functor $\mathcal{U} : \mathbf{E} \longrightarrow \mathbf{Set}$ is called topological , or that $\mathbf{E}$ is a topological category over $\mathbf{Set}$, (the category of set), if $\mathcal{U}$ is concrete, i.e., faithful and amnestic (i.e., if $\mathcal{U}(f) = id$ and f is an isomorphism, then $f = id$), has small (i.e., set) fibers, and every $\mathcal{U}$-source has an initial lift or, equivalently , each $\mathcal{U}$-sink has a final lift.

Note that a topological functor $\mathcal{U} : \mathbf{E} \longrightarrow \mathbf{Set}$ has a left adjoint , is called discrete functor, and $\mathcal{U}$ has a right adjoint is called indiscrete functor. Recall, in [1] or [2], that an object $X \in \mathbf{E}$ is indiscrete if and only if every map $\mathcal{U}(Y) \to \mathcal{U}(X)$ lifts to a map $Y \to X$ for each object $Y \in \mathbf{E}$ and an object $X \in \mathbf{E}$ is discrete if and only if every map $\mathcal{U}(X) \to \mathcal{U}(Y)$ lifts to a map $X \to Y$ for each object $Y \in \mathbf{E}$.

Let X be a set and $p \in X$. Let $X \vee_p X$ be the wedge at p [4], i.e., two disjoint copies of X identified at p, in other words, the pushout of $p : 1 \to X$ along itself (where 1 is a terminal object in $\mathbf{Set}$). More expressly, if i_1 and $i_2 : X \to X \vee_p X$ denote the inclusion of X as the first and second factor, respectively, then $i_1 p = i_2 p$ is a pushout diagram. A point x in $X \vee_p X$ will be denoted by $x_1 (x_2)$ if x is in the first (resp. the second) component of $X \vee_p X$.

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The fold map at p, $\nabla_p : X \vee_p X \to X$ is defined as $\nabla_p(x_i) = x$ for $i = 1, 2$. The principal p-axis map $A_p : X \vee_p X \to X^2$ is defined as $A_p(x_1) = (x, p)$ and $A_p(x_2) = (p, x)$ [4] or [8].
Note that the principal p-axis map (i.e.,A_p) and fold p-axis map (i.e., ∇_p) are the unique maps appearing from the above pushout diagram for which $A_p i_1 = (id, p) : X \to X^2$, $A_p i_2 = (p, id) : X \to X^2$, and $\nabla_p i_j = id$; $j = 1, 2$, respectively.

Definition 2.1. Let (X, τ) be a topological space and $p \in X$.

1. For each point x distinct from p, there exists a neighborhood of p missing x or there exists a neighborhood of x missing p, then (X, τ) is said to be T_0 at p [8].
2. If a topological space (X, τ) is T_0 at p for all points p, then (X, τ) is called T_0 [8].

Theorem 2.1. *A topological space (X, τ) is T_0 at p if and only if the initial topology induced by $\{A_p : X \vee_p X \to (X^2, \tau_*)$ and $\nabla_p : X \vee_p X \to (X, P(X))\}$ is discrete where τ_* is the product topology on X^2.*

Proof. It is given in [8]. □

In the view of Theorem 2.1, Baran [4] gave the following definition.

Definition 2.2. (cf. [4]) Let $U : E \to$ **Set** be topological, X an object in E with $U(X) = B$ and p be a point in B.
X is $\overline{T_0}$ at p if and only if the initial lift of the U-source $\{A_p : B \vee_p B \to U(X^2) = B^2$ and $\nabla_p : B \vee_p B \to UD(B) = B\}$ is discrete, where D is the discrete functor which is a left adjoint to U.

3. Local T_0 Approach Spaces

Definition 3.1. (cf. [10], [11] or [13]) Let X be a set and 2^X be power set of X. A map $\delta : X \times 2^X \to [0, \infty]$ is called distance on X if δ satisfies the followings:

(i) $\forall A \subseteq X$ and $\forall x \in A$, $\delta(x, A) = 0$

(ii) $\forall x \in X$ and $\emptyset$, the empty set, $\delta(x, \emptyset) = \infty$

(iii) $\forall x \in X, \forall A, B \subseteq X$, $\delta(x, A \cup B) = \min(\delta(x, A), \delta(x, B))$

(iv) $\forall x \in X, \forall A \subseteq X, \forall \epsilon \in [0, \infty]$, $\delta(x, A) \leq \delta(x, A^{(\epsilon)}) + \epsilon$, where $A^{(\epsilon)} = \{x \in X | \delta(x, A) \leq \epsilon\}$.

The pair (X, δ) is called distance-approach spaces.

Recall [16] that an extended pseudo-quasi metric on a set X is a map $d : X \times X \to [0, \infty]$ providing for all $x \in X$, $d(x, x) = 0$ and for all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

Definition 3.2. (cf. [10], [11]) Let X be a non-empty set and let $pqMet^\infty(X)$ be the set of all extended pseudo- quasi metrics on X , $\mathfrak{D} \subseteq pqMet^\infty(X)$ and $d \in pqMet^\infty(X)$, then
(i) $\mathfrak{D}$ is called ideal if it is closed under the formation of finite suprema, i.e., if $d, d' \in \mathfrak{D}$ there exists $d'' \in \mathfrak{D}$ such that $d \bigvee d' \leq d''$.
(ii) $\mathfrak{D}$ dominates d if $\forall x \in X, \epsilon > 0$ and $\omega < \infty$ there exists a $d_x^{\epsilon,\omega} \in \mathfrak{D}$ such that $d(x, .) \wedge \omega \leq d_x^{\epsilon,\omega}(x, .) + \epsilon$ and if $\mathfrak{D}$ dominates d then $\mathfrak{D}$ is called saturated.
If $\mathfrak{D}$ is an ideal in $pqMet^\infty(X)$ and saturated, then $\mathfrak{D}$ is called gauge. The pair $(X, \mathfrak{D})$ is called gauge-approach spaces.

