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W-Curves in Lorentz-Minkowski Space

Emre Öztürk* and Yusuf Yaylı

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Abstract
In this paper, we investigate the chord properties of the non-null W-curves in Lorentz-Minkowski space. We give the general equation form for W-curves in (2n+1)-dimension. We define some special curves and give the relations between these curves and isoparametric surfaces. Finally we obtain the geodesics of the pseudospherical cylinder and pseudohyperbolic cylinder in 4-dimensional space.

Keywords: *W-curve; isoparametric surface; Lorentz-Minkowski space*

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****Corresponding author***

1. Introduction

Let $\gamma : I \to \mathbb{R}^n$ be a regular curve in Euclidean space $\mathbb{R}^n$ (i.e. $\|\gamma'\|$ is nowhere zero), where I is an interval in $\mathbb{R}$. The curve γ is called a Frenet curve of rank r if $\gamma'(t), \gamma''(t), ..., \gamma^{(r)}(t)$ are linear independent and $\gamma'(t), \gamma''(t), ..., \gamma^{(r+1)}$ are no longer linear independent for all t in I. In this case, $Im(\gamma)$ lies in an $r-$dimensional Euclidean subspace of $\mathbb{R}^n$. For each unit speed Frenet curve of rank r there occur an associated orthonormal $r-$frame $\{V_1, V_2, ..., V_r\}$ along γ, the Frenet $r-$frame, and $r-1$ functions $\kappa_1, \kappa_2, ..., \kappa_{r-1} : I \longrightarrow \mathbb{R}$, and the Frenet curvatures, such that

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ ... \\ V_r' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & ... & 0 \\ -\kappa_1 & 0 & \kappa_2 & ... & 0 \\ 0 & -\kappa_2 & 0 & ... & 0 \\ ... & & & & \kappa_{r-1} \\ 0 & 0 & ... & -\kappa_{r-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ ... \\ V_r \end{bmatrix} \tag{1.1}$$

[9]. Equation (1.1) is called by Frenet-Serret formulas. The curve γ is called a $W-$curve or curve with constant curvature, if it has constant Frenet curvatures. $W-$curves are the orbits of the instanenous space motions. According to relativity theory in Physics, the free particles move on these curves in space. In [8], some of surfaces whose geodesics are $W-$curves investigated in $\mathbb{R}^4$. They used "helical surface" term whose geodesics have the same constant curvatures at any point on this surfaces. In ([10], Chapter 33), $W-$curves are considered as a solution of the system of linear homogeneous ordinary differential equations of first order with constant coefficients. This equation system is given by taking all Frenet curvatures κ_i, $1 \le i \le r-1$ are constant in (1.1). In [11], the parametric equation of $W-$curves in $\mathbb{R}^{2n+1}$ is given by the following theorem:

Theorem 1.1. *For any unit speed $W-$curve in $\mathbb{R}^{2n+1}$ has the following form,*

$$\gamma(s) = \vec{\gamma}_0 + as\vec{e}_0 + \sum_{i=1}^{n} r_i \left(\cos(a_i s)\, \vec{e}_{2i-1} + \sin(a_i s)\, \vec{e}_{2i}\right)$$

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such that here $\{\vec{e}_0, \vec{e}_1, ..., \vec{e}_{2n}\}$ is orthonormal base of $\mathbb{R}^{2n+1}$. In [2], $\vec{\gamma_0}$ is constant vector, $a \in \mathbb{R}$ and $a_1 < a_2 < ... < a_n$ positive real numbers satisfy the following equation,

$$a^2 + \sum_{i=1}^{k} (r_i a_i)^2 = 1$$

If $a = 0$ then rank of γ is even and this curve lies in $\mathbb{R}^{2n}$ on hypersphere, otherwise γ fully lies in $\mathbb{R}^{2n+1}$.

Let we give the simple idea in ([1], pp. 160–162):

> "Circle can characterize as a closed plane curve such that the chord joining any two points on it meets the curve at the same angle at the two points."

In ([2], Lemma 7.1), the idea above interpreted with complex analysis perspective and they expressed the following:

> "The only plane region Ω (bounded, smooth, simply-connected) whose Szegö kernel coincides with the Cauchy kernel is the disc."

Boas [3], generalized the result above, by the help of Bochner-Martinelli kernel, and he stated that "Ball (hypersphere) is the only bounded C^1 domain in $\mathbb{R}^m$ such that given any two points of the boundary, the chord joining them meets the normals at the two endpoints with equal angles."

In ([3], Proof of Theorem 2, pp. 277-278), the chord property idea used for characterize the isoparametric surfaces (surfaces with constant principal curvatures) in Euclidean space. Boas [4], gave the global characterization theorem and local characterization theorem as a result of chord property and he generalized his [3] work to all isoparametric surfaces. Wegner [5], gave short proof for ([4], Local characterization theorem, pp.120). Differently from ([3],[4] and [5]), in ([6], Proof of Theorem 1, pp. 460-463), $W-$curves are characterized by chord property. They interpreted the idea in ([1], pp. 160–162) and based on this idea they gave the condition

$$\langle X(t) - X(s), T(t) - T(s) \rangle = 0 \tag{\textbf{C}}$$

for all circles. Here, $X(s)$ is a unit speed curve, and $T(s) = X'(s)$ is a unit tangent vector field of the curve. After this, they asked that question,

> "Which Euclidean space curves satisfy condition ($\mathbf{C}$)?"

and they answered above the question by the following theorem:

Theorem 1.2. *Any unit speed smooth curve X in the Euclidean $n-$space is a $W-$curve if and only if the chord joining any two points on it meets the curve at the same angle.*

In ([6], Theorem 1) chord property generalized for the curves in high dimensions and some mathematical analysis technics are used for the proof. Unlike the [6], in ([7], Theorem B) the condition ($\mathbf{C}$) is interpreted by the help of linear algebraic technics and elementary proof is given for the Theorem 1.2, in $\mathbb{R}^n$. They showed that, the unit tangent vector field of the curves whose are satisfy condition ($\mathbf{C}$), can be presented with $T(s) = AX(s) + b$ for any constant skew-symmetric matrix A and constant vector b. They showed the derivations of $\left\|X^{(k)}(s)\right\|$ are constant for all $k, 1 \leq k \leq n$. Also they gave the chord property for surfaces given by the following idea:

> "For a hypersphere M of Euclidean space, the chord joining any two points on it meets the sphere at the same angle at the two points, that is, the sphere satisfies the condition."

