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Extended Kudryashov Method for Fractional Nonlinear Differential Equations

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Abstract
In this study, we have propesed the extended Kudryashov method to obtain the exact solutions of nonlinear fractional differential equations. Definiton of modified Riemann Liouville sense fractional derivative is used and the proposed method is applied to two nonlinear fractional differential equations. Analytical solutions including hyperbolic functions are obtained.

Keywords: *Fractional nonlinear differential equations, extended Kudryashov method; the space time fractional Zakharov Kuznetsov Benjamin Bona Mahony; the space time fractional Fokas equation.*

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1. Introduction

Fractional calculus is one of the most multidisciplinary field of mathematics. Many processes in physics and engineering are modeled more assertively by fractional derivatives than conventional integer order derivatives. Miller and Rose [21] made mention of nearly every field of science has application of fractional derivatives. It is well known that many real systems are fractional in nature, hence, it is more efficient to model them under favor of fractional order than integer order systems.

Fractional differential equations are the generalization of classical differential equations with integer order. So, in recent years, fractional differential equations become the realm of physicists and mathematicians who investigate the expediency of such non-integer order derivatives in different areas of physics and mathematics. It has been found that the behavior of many physical systems can be properly defined by using fractional order differential equations. For instance, heat conduction systems, nonlinear chaotic systems, viscoelasticity, plasma waves, acustic gravity waves, diffusion processes are governed by the fractional differential equations such as in [3, 6, 21, 25] and the reference therein. Many research works have proposed powerful techniques for solving fractional evolution equations, such as $G'/G-$expansion method [4, 5], Exp$-$function method [12, 29], first integral method[18], sub$-$equation method [2, 13, 27], trial equation method [8, 22, 24], rational function method [1], sub$-$equation method [27, 30], complex transform method [19, 20] and others.

The spearheading work of Kudryashov [17], introduced Kudryashov method for reliable treatment of nonlinear wave equations. The practicable Kudryashov method is widely used for both integer order and fractional order evolution equations by many researchers such as in [7, 9–11, 23, 26, 28] and the reference therein. In this paper we propose extended Kudryashov method for fractional evolution equations based upon homogenous balance principle by means of traveling wave transformation. In this method, by using the transformation $\xi = \frac{kx^{\beta}}{\Gamma(1+\beta)} + \frac{ly^{\gamma}}{\Gamma(1+\gamma)} + \frac{mz^{\delta}}{\Gamma(1+\delta)} + \cdots + \frac{ct^{\alpha}}{\Gamma(1+\alpha)}$, a given fractional differential equation turn into fractional ordinary differential equation whose solutions are in the form $U(\xi, Y) = \sum_{i=0}^{N} a_i Y^i(\xi)$, where $Y(\xi)$ satisfies the fractional

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Riccati equation $D_{\xi}^{\alpha}Y = Y^3 - Y$. The main merit of this method over the other classical methods is that gives more solutions with some parameters which effects both (either) speed and (or) amplitude of waves. By choosing convenient parameter, solutions can be turned into certain solutions obtained by existing methods.

The aim of this work is to find solitons and soliton-like solutions of the space time fractional Zakharov Kuznetsov Benjamin Bona Mahony and the space time fractional Fokas equations. The rest of paper is arranged as follows. Section 2 gives definition of Gamma function and an overview of modified Riemann Liouville fractional derivative. Section 3 presents the algorithmic procedure of extended Kudryashov method. We construct traveling wave solutions of space time fractional Zakharov Kuznetsov Benjamin Bona Mahony and the space time fractional Fokas equations in Section 4 to attest the effectiveness of the proposed method. We finish with Section 5 providing conclusions.

2. Preliminaries

Definition 1: A real function $f(t)$, $t > 0$, is said to be in the space C_κ, $\kappa \in R$, if there exists areal number $p > \kappa$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, and it is said to be in the space C_κ^m if $f^m \in C_\kappa, m \in N$ [26,27].