The transition from gauge-approach space to distance-approach space is provided by

$$\delta(x, A) = \sup_{d \in \mathfrak{D}} \inf_{a \in A} d(x, a)$$

and conversely, from distance-approach space to gauge-approach is given as [11]

$$\mathfrak{D} = \{d | \forall A \subset X : \inf_{a \in A} d(., a) \leq \delta(., a)\}.$$

Definition 3.3. Let (X,δ) and (X',δ') be distance-approach spaces (resp. $(X,\mathfrak{D})$ and $(X',\mathfrak{D}')$ be gauge-approach spaces). For all $x \in X$ and $A \subset X$, if $\delta'(f(x), f(A)) \leq \delta(x,A)$ (resp. for all $d' \in \mathfrak{D}$, $d'(f\times f) \in \mathfrak{D}$), then $f : (X,\delta) \to (X',\delta')$ is called a contraction map [12] or [11].

The category **App** of approach spaces has as objects the pairs (X,δ) distance-approach spaces or $(X,\mathfrak{D})$ gauge-approach spaces, and as morphisms contraction maps. Note that **App** is a topological category over **Set** [10] or [11].

Remark 3.1. (i) A source $\{f_i : (X,\delta) \to (X_i,\delta_i)\}$ is initial in distance-approach space if and only if for all $x \in X, A \subseteq X$, $\delta(x,A) = sup_{\mathcal{P}\in R(A)}\ min_{P\in\mathcal{P}}\ sup_{i\in I}\ \delta_i(f_i(x), f_i(P))$ where $R(A)$ is the set of finite partitions of A with subsets of A [10] or [11].

(ii) A source $\{f_i : (X,\delta) \to (X_i,\delta_i)\}$ is initial in gauge-approach space if and only if for any $i \in I, \mathcal{H}_i$ is a basis for gauge in X_i, then initial gauge on X is defined by [11]

$$\mathcal{H} = \{\sup_{i\in K} d_i \circ (f_i \times f_i) : K \in 2^{(I)}, \forall i \in K, d_i \in \mathcal{H}_i\}.$$

(iii) The discrete distance-approach structure δ on X is given as for all $x \in X$ and $A \subseteq X$

$$\delta(x,A) = \begin{cases} 0, & x \in A \\ \infty, & x \notin A \end{cases}$$

[10] or [11].

(iv) The discrete gauge-approach structure $\mathfrak{D}$ on X is $\mathfrak{D} = pqMet^{\infty}(X)$ (all extended pseudo-quasi metric space) [11].

Example 3.1. Every metric space is an approach space [11].

Let (X,d) be a metric space, $A \subseteq X$ and $\delta_d : X \times 2^X \longrightarrow [0,\infty]$ be a function defined as $\delta_d(x,A) = \inf_{a\in A} d(x,a)$. It is easy to show that δ_d is the distance-approach structure on X.

Example 3.2. Every topological space is an approach space [11].

Let (X,τ) be topological space and $A \subseteq X$. Define the function $\delta_\tau : X \times 2^X \longrightarrow [0,\infty]$ by

$$\delta_\tau(x,A) = \begin{cases} 0, & x \in \overline{A} \\ \infty, & x \notin \overline{A} \end{cases}$$

where $\overline{A}$ is the closure of A. It can be easily seen that δ_τ is the distance-approach structure on X.

Theorem 3.1. *A distance-approach space (X,δ) is $\overline{T_0}$ at p if and only if for all $x \in X$ with $x \neq p$, $\delta(x,\{p\}) = \infty$ or $\delta(p,\{x\}) = \infty$.*

Proof. Let $x \in X$, $x \neq p$, $u = x_1 \in X \vee_p X$ and $A = \{p, x_2\} \subset X \vee_p X$. Let $\overline{\delta}$ be an initial structure on the wedge $X \vee_p X$ induced by $A_p : X \vee_p X \to U(X^2,\delta^2) = X^2$ and $\nabla_p : X \vee_p X \to U(X,\delta_{dis}) = X$ where δ^2 is the product structure on X^2 induced by $\pi_1,\pi_2 : X^2 \to X$ projection maps and δ_{dis} is the discrete structure on X. Let $\mathcal{P}_1 = \{A\}$ and $\mathcal{P}_2 = \{\{p\},\{x_2\}\}$ be the partitions of A. Note that

$$\begin{aligned}
&\min_{B\in\mathcal{P}_1} \sup\{\delta_{dis}(\nabla_p(u),\nabla_p(B)), \delta(\pi_1 A_p(u), \pi_1 A_p(B)), \delta(\pi_2 A_p(u), \pi_2 A_p(B))\} \\
&= \sup\{\delta_{dis}(x,\{x,p\}), \delta(x,\{p\}), \delta(p,\{p,x\})\} \\
&= \sup\{0, \delta(x,\{p\})\} \\
&= \delta(x,\{p\}),
\end{aligned}$$