In ([7], Theorem A), similarly in ([4], Local Characterization Theorem), they gave the following condition for hypersurfaces M by

$$\langle y - x, G(x) + G(y) \rangle = 0 \tag{\textbf{G}}$$

The condition holds identically for hyperspheres such that in condition ($\mathbf{G}$), $x, y \in M$ and G is Gauss map of M. Then they asked,

> "What are hypersurfaces of Euclidean space which satisfy the condition ($\mathbf{G}$)?"

and answered above the question by "isoparametric surfaces". Also they showed, the Gauss map of isoparametric surfaces can be written as $G(x) = Ax + b$ for constant symmetric matrix A and constant $b \in \mathbb{R}^n_1$ vector.

Unlike the Euclidean space, metric tensor of the Lorentz-Minkowski space has also negative definite. Therefore this space has more type of vectors, curves and surfaces than the Euclidean space. Let give the definition of the metric tensor (inner product) of Lorentz-Minkowski space via [12]:

Definition 1.1. Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be a two vector distinct from zero in $n-$dimensional real vector space $\mathbb{R}^n$. Following inner product,

$$\langle X, Y\rangle = -x_1y_1 + \sum_{i=2}^{n} x_iy_i$$

is called by Lorentzian inner product of X and Y. Here, $\langle,\rangle$ is called metric tensor of vector space and $(\mathbb{R}^n, \langle,\rangle)$ is called Lorentz-Minkowski space and showed by $\mathbb{R}^n_1$ or simply $\mathbb{L}^n$.

$W-$curves are studied by ([11], [13], [15], [20]) in Lorentz-Minkowski space. Walrave [15] presented the classification of the $W-$curves and he gave the relations between the curvature and torsion of the $W-$curves in $3-$dimensional Minkowski space. Differently from Euclidean space, in [15], shown that some $W-$curves has cubic polynomial parametric forms in $3-$dimensional Minkowski space. In ([13], Section 3-4), spacelike $W-$curves are considered in $3-$dimensional Minkowski space and they gave some chracterizations for spacelike $W-$curves whose image lies on the pseudohyperbolical space $\mathbb{H}^2_0$ and Lorentzian sphere $\mathbb{S}^2_1$ by using the position vectors of the curve. In ([11], Section 3), all spacelike $W-$curves are classified and specified where the curve lies, in $4-$dimensional Minkowski space. In [20], the equations of the curves with constant curvatures calculated through the interpreting the mapping of helicoidal vector fields, and elements of Lie algebra in $\mathbb{R}^{2n+1}_1$. It is done by classifying rank of the following matrix:

$$Q = \begin{bmatrix} w & v \\ 0 & 0 \end{bmatrix}$$

where $w \in \mathbb{R}^{2n+1}_{2n+1}$ is semi skew-symmetric matrix, $v \in \mathbb{R}^{2n+1}_1$ is coloumn vector and Q is the matrix of the helicoidal vector field X with respect to orthonormal base $\{0; u_1, u_2, ..., u_{2n+1}\}$ in $\mathbb{R}^{2n+1}_1$. Ünal [20], considered $W-$curves as a solutions of the following differential equation system:

$$\begin{bmatrix} X(M) \\ 0 \end{bmatrix} = \begin{bmatrix} w & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M \\ 1 \end{bmatrix}$$

where $M = (x_1, x_2, ..., x_{2n+1}) \in \mathbb{R}^{2n+1}_1$ and X is the helicoidal vector field in Lorentz space. Ünal [20] considered the M, as a position vector of the curve $\alpha = (\alpha_1(t), \alpha_2(t), ..., \alpha_{2n+1}(t))$ and $X(M) = \dfrac{d\alpha}{dt}$.

In this study, we called the $\mathbf{C}-$curve which satisfies the condition $(\mathbf{C})$ in Lorentz-Minkowski space. We consider the chord properties of $W-$curves and we get some results about the $W-$curves and $\mathbf{C}-$curves. We define the $\mathbf{N}-$curves through the chord property and we investigate the relation between $\mathbf{C}-$curves and $\mathbf{N}-$curves. We give the theorem which state the relation of $\mathbf{N}-$curves and isoparametric surfaces. Finally we get the geodesics of $\mathbb{S}^2_1 \times \mathbb{R}$ and $\mathbb{H}^2_1 \times \mathbb{R}$, and we show that these geodesics are the $\mathbf{N}-$curves in Lorentz-Minkowski space.

2. Preliminary

In Lorentz-Minkowski space, vectors have different causal characters such that if $\langle v, v\rangle > 0$ or $v = 0$, $\langle v, v\rangle < 0$ and $\langle v, v\rangle = 0$ $(v \neq 0)$ then v is called by spacelike, timelike and lightlike (or null) vector respectively for $v \in \mathbb{R}^n_\nu$. The number of timelike vectors at the orthonormal basis of the vector space is called by the index of the space. This number is usually denoted by ν and in particular, if the index of space is 1 then the space is indicated by $\mathbb{R}^n_1$ and called Lorentz vector space. Norm of any vector is given with $\|v\| = \sqrt{|\langle v, v\rangle|}$ [17]. For any regular $\gamma : I \to \mathbb{R}^n_1$ curve if $\langle \gamma', \gamma'\rangle > 0$, $\langle \gamma', \gamma'\rangle < 0$ or $\langle \gamma', \gamma'\rangle = 0$ then the curve is called by spacelike, timelike or lightlike (null) curve respectively [16]. If $\langle \gamma'(t), \gamma'(t)\rangle = \mp 1$ for all $t \in I$ then γ called by unit speed curve. Let index of $n-$dimensional vector space of V be $0 \leq \nu \leq n$. If $\varepsilon_1 = \varepsilon_2 = ... = \varepsilon_\nu = -1$ and $\varepsilon_{\nu+1} = \varepsilon_{\nu+2} = ... = \varepsilon_n = 1$ then the diagonal matrix $(\delta_{ij}\varepsilon_j)$ is called by the sign matrix of V such that