Definition 2: The gamma function $\Gamma(\alpha)$ is defined by the integral

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{(\alpha-1)} dt.$$

$\Gamma(\alpha)$ generates the factorial $n!$ and allows n to take also non-integer values and also gamma function can be represented also by limit

$$\Gamma(\alpha) = \lim_{n\to\infty} \frac{n! n^\alpha}{\alpha(\alpha+1)...(\alpha+n)}.$$

Definition 3: The modified Riemann-Liouville derivative is defined as [26,27]:

$$D_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-\xi)^{-\alpha-1} \left[f(\xi) - f(0)\right] d\xi, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-\xi)^{-\alpha} \left[f(\xi) - f(0)\right] d\xi, & 0 < \alpha < 1, \\ (f^{(n)}(x))^{\alpha-n}, & n \le \alpha < n+1, \ \ n \ge 1. \end{cases} \tag{2.1}$$

where

$$D_x^\alpha f(x) := \lim_{h\downarrow 0} h^{-\alpha} \sum_{k=0}^{\infty} (-1)^k f[x + (\alpha - k)h]. \tag{2.2}$$

In addition, some basic properties for the modified Riemann-Liouville derivative are given in [14–16] as follows:

$$D_t^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, \quad \gamma > 0, \tag{2.3}$$

$$D_t^\alpha (f(t)g(t)) = g(t) D_t^\alpha f(t) + f(t) D_t^\alpha g(t), \tag{2.4}$$

$$D_t^\alpha f[g(t)] = f_g'[g(t)] D_t^\alpha g(t) = D_g^\alpha f[g(t)](g'(t))^\alpha, \tag{2.5}$$

which are direct results of the equality $D^\alpha x(t) = \Gamma(1+\alpha) Dx(t)$ which holds for non-differentiable functions. In the above formulas, $f(t)$ is nondifferentiable function in Eq.(2.4), $g(t)$ is nondifferentiable function both in Eq.(2.4) and the right-side of Eq.(2.5), also differentiable in the left side of Eq.(2.5). $f(g)$ is differentiable in the right side of Eq.(2.5) and nondifferentiable in the left side of Eq.(2.5).

3. The extended Kudryashov method

We summarize the main steps of the extended Kudryashov method as follows:
For a given nonlinear FDEs for a function u of independent variables, $X = (x, y, z, \dots, t)$:

$$P\left(u, u_t, u_x, u_y, u_z, \dots, D_t^\alpha u, D_x^\alpha u, D_y^\alpha u, D_z^\alpha u, \dots\right) = 0. \tag{3.1}$$

where $D_t^\alpha u, D_x^\alpha u, D_y^\alpha u$ and $D_z^\alpha u$ are the modified Riemann-Liouville derivatives of u with respect to t, x, y and z. P is a polynomial in $u = u(x, y, z, \dots, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.
Step 1. We seek the traveling wave solutions of Eq.(3.1) by using the transformations in the form:

$$u(x, y, z, \dots, t) = u(\xi, Y), \quad \xi = \frac{kx^\beta}{\Gamma(1+\beta)} + \frac{ny^\gamma}{\Gamma(1+\gamma)} + \frac{mz^\delta}{\Gamma(1+\delta)} + \dots + \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}, \tag{3.2}$$

where k, n, m and λ are arbitrary constants. Using Eq.(2.3) and the first equality of Eq.(2.5), we obtain $D_x^\beta u = D_x^\beta u(\xi, Y) = u_\xi D_x^\beta \xi = k u_\xi$, $D_y^\gamma u = D_y^\gamma u(\xi, Y) = u_\xi D_y^\gamma \xi = n u_\xi$, ... , $D_t^\alpha u = D_t^\alpha u(\xi, Y) = u_\xi D_t^\alpha \xi = \lambda u_\xi$. Then Eq.(3.1) reduces to the following nonlinear ordinary differential equation of the form:

$$G(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}, \dots) = 0. \tag{3.3}$$