and

$$\begin{aligned}
&\min_{B\in\mathcal{P}_2} \sup\{\delta_{dis}(\nabla_p(u),\nabla_p(B)), \delta(\pi_1 A_p(u), \pi_1 A_p(B)), \delta(\pi_2 A_p(u), \pi_2 A_p(B))\} \\
&= \min\{\sup\{\delta_{dis}(x,\{p\}), \delta(x,\{p\}), \delta(p,\{x\})\}, \sup\{\delta_{dis}(x,\{x\}), \delta(x,\{p\}), \delta(p,\{x\})\}\} \\
&= \min_{B\in\mathcal{P}_2} \sup\{\delta_{dis}(\nabla_p(u),\nabla_p(B)), \delta(\pi_1 A_p(u), \pi_1 A_p(B)), \delta(\pi_2 A_p(u), \pi_2 A_p(B))\} \\
&= \min\{\sup\{\delta_{dis}(x,\{p\}), \delta(x,\{p\}), \delta(p,\{x\})\}, \sup\{\delta_{dis}(x,\{x\}), \delta(x,\{p\}), \delta(p,\{x\})\}\} \\
&= \min\{\infty, \sup\{\delta(x,\{p\}), \delta(p,\{x\})\}\} \\
&= \sup\{\delta(x,\{p\}), \delta(p,\{x\})\}.
\end{aligned}$$

since $\delta_{dis}(x,\{p\}) = \infty$ and $x \neq p$. Since $u \notin A$ and (X,δ) is $\overline{T_0}$ at p,

$$\infty = \overline{\delta}(u,A) = \sup_{\mathcal{P}\in R(A)} \min_{B\in\mathcal{P}} \sup\{\delta_{dis}(\nabla_p(u),\nabla_p(B)), \delta(\pi_1 A_p(u), \pi_1 A_p(B)),$$

$$\delta(\pi_2 A_p(u), \pi_2 A_p(B))\} = \sup\{\delta(x,\{p\}), \sup\{\delta(x,\{p\}), \delta(p,\{x\})\}\}$$

and consequently, we have either $\delta(x,\{p\}) = \infty$ or $\delta(p,\{x\}) = \infty$.

Conversely let $u \in X\vee_p X$, $A \subset X\vee_p X$ and for all $x \in X$ with $x \neq p$, $\delta(x,\{p\}) = \infty$ or $\delta(p,\{x\}) = \infty$. Let $\overline{\delta}$ be an initial structure on the wedge $X\vee_p X$ induced by $A_p : X\vee_p X \to U(X^2,\delta^2) = X^2$ and $\nabla_p : X\vee_p X \to U(X,\delta_{dis}) = X$ where δ^2 is the product structure on X^2 and δ_{dis} is the discrete structure on X, and $\pi_1, \pi_2 : X^2 \to X$ are projection maps.

If $A = \emptyset$, then $\overline{\delta}(u,A) = \overline{\delta}(u,\emptyset) = \infty$. Suppose $A \neq \emptyset$. Let $\mathcal{P} = \{A_1, A_2, ..., A_n\}$ be any finite partition of A. If $\nabla_p(u) = p \in \nabla_p(A_k)$, then $u = p_1 = p_2 \in A_k \subset A$ for some $k \in \{1,2,...,n\}$. Note that

$$\begin{aligned}\delta_{dis}(\nabla_p(u),\nabla_p(A_k)) &= \delta_{dis}(p,\nabla_p(A_k))\\ &= 0,\end{aligned}$$

$$\begin{aligned}\delta(\pi_1 A_p(u), \pi_1 A_p(A_k)) &= \delta(p, \pi_1 A_p(A_k))\\ &= 0\\ &= \delta(\pi_2 A_p(u), \pi_2 A_p(A_k)),\end{aligned}$$

and

$$\begin{aligned}\delta_{dis}(\nabla_p(u),\nabla_p(A_i)) &= \delta_{dis}(p,\nabla_p(A_i))\\ &= \infty\end{aligned}$$

since $p \notin \nabla_p(A_i)$ for $i \neq k$, $i = 1,2,...,n$,

$$\delta(\pi_1 A_p(u), \pi_1 A_p(A_i)) = \delta(p, \pi_1 A_p(A_i))$$

and

$$\delta(\pi_2 A_p(u), \pi_2 A_p(A_i)) = \delta(p, \pi_2 A_p(A_i)).$$

By Remark 3.1 (i),

$$\begin{aligned}\overline{\delta}(u,A) &= \sup_{\mathcal{P}\in R(A)} \min_{B\in\mathcal{P}} \sup\{\delta_{dis}(\nabla_p(u),\nabla_p(B)), \delta(\pi_1 A_p(u), \pi_1 A_p(B)), \delta(\pi_2 A_p(u), \pi_2 A_p(B))\}\\ &= \sup\{\min\{0, \sup\{\infty, \delta(p, \pi_1 A_p(A_i)), \delta(p, \pi_2 A_p(A_i))\}\}\}\\ &= \sup\{\min\{0,\infty\}\}\\ &= 0\end{aligned}$$

and consequently, $\overline{\delta}(u,A) = 0$.