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \quad 1 \leq i, j \leq n$$

is Kronecker delta [17]. For any vectors $\overrightarrow{u} = (u_1, u_2, u_3)$ and $\overrightarrow{v} = (v_1v_2, v_3)$ in $\mathbb{R}^3_1$, Lorentzian cross product of $\overrightarrow{u}$ and $\overrightarrow{v}$ is defined by

$$\overrightarrow{u} \times \overrightarrow{v} = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2, u_1v_3 - u_3v_1, u_2v_1 - u_1v_2)$$

[14]. Frenet formulas for any spacelike or timelike γ curve with $\langle \gamma'', \gamma'' \rangle \neq 0$ in $\mathbb{R}^3_1$ are given by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa\eta & 0 \\ -\kappa\epsilon & 0 & -\tau\epsilon\eta \\ 0 & -\tau\eta & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.1}$$

where $\{T, N, B\}$ is the Frenet frame of the curve, that is $N = \dfrac{T'}{\|T'\|}$ and $B = T \times N$. Also $\langle T, T \rangle = \epsilon$, $\langle N, N \rangle = \eta$ and $\langle B, B \rangle = -\epsilon\eta$ [16]. Let we give the Frenet formulas for the curve in $\mathbb{R}^n_1$ by the following definition:

Definition 2.1. [12] Let $\gamma : I \to \mathbb{R}^n_1$ be a unit speed non-null curve, and κ_i be $i-$th Frenet curvature function of the curve. Frenet frame and Frenet formulas of the curve is given by $\{V_1, V_2, ..., V_n\}$ and

$$\begin{array}{lllll} V_1' & = & \nabla_{V_1} V_1 & = & \varepsilon_2 \kappa_1 V_2 \\ V_2' & = & \nabla_{V_1} V_2 & = & -\varepsilon_1 \kappa_1 V_1 + \varepsilon_3 \kappa_2 V_3 \\ . & & . & & . \\ . & & . & & . \\ . & & . & & . \\ V_i' & = & \nabla_{V_1} V_i & = & -\varepsilon_{i-1} \kappa_{i-1} V_{i-1} + \varepsilon_{i+1} \kappa_i V_{i+1} \\ V_{i+1}' & = & \nabla_{V_1} V_{i+1} & = & -\varepsilon_i \kappa_i V_i \end{array} \tag{2.2}$$

respectively. In (2.2), $\varepsilon_i = \langle V_i, V_i \rangle = \mp 1$ and ∇ is the Levi-Civita (Riemann) connection of $\mathbb{R}^n_1$.

Due to equation system of (2.2), $i-$th Frenet curvature function of any unit speed non-null curve is given by

$$\kappa_i(s) = \langle V_i'(s), V_{i+1}(s) \rangle$$

for all $i, 1 \leq i \leq n-1$.

Definition 2.2. [17] Let $n \geq 2$ and $0 \leq \nu \leq n$ then,

(1) The pseudosphere of radius $r > 0$ in $\mathbb{R}^{n+1}_\nu$ is the hyperquadric

$$\mathbb{S}^n_\nu(r) = \left\{ P \in \mathbb{R}^{n+1}_\nu : \langle P, P \rangle = r^2 \right\}$$

with dimension n and index ν.

(2) The pseudohyperbolic space of radius $r > 0$ in $\mathbb{R}^{n+1}_{\nu+1}$ is the hyperqadric

$$\mathbb{H}^n_\nu(r) = \left\{ P \in \mathbb{R}^{n+1}_{\nu+1} : \langle P, P \rangle = -r^2 \right\}$$

with dimension n and index ν.

3. Chord Properties and General Form of the W-Curves

In this section, differently from ([6],[7]) we consider the chord property of the curves in Lorentz-Minkowski space. We define special curves and investigate the relation between these curves and isoparametric surfaces. In ([7], Theorem B, Theorem A), unit tangent vector field of the $W-$curves and unit normal vector field of the isoparametric surfaces are given by skew-symmetric and symmetric matrix in Euclidean space, respectively. Analogously, we give the unit tangent vector field of the $W-$curves and unit normal vector field of the isoparametric surfaces by semi skew-symmetric and semi symmetric matrix in Lorentz-Minkowski space, respectively. Inspiration by chord property of curves and surfaces, we give a new definition for curves. We called this curves by $\mathbf{N}-$curves. We see that these curves corresponds to a geodesics of the isoparametric surface in Lorentz-Minkowski (also Euclidean) space. Also all $\mathbf{C}-$curves are the $\mathbf{N}-$curves in Lorentz-Minkowski (also Euclidean) space. Here, we except the Euclidean space at our results.

Definition 3.1. Let $X = X(s) : I \to \mathbb{R}^n_1$ be a unit speed non-null curve. For all $t \in I$, if

$$\langle X(t) - X(s), T(t) - T(s) \rangle = 0$$

then X is called by $\mathbf{C}-$curve in Lorentz-Minkowski space.

Theorem 3.1. *Let $X = X(s) : I \to \mathbb{R}_1^n$ be a unit speed non-null* **C**$-$*curve. Unit tangent vector field of X is given by*

$$T(s) = AX(s) + b \tag{3.1}$$

where A is constant semi skew-symmetric matrix, and $b \in \mathbb{R}_1^n$ is constant column vector.

Proof. Let define the matrix A in ([7], Lemma 3.1) by

$$A^T = \varepsilon \left([B_1, B_2, ..., B_n] [A_1, A_2, ..., A_n]^{-1} \right) \varepsilon$$

where $\varepsilon = diag(-1, 1, ..., 1)$ is the sign matrix of $\mathbb{R}_1^n$ and A^T is the transpose matrix of A. By the help of similar technics in ([7], Proof of Theorem A and Theorem B), we get the intended. □

Here note that, (3.1) shows that we can calculate the equations of **C**$-$curves by the help of any semi skew-symmetric matrix in Lorentz-Minkowski space. If we consider the equation $X'(s) = AX(s) + b$, similarly in Kinematics, we can see that these curves corresponds to integral curves of its unit tangent vector field.