Step 2. We assume that the reduced equation has the following solution:

$$u(\xi, Y) = \sum_{i=0}^{N} a_i Y^i(\xi) \tag{3.4}$$

where $Y(\xi) = \frac{\pm 1}{\sqrt{1 \pm e^{2\xi}}}$ and the function Y is the solution of equation

$$Y_\xi(\xi) = Y^3(\xi) - Y(\xi). \tag{3.5}$$

Step 3. According to the method, solution of Eq.(3.3) can be expanded in the form

$$u(\xi, Y) = a_N Y^N + \cdots . \tag{3.6}$$

Analogously as in the classical Kudryashov method, we balance the highest order nonlinear terms in Eq.(3.3) to find out the value of the pole order N. Supposing $u^l(\xi, Y) u^{(s)}(\xi)$ and $(u^{(p)}(\xi, Y))^r$ are the highest order nonlinear terms of Eq.(3.3) and balancing the highest order nonlinear terms we have:

$$N = \frac{2(s - rp)}{r - l - 1}. \tag{3.7}$$

Step 4. Substituting Eq.(3.4) into Eq.(3.3) and equating the coefficients of Y^i to zero, we obtain a system of algebraic equations. Solving this system, we procure the exact solutions of Eq.(3.1). And the obtained solutions can depend on hyperbolic functions.

4. Examples

In this section, we will apply the extended Kudryashov method to the space-time Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZKBBM) equation and the space-time fractional Fokas equation .

4.1 Space-Time Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZKBBM) Equation

Let us consider the space-time Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZKBBM) equation

$$D_t^\alpha u + D_x^\alpha u - 2au D_x^\alpha u - b D_t(D_x^{2\alpha} u) = 0. \tag{4.1}$$

$$t > 0, \quad 0 < \alpha \leq 1.$$

where a and b are arbitrary constants. It arises as description of unidirectional propagation of long waves in certain nonlinear dispersive systems and small wave amplitude in the large wavelength regime.
Firstly, we take the following transformations

$$u(x,t) = u(\xi), \quad \xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)} \tag{4.2}$$

where $k, c \neq 0$ are constants.
Using property (2.3) and considering the wave transformation Eq.(4.2), Eq.(4.1) becomes an ordinary differential equation

$$(c+k)u - aku^2 - bck^2u'' = 0, \tag{4.3}$$

Suppose that the solutions of Eq.(4.3) can be expressed as follows:

$$u(\xi, Y) = \sum_{i=0}^{N} a_i Y^i(\xi)$$

where $Y(\xi) = \frac{\pm 1}{\sqrt{1 \pm e^{2\xi}}}$ satisfies $Y_\xi(\xi) = Y^3(\xi) - Y(\xi)$. Then, considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq.(4.3), we find

$$N = 4.$$

Hence, we have

$$u(\xi, Y) = a_0 + a_1 Y(\xi) + a_2 Y^2(\xi) + a_3 Y^3(\xi) + a_4 Y^4(\xi) \tag{4.4}$$

and substituting derivatives of $u(\xi, Y)$ with respect to ξ we obtain

$$\begin{aligned} u_\xi &= 4a_4Y^6(\xi) + 3a_3Y^5(\xi) + (2a_2 - 4a_4)Y^4(\xi) + (a_1 - 3a_3)Y^3(\xi) \\ &- 2a_2Y^2(\xi) - a_1Y(\xi), \\ u_{\xi\xi} &= 24a_4Y^8(\xi) + 15a_3Y^7(\xi) + (8a_2 - 40a_4)Y^6(\xi) + (3a_1 - 24a_3)Y^5(\xi) \\ &+ (16a_4 - 12a_2)Y^4(\xi) + (9a_3 - 4a_1)Y^3(\xi) + 4a_2Y^2(\xi) + a_1Y(\xi). \end{aligned}$$

substituting the obtained derivatives and Eq.(4.4) into Eq.(4.3) and collecting the coefficient of each power of $Y(\xi)$, setting each of the coefficients to zero, a set of algebraic equations are obtained. By means of the symbolic software Mathematica, the set of algebraic equations yields the following solutions.
Case 1:

$$\begin{gathered} a_0 = \frac{c+k}{ak}, \quad a_1 = 0, \quad a_2 = -6\frac{c+k}{ak}, \quad a_3 = 0 \\ a_4 = 6\frac{c+k}{ak}, \quad k = k, \quad c = c, \quad b = -\frac{c+k}{4ck^2}. \end{gathered}$$