If $\nabla_p(u) = p \notin \nabla_p(A_k)$, then $u = p_1 = p_2 \notin A_k \subset A$ for all $k \in \{1,2,...,n\}$.

$$\begin{aligned}\delta_{dis}(\nabla_p(u),\nabla_p(A_k)) &= \delta_{dis}(p,\nabla_p(A_k))\\ &= \infty\end{aligned}$$

since $\nabla_p(u) = p \notin \nabla_p(A_k)$, and consequently $\overline{\delta}(u,A) = \infty$.

Suppose that $\nabla_p(u) = x$ for some $x \in X$ with $x \neq p$. It follows that $u = x_1$ or $u = x_2$. If $u \notin A$, then $u \notin A_k$ for all $k \in \{1,2,...,n\}$ and $\nabla_p(u) = x \notin \nabla_p(A_k)$. It follows that

$$\begin{aligned}\delta_{dis}(\nabla_p(u),\nabla_p(A_k)) &= \delta_{dis}(x,\nabla_p(A_k))\\ &= \infty\end{aligned}$$

and consequently, $\overline{\delta}(u,A) = \infty$.

Suppose that $u = x_1, x_2 \in A$, then $\exists k, m \in \{1, 2, ..., n\}$ such that $x_1 \in A_k$ and $x_2 \in A_m$. Note that

$$\begin{aligned} \delta_{dis}(\nabla_p(x_1), \nabla_p(A_k)) &= \delta_{dis}(x, \nabla_p(A_k)) \\ &= 0 \end{aligned}$$

since $x \in \nabla_p(A_k))$,

$$\begin{aligned} \delta(\pi_1 A_p(x_1), \pi_1 A_p(A_k)) &= \delta(x, \pi_1 A_p(A_k)) \\ &= 0 \end{aligned}$$

since $\pi_1 A_p(x_1) = x \in \pi_1 A_p(A_k)$,

$$\begin{aligned} \delta(\pi_2 A_p(x_1), \pi_2 A_p(A_k)) &= \delta(p, \pi_2 A_p(A_k)) \\ &= 0 \end{aligned}$$

since $\pi_2 A_p(x_1) = p \in \pi_2 A_p(A_k)$, and

$$\begin{aligned} \delta_{dis}(\nabla_p(x_2), \nabla_p(A_m)) &= \delta_{dis}(x, \nabla_p(A_m)) \\ &= 0 \end{aligned}$$

since $x \in \nabla_p(A_m))$,

$$\begin{aligned} \delta(\pi_1 A_p(x_2), \pi_1 A_p(A_m)) &= \delta(p, \pi_1 A_p(A_m)) \\ &= 0 \end{aligned}$$

since $\pi_1 A_p(x_2) = p \in \pi_1 A_p(A_m)$,

$$\begin{aligned} \delta(\pi_2 A_p(x_2), \pi_2 A_p(A_k)) &= \delta(x, \pi_2 A_p(A_m)) \\ &= 0 \end{aligned}$$

since $\pi_2 A_p(x_2) = x \in \pi_2 A_p(A_m)$, and

$$\begin{aligned} \delta_{dis}(\nabla_p(u), \nabla_p(A_j)) &= \delta_{dis}(x, \nabla_p(A_j)) \\ &= \infty \end{aligned}$$

since $x \notin \nabla_p(A_j)$ for $j \neq k$ and $j \neq m$, and

$$\delta(\pi_1 A_p(u), \pi_1 A_p(A_j)) = \delta(x, \pi_1 A_p(A_j)),$$

$$\delta(\pi_2 A_p(u), \pi_2 A_p(A_j)) = \delta(p, \pi_2 A_p(A_j)).$$

It follows that

$$\begin{aligned} \overline{\delta}(u, A) &= \sup_{\mathcal{P} \in R(A)} \min_{B \in \mathcal{P}} \sup\{\delta_{dis}(\nabla_p(u), \nabla_p(B)), \delta(\pi_1 A_p(u), \pi_1 A_p(B)), \delta(\pi_2 A_p(u), \pi_2 A_p(B))\} \\ &= \sup\{\min\{0, \sup\{\infty, \delta(p, \delta(x, \pi_1 A_p(A_j)), \delta(p, \pi_2 A_p(A_j))\}\}\} \\ &= \sup\{\min\{0, \infty\}\} = 0. \end{aligned}$$

Suppose that $u = x_1 \notin A$, and $x_2 \in A$. Let $\mathcal{P}_1 = \{\{x_2\}, A_1, A_2, ..., A_n\}$ be any partition of A such that $A_k \subset A$, $k \in \{1, 2, ..., n\}$.

$$\begin{aligned} \delta_{dis}(\nabla_p(u), \nabla_p(\{x_2\})) &= \delta_{dis}(x, \{x\}) \\ &= 0, \end{aligned}$$

$$\delta(\pi_1 A_p(u), \pi_1 A_p(\{x_2\})) = \delta(x, \{p\})$$

and

$$\delta(\pi_2 A_p(u), \pi_2 A_p(\{x_2\})) = \delta(p, \{x\}),$$

and

$$\delta_{dis}(\nabla_p(u),\nabla_p(A_k))=\delta_{dis}(x,\nabla_p(A_k))=\infty$$

since $x\notin\nabla_p(A_k)$, $k\in\{1,2,...,n\}$,

$$\delta(\pi_1A_p(u),\pi_1A_p(A_k))=\delta(x,\pi_1A_p(A_k))$$

and

$$\delta(\pi_2A_p(u),\pi_2A_p(A_k))=\delta(p,\pi_2A_p(A_k)).$$

Note that

$$\begin{aligned}&\min_{B\in\mathcal{P}_1}\sup\{\delta_{dis}(\nabla_p(u),\nabla_p(B)),\delta(\pi_1A_p(u),\pi_1A_p(B)),\delta(\pi_2A_p(u),\pi_2A_p(B))\}\\ =\ &\min\{\sup\{0,\delta(x,\{p\}),\delta(p,\{x\})\},\end{aligned}$$