Corollary 3.1. *Let $X = X(s) : I \to \mathbb{R}_1^n$ be a non-null unit speed* **C**$-$*curve in Lorentz-Minkowski space. Norm of the high order derivations of the curve, $\left\| X^{(k)}(s) \right\|$ are constant, for all $k, 1 \leq k \leq n$.*

Proof. It can be easily seen as a result of Definition 3.1 and Theorem 3.2. □

Corollary 3.3 shows the important detail for the motions that the particles has constant accelerate in space whose are moves on the non-null unit speed **C**$-$curve, because of $\|X''(s)\|$ is constant.

Theorem 3.2. *Let $X = X(s) : I \to \mathbb{R}_1^n$ be a non-null unit speed* **C**$-$*curve. Frenet curvatures of X are constant, in other words, X is a $W-$curve.*

Proof. Suppose that X is a non-null unit speed **C**$-$curve. From Corollary 3.3, we get easily $\left\langle X^{(i)}(s), X^{(j)}(s) \right\rangle$ is constant when $j - i$ even, otherwise $\left\langle X^{(i)}(s), X^{(j)}(s) \right\rangle$ is zero. Let $\left\{ X'(s), X''(s), ..., X^{(r)}(s) \right\}$ be linear independent for $1 \leq r \leq n-1$, on interval I and $Sp \left\{ X'(s), X''(s), ..., X^{(r)}(s) \right\}$ be a non-degenerate space. By Gramm-Schmidt's orthogonalization method on $\left\{ X'(s), X''(s), ..., X^{(r)}(s) \right\}$ we define,

$$E_i(s) = X^{(i)}(s) - \sum_{j<i} \epsilon_j \frac{\left\langle X^{(i)}(s), E_j(s) \right\rangle}{\|E_j(s)\|^2} E_j(s)$$

where we consider $1 \leq i \leq n-1$ and $\epsilon_j = 1$ if $E_j(s)$ is spacelike, otherwise $\epsilon_j = -1$ if $E_j(s)$ is timelike. Easily we get $\left\langle X^{(i)}(s), E_j(s) \right\rangle$ and $\|E_j(s)\|^2$ are constant and so $E_i \in Sp \left\{ X'(s), X''(s), ..., X^{(r)}(s) \right\}$ and $\|E_i(s)\|$ is constant. Therefore $i-$th Frenet vector of the curve is given by

$$V_i(s) = c_1 X'(s) + c_2 X''(s) + ... + c_i X^{(i)}(s)$$

where $V_i(s) = \dfrac{E_i(s)}{\|E_i(s)\|}$ and $c_i \in \mathbb{R}$. Due to Definition 2.1 we get $\kappa_i(s)$ are constant for all $i, 1 \leq i \leq n-1$, so X is a $W-$curve. □

Definition and applications of the linear vector fields in Euclidean space, are given in ([19], Definition 5.24). In [18], the integral curves of the linear vector fields are investigated in Euclidean space. In [20], helicoidal vector fields are considered as a linear vector fields and investigated relation between these vector fields and instantaneous motions. $W-$curves are considered as an integral curves of the helicoidal vector field and generalized in the $\mathbb{R}_1^{2n+1}$. Let we give the following theorem first:

Theorem 3.3. *([19], Theorem 5.26) Let A be a linear mapping in $\mathbb{R}^3$ given by a skew-symmetric matrix in an orthonormal base. Then it is possible to find an orthonormal base in $\mathbb{R}^3$ such that the matrix of the mapping A assumes the form*

$$\Psi = \begin{bmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\lambda \in \mathbb{R}$.

Ünal [20] considered the Ψ as a matrix of helicoidal vector fields in $\mathbb{R}_1^{2n+1}$. In ([20], Theorem 5.4.4) the following differential equation system is given by

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ . \\ . \\ . \\ . \\ x_{2n-1}'(t) \\ x_{2n}'(t) \\ x_{2n+1}'(t) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \lambda_1 & 0 & . & . & . & 0 & 0 & 0 & 0 & b_1 \\ \lambda_1 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & \lambda_2 & 0 & . & . & . & 0 & 0 & b_3 \\ . & . & -\lambda_2 & 0 & 0 & . & . & . & 0 & 0 & . \\ . & . & 0 & 0 & 0 & & & & . & . & . \\ . & . & . & & & . & & & . & . & . \\ 0 & . & . & . & . & . & . & & . & . & . \\ 0 & 0 & . & . & . & 0 & 0 & 0 & \lambda_n & 0 & b_{2n-1} \\ 0 & 0 & 0 & . & . & . & 0 & -\lambda_n & 0 & 0 & b_{2n} \\ 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & b_{2n+1} \\ 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ . \\ . \\ . \\ . \\ x_{2n-1}(t) \\ x_{2n}(t) \\ x_{2n+1}(t) \\ 1 \end{bmatrix} \tag{3.2}$$

such that for all $i, 1 \leq i \leq 2n+1$ and $j, 1 \leq j \leq n$, $b_i, \lambda_j \in \mathbb{R}$ and $\alpha = \alpha(t) = (x_1(t), x_2(t), ..., x_{2n+1}(t))$ is the integral curve of the helicoidal vector field in $\mathbb{R}_1^{2n+1}$. Explicitly, statement of (3.2) is given by

$$\frac{dx_1}{dt} = \lambda_1 x_2 + b_1, \frac{dx_2}{dt} = \lambda_1 x_1 + b_2, \frac{dx_{2n+1}}{dt} = b_{2n+1}$$

otherwise for all $i, 3 \leq i \leq 2n$,

$$\frac{dx_i}{dt} = \begin{cases} \lambda_{\frac{i+1}{2}} x_{i+1} + b_i, \ i \text{ is odd} \\ -\lambda_{\frac{i}{2}} x_{i-1} + b_i, \ i \text{ is even} \end{cases}$$

Ünal [20] considered (3.2) as a system of linear homogeneous ordinary differential equations of first order with $\lambda_1 = \lambda_2 = ... = \lambda_n = 1$.

Remark 3.1. According to Theorem 3.2, $\mathbf{C}-$curves are the integral curve of the unit tangent vector field of the curve. So we can consider the (3.1) and (3.2) are equivalent.

Differently from ([20], Theorem 5.4.4), we give general form for (arbitrary λ_i) unit speed $W-$curves and we sign where the curve lies in Lorentz-Minkowski space. We consider general form of the curve with arbitrary $\lambda_i \in \mathbb{R}$ and $b = (b_1, b_2, ..., b_{2n+1}) = 0$ in (3.2).