By means of the obtained coefficients, solutions of Eq.(4.1) are in the form:

$$\begin{aligned} u_1(x,t) &= \tfrac{c+k}{ak}\left[1 - \frac{3}{2cosh^2\left(\frac{kx^\alpha - \frac{k}{4k^2b+1}t^\alpha}{\Gamma(1+\alpha)}\right)}\right], \\ u_2(x,t) &= \tfrac{c+k}{ak}\left[1 - \frac{3}{2sinh^2\left(\frac{kx^\alpha - \frac{k}{4k^2b+1}t^\alpha}{\Gamma(1+\alpha)}\right)}\right]. \end{aligned}$$

Case 2:

$$\begin{gathered} a_0 = 0, \quad a_1 = 0, \quad a_2 = 6\frac{c+k}{ak}, \quad a_3 = 0 \\ a_4 = -6\frac{c+k}{ak}, \quad k = k, \quad c = c, \quad b = \frac{c+k}{4ck^2}. \end{gathered}$$

Inserting the obtained coefficients into Eq.(4.4) we reach the solution of Eq.(4.1)

$$u_3(x,t) = \frac{3(c+k)}{2ak}\left[\frac{1}{cosh^2\left(\frac{kx^\alpha - \frac{k}{2k^2b-1}t^\alpha}{\Gamma(1+\alpha)}\right)}\right],$$

$$u_4(x,t) = \frac{3(c+k)}{2ak}\left[\frac{1}{sinh^2\left(\frac{kx^\alpha - \frac{k}{2k^2b-1}t^\alpha}{\Gamma(1+\alpha)}\right)}\right].$$

Case 3:

$$a_0 = \frac{c+k}{ak}, \quad a_1 = 0, \quad a_2 = 6\frac{c+k}{ak}, \quad a_3 = 0$$
$$a_4 = -6\frac{c+k}{ak}, \quad k = k, \quad c = c, \quad b = \frac{c+k}{4ck^2}.$$

From the above coefficients, we obtain the following solutions of Eq.(4.1)

$$u_5(x,t) = \frac{c+k}{ak}\left[1 + \frac{3}{2cosh^2\left(\frac{kx^\alpha + \frac{k}{4k^2b+1}t^\alpha}{\Gamma(1+\alpha)}\right)}\right],$$

$$u_6(x,t) = \frac{c+k}{ak}\left[1 + \frac{3}{2sinh^2\left(\frac{kx^\alpha + \frac{k}{4k^2b+1}t^\alpha}{\Gamma(1+\alpha)}\right)}\right].$$

Case 4:

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = \frac{24bk^2}{a(4bk^2-1)}, \quad a_3 = 0$$
$$a_4 = -\frac{24bk^2}{a(4bk^2-1)}, \quad k = k, \quad c = \frac{k}{4bk^2-1}.$$

Using the foregoing coefficients, we obtain the solutions of Eq.(4.1) as follows:

$$u_7(x,t) = \frac{6bk^2}{4bk^2-1}\left[\frac{1}{cosh^2\left(\frac{kx^\alpha - \frac{k}{4bk^2-1}t^\alpha}{\Gamma(1+\alpha)}\right)}\right],$$

$$u_8(x,t) = \frac{6bk^2}{4bk^2-1}\left[\frac{1}{sinh^2\left(\frac{kx^\alpha - \frac{k}{4bk^2-1}t^\alpha}{\Gamma(1+\alpha)}\right)}\right].$$

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Case 5:

$$a_0 = \frac{4bk^2}{a(1+4bk^2)}, \quad a_1 = 0, \quad a_2 = -\frac{24bk^2}{a(1+4bk^2)}, \quad a_3 = 0$$
$$a_4 = \frac{24bk^2}{a(1+4bk^2)}, \quad k = k, \quad c = -\frac{k}{1+4bk^2}.$$