$$\begin{aligned}\sup\{\infty,\delta(x,\pi_1A_p(B)),\delta(p,\pi_2A_p(B))\}\} &= \min\{\sup\{\delta(x,\{p\}),\delta(p,\{x\})\},\infty\}\\ &= \sup\{\delta(x,\{p\}),\delta(p,\{x\})\}.\end{aligned}$$

Let $\mathcal{P}_2=\{A_1,A_2,...,A_n\}$ be the partition of A such that $A_k\subset A$, and $x_2\in A_k$ for some $k\in\{1,2,...,n\}$.

$$\begin{aligned}\delta_{dis}(\nabla_p(u),\nabla_p(A_k)) &= \delta_{dis}(x,\nabla_p(A_k))\\ &= 0\end{aligned}$$

since $x\in\nabla_p(A_k)$,

$$\delta(\pi_1A_p(u),\pi_1A_p(A_k))=\delta(x,\pi_1A_p(A_k))$$

and

$$\delta(\pi_2A_p(u),\pi_2A_p(A_k))=\delta(p,\pi_2A_p(A_k)).$$

For $j\neq k$, $j=1,2,...,n$,

$$\begin{aligned}\delta_{dis}(\nabla_p(u),\nabla_p(A_j)) &= \delta_{dis}(x,\nabla_p(A_j))\\ &= \infty\end{aligned}$$

since $x\notin\nabla_p(A_j)$,

$$\delta(\pi_1A_p(u),\pi_1A_p(A_j))=\delta(x,\pi_1A_p(A_j))$$

and

$$\delta(\pi_2A_p(u),\pi_2A_p(A_j))=\delta(p,\pi_2A_p(A_j)).$$

Note that

$$\begin{aligned}&\min_{B\in\mathcal{P}_2}\sup\{\delta_{dis}(\nabla_p(u),\nabla_p(B)),\delta(\pi_1A_p(u),\pi_1A_p(B)),\delta(\pi_2A_p(u),\pi_2A_p(B))\}\\ =\ &\min\{\sup\{0,\delta(x,\pi_1A_p(A_k)),\delta(p,\pi_2A_p(A_k))\},\sup\{\infty,\delta(x,\pi_1A_p(A_j)),\delta(p,\pi_2A_p(A_j))\}\}\\ =\ &\min\{\sup\{\delta(x,\pi_1A_p(A_k)),\delta(p,\pi_2A_p(A_k))\},\infty\}\\ =\ &\sup\{\delta(x,\pi_1A_p(A_k)),\delta(p,\pi_2A_p(A_k))\}.\end{aligned}$$

By Remark 3.1 (i)

$$\begin{aligned}\overline{\delta}(u,A) &= \sup_{\mathcal{P}\in R(A)}\min_{B\in\mathcal{P}}\sup\{\delta_{dis}(\nabla_p(u),\nabla_p(B)),\delta(\pi_1A_p(u),\pi_1A_p(B)),\delta(\pi_2A_p(u),\pi_2A_p(B))\}\\ &= \sup\{\min_{B\in\mathcal{P}_1}\sup\{\delta_{dis}(\nabla_p(u),\nabla_p(B)),\delta(\pi_1A_p(u),\pi_1A_p(B)),\delta(\pi_2A_p(u),\pi_2A_p(B))\},\\ &\quad\min_{B\in\mathcal{P}_2}\sup\{\delta_{dis}(\nabla_p(u),\nabla_p(B)),\delta(\pi_1A_p(u),\pi_1A_p(B)),\delta(\pi_2A_p(u),\pi_2A_p(B))\}\}\\ &= \sup\{\sup\{\delta(x,\{p\}),\delta(p,\{x\})\},\sup\{\delta(x,\pi_1A_p(A_k)),\delta(p,\pi_2A_p(A_k))\}\}\\ &= \infty\end{aligned}$$

since by assumption $\delta(x,\{p\})=\infty$ or $\delta(p,\{x\})=\infty$.

Suppose $u = x_2 \notin A$, and $x_1 \in A$. Let $\mathcal{P}_1 = \{\{x_1\}, A_1, A_2, ..., A_n\}$ be the partition of A such that $A_k \subset A$, $k \in \{1, 2, ..., n\}$.