Theorem 3.4. *Let the $Sp\left\{\alpha'(s), \alpha''(s), ..., \alpha^{(2n+1)}(s)\right\}$ be a non-degenere space. Due to (3.2), general parametric form of the unit speed non-null curve $\alpha = \alpha(s)$ in $\mathbb{R}_1^{2n+1}$ is given by*

$$\begin{aligned} \alpha(s) = (&a_1 \sinh \lambda_1 s + b_1 \cosh \lambda_1 s, a_1 \cosh \lambda_1 s + b_1 \sinh \lambda_1 s, \\ &a_2 \sin \lambda_2 s + b_2 \cos \lambda_2 s, a_2 \cos \lambda_2 s - b_2 \sin \lambda_2 s, ..., \\ &a_n \sin \lambda_n s + b_n \cos \lambda_n s, a_n \cos \lambda_n s - b_n \sin \lambda_n s, cs) \end{aligned} \tag{3.3}$$

such that $a_i, b_i, \lambda_i, c \in \mathbb{R}$ for all $i, 1 \leq i \leq n$ and

$$-a_1^2\lambda_1^2 + b_1^2\lambda_1^2 + c^2 + \sum_{i=2}^{n} \left(a_i^2 + b_i^2\right) \lambda_i^2 = \mp 1$$

In (3.3), α is a $\mathbf{C}-$curve. If $c = 0$, α lies on $\mathbb{S}_1^{2n}$ pseudosphere or on $\mathbb{H}_1^{2n}$ pseudohyperbolic space, otherwise α fully lies in $\mathbb{R}_1^{2n+1}$.

Proof. Let α is given by (3.3). Unit speed tangent vector field of this curve is written by

$$\begin{aligned} T(s) = (&a_1\lambda_1 \cosh \lambda_1 s + b_1\lambda_1 \sinh \lambda_1 s, a_1\lambda_1 \sinh \lambda_1 s + b_1\lambda_1 \cosh \lambda_1 s, \\ &a_2\lambda_2 \cos \lambda_2 s - b_2\lambda_2 \sin \lambda_2 s, -a_2\lambda_2 \sin \lambda_2 s - b_2\lambda_2 \cos \lambda_2 s, ..., \\ &a_n\lambda_n \cos \lambda_n s - b_n\lambda_n \sin \lambda_n s, -a_n\lambda_n \sin \lambda_n s - b_n\lambda_n \cos \lambda_n s, c) \end{aligned} \tag{3.4}$$

Due to (3.3) and (3.4), easily we get

$$\langle \alpha(t) - \alpha(s), T(t) - T(s) \rangle = 0$$

by straightforward calculations. Therefore α is a $\mathbf{C}-$curve. If $c=0$ then we write

$$\langle \alpha(s), \alpha(s)\rangle = a_1^2 - b_1^2 + \sum_{i=2}^{n}\left(a_i^2+b_i^2\right) = \mp r^2$$

Due to Definition 2.2, α lies on $\mathbb{S}_1^{2n}(r)$ or $\mathbb{H}_1^{2n}(r)$. Otherwise it fully lies in $\mathbb{R}_1^{2n+1}$. □

Corollary 3.2. *In (3.3), $\alpha(s)$ is a $W-$curve.*

Proof. It can be easily seen by Theorem 3.4 and Theorem 3.7. □

Remark 3.2. Unlike the Euclidean space, we have to note that there are also some $W-$curves in $\mathbb{R}_1^3$ whose have a polynomial form such as

$$\gamma(s) = \left(\frac{1}{6}\lambda\left|\lambda\right|s^3, -\frac{1}{6}\lambda^2 s^3 + s, \frac{1}{2}\left|\lambda\right|s^2\right)$$

and

$$\gamma(s) = \left(\frac{1}{6}\lambda^2 s^3 + s, \frac{1}{2}\lambda s^2, \frac{1}{6}\left|\lambda\right|\lambda s^3\right)$$

for $\lambda \in \mathbb{R}$[15]. These curves can get easily by substitute

$$A = \begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & |\lambda| \\ \lambda & -|\lambda| & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & \lambda & 0 \\ \lambda & 0 & |\lambda| \\ 0 & -|\lambda| & 0 \end{bmatrix}$$

in Theorem 3.2, respectively.

Definition 3.2. Let $I \subset \mathbb{R}$, $\gamma = \gamma(s) : I \to \mathbb{R}_1^n$ be a unit speed curve and N be a unit normal vector field of γ. If for all $t \in I$

$$\langle \gamma(t) - \gamma(s), N(t) + N(s)\rangle = 0$$

then γ is called by $\mathbf{N}-$curve in Lorentz-Minkowski space.

Theorem 3.5. *$\gamma = \gamma(s)$ be a unit speed $\mathbf{C}-$curve. This curve is a $\mathbf{N}-$curve and unit normal vector field of γ written by*

$$N(s) = B\gamma(s) + c$$

for constant semi symmetric B matrix and constant coloumn c vector.

Proof. Let γ be a unit speed $\mathbf{C}-$curve. By Theorem 3.2 we write

$$T(s) = A\gamma(s) + b \tag{3.5}$$

for semi skew-symmetric A matrix and $b \in \mathbb{R}_1^n$ coloumn vector. If we differentiate (3.5) according to arclenght s, we get

$$T'(s) = A\gamma'(s) = AT(s) \tag{3.6}$$

Through the (2.1) we write

$$T'(s) = \kappa(s)\eta N(s) \tag{3.7}$$

such that $\eta = \langle N, N\rangle$. By Theorem 3.4, $\kappa(s) = \kappa \neq 0$ is constant. Due to (3.6) and (3.7) we write

$$N(s) = \frac{1}{\kappa\eta}AT(s) \tag{3.8}$$

By the help of (3.8) and semi skew-symmetric A matrix we get

$$\langle \gamma(t) - \gamma(s), N(t) + N(s)\rangle = 0$$

Therefore γ is a $\mathbf{N}-$curve. From (3.5) and (3.8) we write

$$N(s) = \frac{1}{\kappa\eta}A^2\gamma(s) + \frac{1}{\kappa\eta}Ab \tag{3.9}$$

A^2 is semi symmetric matrix because of A is semi skew-symmetric. Consider $\frac{1}{\kappa\eta}A^2 = B$ and $\frac{1}{\kappa\eta}Ab = c$ in (3.9) and we write