$$\begin{aligned}\delta_{dis}(\nabla_p(u), \nabla_p(\{x_1\})) &= \delta_{dis}(x, \{x\}) \\ &= 0,\end{aligned}$$

$$\delta(\pi_1 A_p(u), \pi_1 A_p(\{x_1\})) = \delta(p, \{x\})$$

and

$$\delta(\pi_2 A_p(u), \pi_2 A_p(\{x_1\})) = \delta(x, \{p\}),$$

and

$$\begin{aligned}\delta_{dis}(\nabla_p(u), \nabla_p(A_k)) &= \delta_{dis}(x, \nabla_p(A_k)) \\ &= \infty\end{aligned}$$

since $x \notin \nabla_p(A_k)$, $k \in \{1, 2, ..., n\}$,

$$\delta(\pi_1 A_p(u), \pi_1 A_p(A_k)) = \delta(p, \pi_1 A_p(A_k))$$

and

$$\delta(\pi_2 A_p(u), \pi_2 A_p(A_k)) = \delta(x, \pi_2 A_p(A_k)).$$

Note that

$$\begin{aligned}&\min_{B \in \mathcal{P}_1} \sup\{\delta_{dis}(\nabla_p(u), \nabla_p(B)), \delta(\pi_1 A_p(u), \pi_1 A_p(B)), \delta(\pi_2 A_p(u), \pi_2 A_p(B))\} \\ = \;& \min\{\sup\{0, \delta(x, \{p\}), \delta(p, \{x\})\}, \sup\{\infty, \delta(p, \pi_1 A_p(B)), \delta(x, \pi_2 A_p(B))\}\} \\ = \;& \min\{\sup\{\delta(x, \{p\}), \delta(p, \{x\})\}, \infty\} \\ = \;& \sup\{\delta(x, \{p\}), \delta(p, \{x\})\}.\end{aligned}$$

Let $\mathcal{P}_2 = \{A_1, A_2, ..., A_n\}$ be any partition of A such that $A_k \subset A$, and $x_1 \in A_k$ for some $k \in \{1, 2, ..., n\}$.

$$\begin{aligned}\delta_{dis}(\nabla_p(u), \nabla_p(A_k)) &= \delta_{dis}(x, \nabla_p(A_k)) \\ &= 0\end{aligned}$$

since $x \in \nabla_p(A_k)$,

$$\delta(\pi_1 A_p(u), \pi_1 A_p(A_k)) = \delta(p, \pi_1 A_p(A_k))$$

and

$$\delta(\pi_2 A_p(u), \pi_2 A_p(A_k)) = \delta(x, \pi_2 A_p(A_k)).$$

For $j \neq k$, $j = 1, 2, ..., n$,

$$\delta_{dis}(\nabla_p(u), \nabla_p(A_j)) = \delta_{dis}(x, \nabla_p(A_j)) = \infty$$

since $x \notin \nabla_p(A_j)$,

$$\delta(\pi_1 A_p(u), \pi_1 A_p(A_j)) = \delta(p, \pi_1 A_p(A_j))$$

and

$$\delta(\pi_2 A_p(u), \pi_2 A_p(A_j)) = \delta(x, \pi_2 A_p(A_j)).$$

Note that

$$\begin{aligned}&\min_{B \in \mathcal{P}_2} \sup\{\delta_{dis}(\nabla_p(u), \nabla_p(B)), \delta(\pi_1 A_p(u), \pi_1 A_p(B)), \delta(\pi_2 A_p(u), \pi_2 A_p(B))\} \\ = \;& \min\{\sup\{0, \delta(p, \pi_1 A_p(A_k)), \delta(x, \pi_2 A_p(A_k))\}, \sup\{\infty, \delta(p, \pi_1 A_p(A_j)), \delta(x, \pi_2 A_p(A_j))\}\} \\ = \;& \min\{\sup\{\delta(p, \pi_1 A_p(A_k)), \delta(x, \pi_2 A_p(A_k))\}, \infty\} \\ = \;& \sup\{\delta(p, \pi_1 A_p(A_k)), \delta(x, \pi_2 A_p(A_k))\}.\end{aligned}$$

By Remark 3.1 (i)

$$\begin{aligned}
\overline{\delta}(u,A) &= \sup_{\mathcal{P}\in R(A)} \min_{B\in\mathcal{P}} \sup\{\delta_{dis}(\nabla_p(u),\nabla_p(B)),\delta(\pi_1A_p(u),\pi_1A_p(B)),\delta(\pi_2A_p(u),\pi_2A_p(B))\}\\
&= \sup\{\min_{B\in\mathcal{P}_1} \sup\{\delta_{dis}(\nabla_p(u),\nabla_p(B)),\delta(\pi_1A_p(u),\pi_1A_p(B)),\delta(\pi_2A_p(u),\pi_2A_p(B))\},\\
&\quad \min_{B\in\mathcal{P}_2} \sup\{\delta_{dis}(\nabla_p(u),\nabla_p(B)),\delta(\pi_1A_p(u),\pi_1A_p(B)),\delta(\pi_2A_p(u),\pi_2A_p(B))\}\}\\
&= \sup\{\sup\{\delta(x,\{p\}),\delta(p,\{x\})\}, sup\{\delta(p,\pi_1A_p(A_k)),\delta(x,\pi_2A_p(A_k))\}\}\\
&= \infty
\end{aligned}$$

since by assumption $\delta(x,\{p\})=\infty$ or $\delta(p,\{x\})=\infty$. Hence, for all $u\in X\vee_p X$ and $A\subset X\vee_p X$, we have

$$\overline{\delta}(u,A)=\begin{cases}0, & u\in A\\ \infty, & u\notin A\end{cases}$$

i.e., by Remark 3.1 (iii) $\overline{\delta}$ is the discrete structure and by Definition 2.2 (X,δ) is $\overline{T_0}$ at p. □

Theorem 3.2. *A gauge-approach space $(X,\mathfrak{D})$ is $\overline{T_0}$ at p if and only if for all $x\in X$ with $x\neq p$, $\exists d\in\mathfrak{D}$ such that $d(x,p)=\infty$ or $d(p,x)=\infty$.*

Proof. Let $(X,\mathfrak{D})$ be $\overline{T_0}$ at p, $x\in X$ and $x\neq p$. Let $\overline{\mathfrak{D}}$ be the initial gauge structure on $X\vee_p X$ induced by $A_p: X\vee_p X\to U(X^2,\mathfrak{D}^2)=X^2$ and $\nabla_p: X\vee_p X\to U(X,\mathfrak{D}_{dis})=X$ where $\mathfrak{D}^2$ is product structure on X^2 induced by $\pi_1,\pi_2: X^2\to X$ projection maps and $\mathfrak{D}_{dis}$ is discrete structure on X. Assume that $\mathcal{H}_{dis}=\{d_{dis}\}$ is a basis for discrete gauge where d_{dis} is the discrete extended pseudo-quasi metric on X. Let $\mathcal{H}$ be gauge basis of $\mathfrak{D}$ and $d\in\mathcal{H}$, and $\overline{\mathcal{H}}=\{\overline{d_{dis}}\}$ be initial gauge basis of $\overline{\mathfrak{D}}$ where $\overline{d_{dis}}$ is the discrete extended pseudo-quasi metric on $X\vee_p X$. Note that

$$d(\pi_1A_p(x_1),\pi_1A_p(x_2))=d(x,p),$$

$$d(\pi_2A_p(x_1),\pi_2A_p(x_2))=d(p,x),$$

$$d_{dis}(\nabla_p(x_1),\nabla_p(x_2))=d_{dis}(x,x)=0.$$

Since $x_1\neq x_2$, $\overline{d_{dis}}$ is the discrete extended pseudo-quasi metric on $X\vee_p X$ and $(X,\mathfrak{D})$ is $\overline{T_0}$ at p, by Remark 3.1 (ii)

$$\begin{aligned}
\infty &= \overline{d}_{dis}(x_1,x_2)\\
&= \sup\{d_{dis}(\nabla_p(x_1),\nabla_p(x_2)),d(\pi_1A_p(x_1),\pi_1A_p(x_2)),d(\pi_2A_p(x_1),\pi_2A_p(x_2))\}\\
&= \sup\{0,d(x,p),d(p,x)\}
\end{aligned}$$

and consequently, $d(x,p)=\infty$ or $d(p,x)=\infty$.

Conversely, let $\overline{\mathcal{H}}$ be initial gauge basis on $X\vee_p X$ induced by $A_p: X\vee_p X\to U(X^2,\mathfrak{D}^2)=X^2$ and $\nabla_p: X\vee_p X\to U(X,\mathfrak{D}_{dis})=X$ where $\mathfrak{D}_{dis}=pqMet^\infty$ discrete gauge on X and $\mathfrak{D}^2$ be the product structure on X^2 induced by $\pi_1,\pi_2: X^2\to X$ projection maps. Suppose $\overline{d}\in\overline{\mathcal{H}}$ and $u,v\in X\vee_p X$.

1. If $u=v$, then

$$\begin{aligned}
\overline{d}(u,v) &= \sup\{d_{dis}(\nabla_p(u),\nabla_p(u)),d(\pi_1A_p(u),\pi_1A_p(u)),d(\pi_2A_p(u),\pi_2A_p(u))\}\\
&= 0.
\end{aligned}$$

2. If $u\neq v$, if $\nabla_p(u)\neq\nabla_p(v)$ implies $d_{dis}(\nabla_p(u),\nabla_p(v))=\infty$. By Remark 3.1 (ii)

$$\begin{aligned}
\overline{d}(u,v) &= \sup\{d_{dis}(\nabla_p(u),\nabla_p(v)),d(\pi_1A_p(u),\pi_1A_p(v)),d(\pi_2A_p(u),\pi_2A_p(v))\}\\
&= \sup\{\infty,d(\pi_1A_p(u),\pi_1A_p(v)),d(\pi_2A_p(u),\pi_2A_p(v))\}\\
&= \infty.
\end{aligned}$$

3. Suppose $u \neq v$ and $\nabla_p(u) = \nabla_p(v)$. If $\nabla_p(u) = p = \nabla_p(v)$ implies $u = p_1 = p_2 = p = v$, a contradiction since $u \neq v$. If $\nabla_p(u) = x = \nabla_p(v)$ for some $x \in X$ with $x \neq p$, then $u = x_1$ and $v = x_2$ or $u = x_2$ and $v = x_1$ since $u \neq v$. Let $u = x_1$ and $v = x_2$.

$$\begin{aligned} d(\pi_1 A_p(u), \pi_1 A_p(v)) &= d(\pi_1 A_p(x_1), \pi_1 A_p(x_2)) \\ &= d(x,p), d(\pi_2 A_p(u), \pi_2 A_p(v)) \\ &= d(\pi_2 A_p(x_1), \pi_2 A_p(x_2)) \\ &= d(p,x) \end{aligned}$$

and

$$\begin{aligned} d_{dis}(\nabla_p(x_1), \nabla_p(x_2)) &= d_{dis}(x,x) \\ &= 0, \end{aligned}$$

it follows that

$$\begin{aligned} \overline{d}(u,v) &= \overline{d}(x_1,x_2) \\ &= \sup\{d_{dis}(\nabla_p(x_1), \nabla_p(x_2)), d(\pi_1 A_p(x_1), \pi_1 A_p(x_2)), d(\pi_2 A_p(x_1), \pi_2 A_p(x_2))\} \\ &= \sup\{0, d(p,x), d(x,p)\}. \end{aligned}$$

By the assumption $d(x,p) = \infty$ or $d(p,x) = \infty$, we get $\overline{d}(u,v) = \infty$.