$$N(s) = B\gamma(s) + c$$

for constant semi symmetric B matrix and constant coloumn c vector. □

In ([4] and [7]) isoparametric surfaces of Euclidean space characterized by the chord property. Boas [4] gave this property by local characterization theorem for all surfaces M in Euclidean space with $\langle (x-y), \nabla r(x)\rangle = \langle (y-x), \nabla r(y)\rangle$ such that here $x, y \in M$ and ∇r is the unit normal (gradient) vector of M. In [7], this property given with $\langle y-x, G(x)+G(y)\rangle = 0$ such that here $x, y \in M$ and G is the Gauss map of M. In ([7], Theorem A), Gauss map of hypersurface which satisfy the ([7], Condition D) is given with $G(x) = Ax+b$ where A is constant symmetric matrix. In [7] proven that only the isoparametric surfaces ($\mathbb{R}^{m-1}$, $\mathbb{S}^{m-1}$ and $\mathbb{S}^{p-1}\times\mathbb{R}^{m-p}$) have mentioned above properties in $\mathbb{R}^m$.

Theorem 3.6. *In Lorentz-Minkowski space, non-null isoparametric hypersurface M satisfy*

$$\langle Q-P, G(P)+G(Q)\rangle = 0 \tag{3.10}$$

for all $P, Q \in M$.

Proof. According to ([7], Theorem A), all isoparametric surfaces satisfy

$$\langle Q-P, G(P)+G(Q)\rangle = 0$$

in Euclidean space. If we follow similar procedures as in ([7], Proof of Theorem A), we get Lorentzian hyperplanes, pseudosphere $\mathbb{S}_1^{n-1}(r)$, pseudohyperbolic space $\mathbb{H}_1^{n-1}(r)$, pseudospherical cylinder $\mathbb{S}_1^{p-1}(r)\times\mathbb{R}^{n-p}$ and pseudohyperbolic cylinder $\mathbb{H}_1^{p-1}(r)\times\mathbb{R}^{n-p}$ as an isoparametric surface in Lorentz-Minkowski space. It can be easily shown that these surfaces satisfy (3.10). □

Corollary 3.3. *Let M be a non-null hypersurface whose satisfy (3.10) and $x \in M$. Then the Gauss map of M is given by $G(x) = Ax+b$ for constant semi symmetric A matrix and constant $b \in \mathbb{R}_1^n$ vector.*

Proof. We get similar result by straightforward calculations in [7], Proof of theorem A). □

Corollary 3.4. *Let M be an isoparametric hypersurface in $\mathbb{R}_1^n$ Lorentz-Minkowski space. The geodesics of M are the* **N***−curves.*

Proof. For geodesics of any surfaces, unit normal (N) vector field of the geodesic curve and unit normal (n) of the surface are linear dependent so there is a $0 \neq \lambda \in \mathbb{R}$ and $n = \lambda N$. Due to (3.10) we write $\langle Q-P, N(P)+N(Q)\rangle = 0$. Therefore the geodesics of isoparametric surfaces are the **N**−curves. □

4. Geodesics of Pseudospherical Cylinder and Pseudohyperbolic Cylinder

Let $\overline{\nabla}$ be Levi-Civita connection of Lorentz-Minkowski space and ∇ be reduced connection on any hypersurface. According to Lopez [21], Gauss equation of any hypersurface in Lorentz-Minkowski space is written by

$$\overline{\nabla}_X Y = \nabla_X Y + \epsilon \langle AX, Y\rangle \overrightarrow{n} \tag{4.1}$$

where A is the shape operator, $\overrightarrow{n}$ is the unit normal vector field of the surface, X and Y are the vector fields on the surface and $\epsilon = \langle \overrightarrow{n}, \overrightarrow{n}\rangle$. In [22], natural projection of the Euclidean space is given by

$$\pi : \mathbb{R}^3\times\mathbb{R} \to \mathbb{R}^3 \hookrightarrow \mathbb{R}^4,\ \pi(v_1, v_2, v_3, v_4) = (v_1, v_2, v_3, 0) \tag{4.2}$$

Also unit normal vector field of the $\mathbb{S}^2\times\mathbb{R}$ is given by the help of the transformation of π. Therefore, normal vector of $\mathbb{S}^2\times\mathbb{R}$ considered as a normal of $\mathbb{S}^2$. Geodesics γ of this surface are interpreted (via Gauss equation) as a solution of the following equation in Euclidean space:

$$\ddot{\gamma} = \nabla_{\dot{\gamma}}\dot{\gamma} - \Big\langle \pi(\dot{\gamma}), \pi(\dot{\gamma})\Big\rangle \pi(\gamma) \tag{4.3}$$

We get the parametric equations of the geodesics of the $\mathbb{S}_1^2\times\mathbb{R}$ and $\mathbb{H}_1^2\times\mathbb{R}$ in Lorentz-Minkowski space, based on [21] and [22] below:

Theorem 4.1. *Let $\gamma = \gamma(s)$ be a unit speed geodesic curve on $\mathbb{S}_1^2 \times \mathbb{R}$ pseudospherical cylinder. Geodesics of this surface are parallel lines, circle or circular helix if γ is spacelike, otherwise geodesics are hyperbola or hyperbolic helix if γ is timelike.*

Proof. We can consider unit speed geodesic curves as a $\gamma(s) = (x(s), y(s), z(s), t(s))$ on $\mathbb{S}_1^2 \times \mathbb{R}$. Let ∇ be a connection on $\mathbb{S}_1^2 \times \mathbb{R}$. We write with respect to ([17], p.67)

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \tag{4.4}$$

Analogously in (4.2), we get the unit normal vector field $\overrightarrow{n}$ by the help of projection π on $\mathbb{S}_1^2$. Since $\mathbb{S}_1^2$ is timelike we write $\epsilon = \langle \overrightarrow{n}, \overrightarrow{n} \rangle = 1$. As required by (4.1), (4.3) and (4.4) we get

$$\ddot{\gamma} = -\left\langle \pi(\dot{\gamma}), \pi(\dot{\gamma}) \right\rangle \pi(\gamma) \tag{4.5}$$