Let $u = x_2$ and $v = x_1$.

$$\begin{aligned} d(\pi_1 A_p(u), \pi_1 A_p(v)) &= d(\pi_1 A_p(x_2), \pi_1 A_p(x_1)) \\ &= d(p,x), \end{aligned}$$

$$\begin{aligned} d(\pi_2 A_p(u), \pi_2 A_p(v)) &= d(\pi_2 A_p(x_2), \pi_2 A_p(x_1)) \\ &= d(x,p) \end{aligned}$$

and

$$d_{dis}(\nabla_p(x_2), \nabla_p(x_1)) = d_{dis}(x,x) = 0.$$

It follows from Remark 3.1 (ii) that

$$\begin{aligned} \overline{d}(u,v) &= \overline{d}(x_2,x_1) \\ &= \sup\{d_{dis}(\nabla_p(x_2), \nabla_p(x_1)), d(\pi_1 A_p(x_2), \pi_1 A_p(x_1)), d(\pi_2 A_p(x_2), \pi_2 A_p(x_1))\} \\ &= \sup\{0, d(p,x), d(x,p)\}. \end{aligned}$$

By the assumption $d(x,p) = \infty$ or $d(p,x) = \infty$, we get $\overline{d}(u,v) = \infty$.

Therefore, $\forall u, v \in X \vee_p X$, we have

$$\overline{d}(u,v) = \begin{cases} 0, & u = v \\ \infty, & u \neq v \end{cases}$$

and by the assumption $\overline{d}$ is discrete extended pseudo-quasi metric on $X \vee_p X$, i.e., $\overline{\mathcal{H}} = \{\overline{d}\}$, which means $\mathfrak{D}_{dis} = pqMet^{\infty}$. By Definition 2.2 , $(X, \mathfrak{D})$ is $\overline{T_0}$ at p. □

Theorem 3.3. *Let (X, δ) (or $(X, \mathfrak{D})$) be an approach space and $p \in X$. Then, the followings are equivalent.*

1. *(X, δ) is $\overline{T_0}$ at p.*
2. *For all $x \in X$ with $x \neq p$, $\delta(x, \{p\}) = \infty$ or $\delta(p, \{x\}) = \infty$.*
3. *For all $x \in X$ with $x \neq p$, $\exists d \in \mathfrak{D}$ such that $d(x,p) = \infty$ or $d(p,x) = \infty$.*

Proof. It follows from Theorem 3.1 and Theorem 3.2. □

Let (X, δ) be distance-approach spaces and $(X, \mathfrak{D})$ be gauge-approach spaces. Then, topological co-reflection of a distance δ is given by

$$cl_{\tau_\delta}(A) = \{x \in X : \delta(x, A) = 0\}$$

and topological co-reflection of gauge $\mathfrak{D}$ is defined by (X, τ_d) where τ_d stands for topology induced by d extended pseudo-quasi metric [12]. Recall, [13] that a distance-approach space (X, δ)(resp. gauge-approach space $(X, \mathfrak{D})$) is T_0 (we call it as the usual one) if and only if topological space (X, τ_δ) (resp. (X, τ_d)) is T_0.

Theorem 3.4. *Let (X, δ) (or $(X, \mathfrak{D})$) be an approach space. Then, the followings are equivalent.*

1. *(X, δ) is T_0.*
2. *For all $x, y \in X$, $x \neq y$, $\delta(x, \{y\}) > 0$ or $\delta(y, \{x\}) > 0$.*
3. *For all $x, y \in X$, $x \neq y$, $\exists d \in \mathfrak{D}$ such that $d(x, y) > 0$ or $d(y, x) > 0$.*

Proof. It is given in [13]. □

Remark 3.2. By Definition 2.1 (2) and Theorem 3.1, (X, δ) is $\overline{T_0}$ if and only if for all $x, y \in X$ with $x \neq y$, $\delta(x, \{y\}) = \infty$ or $\delta(y, \{x\}) = \infty$. From this fact and Theorem 3.4, $\overline{T_0}$ (in our sense) implies T_0 (in the usual sense) but converse implication is not true. For instance, let $X = \{x, y\}$ with $x \neq y$ and define $\delta(x, \{y\}) = 1 = \delta(y, \{x\})$ and $\delta(x, \{x\}) = 0 = \delta(y, \{y\})$. Clearly, (X, δ) is T_0 (in the usual sense) but it is not $\overline{T_0}$.

Example 3.3. Let $X = \{p, x\}$ be a set, $A \subset X$. Define

$$\delta(p, A) = \begin{cases} 0, & A \neq \emptyset \\ \infty, & A = \emptyset \end{cases}$$

By Theorem 3.1, (X, δ) is not $\overline{T_0}$ at p.

4. Conclusions

In this paper, we gave characterization of local T_0 distance-approach spaces and local T_0 gauge-approach spaces. Moreover, by Theorem 3.1 and 3.2 and Remark 3.2, $\overline{T_0}$ approach space implies the T_0 approach space (in the usual sense) but the converse implication is not true, in general.

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