Solutions of (4.5) changes based on causal character of gedoesic curve γ.
A) Let γ be a spacelike curve. Hence we write

$$-\dot{x}(s)^2 + \dot{y}(s)^2 + \dot{z}(s)^2 + \dot{t}(s)^2 = 1 \tag{4.6}$$

Due to (4.5) and (4.6) we get the differential equation system

$$\begin{aligned} \ddot{x}(s) &= -(1 - \dot{t}(s)^2) x(s) \\ \ddot{y}(s) &= -(1 - \dot{t}(s)^2) y(s) \\ \ddot{z}(s) &= -(1 - \dot{t}(s)^2) z(s) \\ \ddot{t}(s) &= 0 \end{aligned} \tag{4.7}$$

Note that, henceforth, we attach $x_0 = x(0), y_0 = y(0), z_0 = z(0), t_0 = t(0)$ and $u_0 = \dot{x}(0), v_0 = \dot{y}(0), \omega_0 = \dot{z}(0)$. In (4.7), if we consider $t(s) = \zeta_0 s + t_0$, we have three cases based on the values ζ_0.
Case 1. Let $\dot{t}(s) = \zeta_0 = \pm 1$. According to (4.7) we write

$$\begin{aligned} \ddot{x}(s) &= 0 \\ \ddot{y}(s) &= 0 \\ \ddot{z}(s) &= 0 \\ \ddot{t}(s) &= 0 \end{aligned} \tag{4.8}$$

If we solve the equation (4.8), we get

$$\gamma(s) = (u_0 s + x_0, v_0 s + y_0, \omega_0 s + z_0, \pm s + t_0)$$

as a parallel lines of $\mathbb{S}_1^2 \times \mathbb{R}$.
Case 2. Let $\dot{t}(s) = \zeta_0 = 0$. According to (4.7) we write

$$\begin{aligned} \ddot{x}(s) &= -x(s) \\ \ddot{y}(s) &= -y(s) \\ \ddot{z}(s) &= -z(s) \\ \ddot{t}(s) &= 0 \end{aligned} \tag{4.9}$$

If we solve the differential equation system (4.9), we get

$$\gamma(s) = (x_0 \cos s + u_0 \sin s, y_0 \cos s + v_0 \sin s, z_0 \cos s + \omega_0 \sin s, t_0)$$

as a circles whose are contained by the planes through the origin.
Case 3. Let $\dot{t}(s) = \zeta_0 \in (-1, 0) \cup (0, 1)$. According to (4.7) we get

$$\begin{aligned} \gamma(s) \quad = \quad & \left(x_0 \cos(\lambda_0 s) + \frac{u_0}{\lambda_0} \sin(\lambda_0 s), y_0 \cos(\lambda_0 s) + \frac{v_0}{\lambda_0} \sin(\lambda_0 s), \right. \\ & \left. z_0 \cos(\lambda_0 s) + \frac{\omega_0}{\lambda_0} \sin s, \zeta_0 s + t_0 \right) \end{aligned}$$

as a circular helix. Here $\lambda_0 = \sqrt{1-\zeta_0^2}$ and $\lambda_0 \in \mathbb{R}$.
B) Let γ be a timelike curve. Hence we write

$$-\dot{x}(s)^2 + \dot{y}(s)^2 + \dot{z}(s)^2 + \dot{t}(s)^2 = -1 \tag{4.10}$$

Due to (4.5) and (4.10) we get

$$\begin{aligned} \ddot{x}(s) &= (1+\dot{t}(s)^2)x(s) \\ \ddot{y}(s) &= (1+\dot{t}(s)^2)y(s) \\ \ddot{z}(s) &= (1+\dot{t}(s)^2)z(s) \\ \ddot{t}(s) &= 0 \end{aligned} \tag{4.11}$$

If we consider $t(s) = \zeta_0 s + t_0$, we have three cases based on the values ζ_0.
Case 1. Let $\dot{t}(s) = \zeta_0 = \pm 1$. According to (4.11) we write

$$\begin{aligned} \ddot{x}(s) &= 2x(s) \\ \ddot{y}(s) &= 2y(s) \\ \ddot{z}(s) &= 2z(s) \end{aligned} \tag{4.12}$$

If we solve the differential equation system (4.12) we get

$$\begin{aligned} \gamma(s) \;=\; & (x_0 \cosh(\sqrt{2}s) + \frac{u_0}{\sqrt{2}} \sinh(\sqrt{2}s), y_0 \cosh(\sqrt{2}s) + \frac{v_0}{\sqrt{2}} \sinh(\sqrt{2}s), \\ & z_0 \cosh(\sqrt{2}s) + \frac{\omega_0}{\sqrt{2}} \sinh(\sqrt{2}s), \pm s + t_0) \end{aligned}$$

as a hyperbolic helix.
Case 2. Let $\dot{t}(s) = \zeta_0 = 0$. According to (4.11) we write

$$\begin{aligned} \ddot{x}(s) &= x(s) \\ \ddot{y}(s) &= y(s) \\ \ddot{z}(s) &= z(s) \end{aligned} \tag{4.13}$$

If we solve the differential equation system (4.13) we get

$$\begin{aligned} \gamma(s) \;=\; & (x_0 \cosh s + u_0 \sinh s, y_0 \cosh s + v_0 \sinh s, \\ & z_0 \cosh s + \omega_0 \sinh s, t_0) \end{aligned}$$

as a hyperbolas.
Case 3. Let $\zeta_0 \in (-1,0) \cup (0,1)$. According to (4.11) we get

$$\begin{aligned} \gamma(s) \;=\; & (x_0 \cosh(\lambda_0 s) + \frac{u_0}{\lambda_0} \sinh(\lambda_0 s), y_0 \cosh(\lambda_0 s) + \frac{v_0}{\lambda_0} \sinh(\lambda_0 s), \\ & z_0 \cosh(\lambda_0 s) + \frac{\omega_0}{\lambda_0} \sinh s, \zeta_0 s + t_0) \end{aligned}$$

as a hyperbolic helix. Here $\lambda_0 \in \mathbb{R}$ and $\lambda_0 = \sqrt{1+\zeta_0^2}$. □

Theorem 4.2. *Let $\gamma = \gamma(s)$ be a unit speed geodesic curve on $\mathbb{H}_1^2 \times \mathbb{R}$ pseudohyperbolic cylinder. Geodesics of this surface are parallel lines, hyperbola or hyperbolic helix if γ is spacelike, otherwise geodesics are circle or circular helix if γ is timelike.*

Proof. We can consider unit speed geodesic curves as a $\gamma(s) = (x(s), y(s), z(s), t(s))$ on $\mathbb{H}_1^2 \times \mathbb{R}$. Let ∇ be a connection on $\mathbb{H}_1^2 \times \mathbb{R}$. Then we write $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$. Analogously in (4.2) we get the unit normal vector field $\overrightarrow{n}$ by the help of projection on $\mathbb{H}_1^2$. Since $\mathbb{H}_1^2$ is spacelike we write $\epsilon = \langle \overrightarrow{n}, \overrightarrow{n} \rangle = -1$. As required by (4.1) and (4.3) we write

$$\ddot{\gamma} = \left\langle \pi(\dot{\gamma}), \pi(\dot{\gamma}) \right\rangle \pi(\gamma) \tag{4.14}$$

A) Let γ be a spacelike curve. Hence we write

$$-\dot{x}(s)^2 + \dot{y}(s)^2 + \dot{z}(s)^2 + \dot{t}(s)^2 = 1 \tag{4.15}$$

Due to (4.14) and (4.15) we get

$$\begin{aligned}
\ddot{x}(s) &= (1 - \dot{t}(s)^2)x(s) \\
\ddot{y}(s) &= (1 - \dot{t}(s)^2)y(s) \\
\ddot{z}(s) &= (1 - \dot{t}(s)^2)z(s) \\
\ddot{t}(s) &= 0
\end{aligned} \tag{4.16}$$

In (4.16), if we consider $t(s) = \zeta_0 s + t_0$, we have three cases based on the values ζ_0.
Case 1. Let $\dot{t}(s) = \zeta_0 = \pm 1$. In this case we get parallel lines of $\mathbb{H}^2_1 \times \mathbb{R}$.
Case 2. Let $\dot{t}(s) = \zeta_0 = 0$. We get

$$\begin{aligned}
\gamma(s) \quad = \quad & (x_0 \cosh s + u_0 \sinh s, y_0 \cosh s + v_0 \sinh s, \\
& z_0 \cosh s + \omega_0 \sinh s, t_0)
\end{aligned}$$

as a hyperbola.
Case 3. Let $\zeta_0 \in (-1, 0) \cup (0, 1)$. In this case

$$\begin{aligned}
\gamma(s) \quad = \quad & (x_0 \cosh(\lambda_0 s) + \frac{u_0}{\lambda_0} \sinh(\lambda_0 s), y_0 \cosh(\lambda_0 s) + \frac{v_0}{\lambda_0} \sinh(\lambda_0 s), \\
& z_0 \cosh(\lambda_0 s) + \frac{\omega_0}{\lambda_0} \sinh s, \zeta_0 s + t_0)
\end{aligned}$$

is a hyperbolic helix. Here $\lambda_0 \in \mathbb{R}$ and $\lambda_0 = \sqrt{1 - \zeta_0^2}$.
B) Let γ be a timelike curve. Hence we write

$$-\dot{x}(s)^2 + \dot{y}(s)^2 + \dot{z}(s)^2 + \dot{t}(s)^2 = -1 \tag{4.17}$$

Due to (4.14) and (4.17) we get

$$\begin{aligned}
\ddot{x}(s) &= -(1 + \dot{t}(s)^2)x(s) \\
\ddot{y}(s) &= -(1 + \dot{t}(s)^2)y(s) \\
\ddot{z}(s) &= -(1 + \dot{t}(s)^2)z(s) \\
\ddot{t}(s) &= 0
\end{aligned} \tag{4.18}$$

In (4.18), if we consider $t(s) = \zeta_0 s + t_0$, we have three cases based on the values ζ_0.
Case 1. Let $\dot{t}(s) = \zeta_0 = 0$, we get

$$\gamma(s) = (x_0 \cos s + u_0 \sin s, y_0 \cos s + v_0 \sin s, z_0 \cos s + \omega_0 \sin s, t_0)$$

as a circles.
Case 2. Let $\dot{t}(s) = \zeta_0 = \pm 1$ or $\dot{t}(s) = \zeta_0 \in (-1, 0) \cup (0, 1)$ then we get finally

$$\begin{aligned}
\gamma(s) \quad = \quad & (x_0 \cos(\sqrt{2}s) + \frac{u_0}{\sqrt{2}} \sin(\sqrt{2}s), y_0 \cos(\sqrt{2}s) + \frac{v_0}{\sqrt{2}} \sin(\sqrt{2}s), \\
& z_0 \cos(\sqrt{2}s) + \frac{\omega_0}{\sqrt{2}} \sin(\sqrt{2}s), \pm s + t_0)
\end{aligned}$$

or

$$\begin{aligned}
\gamma(s) \quad = \quad & (x_0 \cos(\lambda_0 s) + \frac{u_0}{\lambda_0} \sin(\lambda_0 s), y_0 \cos(\lambda_0 s) + \frac{v_0}{\lambda_0} \sin(\lambda_0 s), \\
& z_0 \cos(\lambda_0 s) + \frac{\omega_0}{\lambda_0} \sin(\lambda_0 s), \zeta_0 s + t_0)
\end{aligned}$$

as a circular helix, respectively. Here $\lambda_0 \in \mathbb{R}$ and $\lambda_0 = \sqrt{1 + \zeta_0^2}$. □

Corollary 4.1. *The curves whose are given in Theorem 4.1 and Theorem 4.2 are the* $\mathbf{N}-$*curves.*

Proof. It can be easily seen from proof of Theorem 3.12 and Corollary 3.13. □

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Affiliations

Emre Öztürk
Address: Turkish Court of Accounts, Çankaya, 06520, Ankara-Turkey.
e-mail: emreozturk1471@gmail.com

Yusuf Yaylı
Address: Ankara University, Faculty of Science, Dept. of Mathematics, 06100, Ankara-Turkey.
e-mail: yayli@science.ankara.edu.tr