By means of the obtained coefficients, solutions of Eq.(4.1) are in the form:

$$u_9(x,t) = \frac{4bk^2}{a(1+4bk^2)}\left[1 - \frac{3}{2cosh^2\left(\frac{kx^\alpha - \frac{k}{4k^2b+1}t^\alpha}{\Gamma(1+\alpha)}\right)}\right],$$

$$u_{10}(x,t) = \frac{4bk^2}{a(1+4bk^2)}\left[1 - \frac{3}{2sinh^2\left(\frac{kx^\alpha - \frac{k}{4k^2b+1}t^\alpha}{\Gamma(1+\alpha)}\right)}\right].$$

4.2 Space−Time Fractional Fokas Equation

Consider the following space−time Fokas equation

$$4\frac{\partial^{2\alpha}u}{\partial t^{\alpha}\partial x_1^{\alpha}}-\frac{\partial^{4\alpha}u}{\partial x_1^{3\alpha}\partial x_2^{\alpha}}+\frac{\partial^{4\alpha}u}{\partial x_1^{\alpha}\partial x_2^{3\alpha}}+12\frac{\partial^{\alpha}u}{\partial x_1^{\alpha}}\frac{\partial^{\alpha}u}{\partial x_2^{\alpha}}+12u\frac{\partial^{2\alpha}u}{\partial x_1^{\alpha}\partial x_2^{\alpha}}-6\frac{\partial^{2\alpha}u}{\partial y_1^{\alpha}\partial y_2^{\alpha}}=0. \tag{4.5}$$

$$t>0,\quad 0<\alpha\leq 1.$$

which is a model for finite amplitude wave packet in fluid dynamics.
For our purpose, we present the following transformations

$$u(x_1,x_2,y_1,y_2,t)=u(\xi),\quad \xi=k_1\frac{x_1^{\alpha}}{\Gamma(1+\alpha)}+k_2\frac{x_2^{\alpha}}{\Gamma(1+\alpha)}+l_1\frac{y_1^{\alpha}}{\Gamma(1+\alpha)}+l_2\frac{y_2^{\alpha}}{\Gamma(1+\alpha)}+c\frac{t^{\alpha}}{\Gamma(1+\alpha)} \tag{4.6}$$

where $k_1,k_2,l_1,l_2,c\neq 0$ are constants. Using property (2.3) and considering the wave transformation Eq.(4.6), Eq.(4.5) can be reduced to the ordinary differential equation,

$$(4ck_1-6l_1l_2)u'+(k_2^3k_1-k_1^3k_2)u'''+12k_1k_2(uu')=0. \tag{4.7}$$

Also we take

$$u(\xi,Y)=\sum_{i=0}^{N}a_iY^i(\xi)$$

where $Y(\xi)=\frac{\pm 1}{\sqrt{1\pm e^{2\xi}}}$ and $Y_\xi(\xi)=Y^3(\xi)-Y(\xi)$. Then by using the homogenous balance formula (3.7) between the highest order derivatives and the nonlinear terms appearing in ODE (4.7), we find

$$N=4.$$

Thus, we have

$$u(\xi,Y)=a_0+a_1Y(\xi)+a_2Y^2(\xi)+a_3Y^3(\xi)+a_4Y^4(\xi) \tag{4.8}$$

and substituting derivatives of $u(\xi,Y)$ with respect to ξ we obtain

$$\begin{aligned}
u_\xi &= 4a_4Y^6(\xi)+3a_3Y^5(\xi)+(2a_2-4a_4)Y^4(\xi)+(a_1-3a_3)Y^3(\xi)\\
&- 2a_2Y^2(\xi)-a_1Y(\xi),\\
u_{\xi\xi} &= 24a_4Y^8(\xi)+15a_3Y^7(\xi)+(8a_2-40a_4)Y^6(\xi)+(3a_1-24a_3)Y^5(\xi)\\
&+ (16a_4-12a_2)Y^4(\xi)+(9a_3-4a_1)Y^3(\xi)+4a_2Y^2(\xi)+a_1Y(\xi),\\
u_{\xi\xi\xi} &= 192a_4Y^{10}(\xi)+105a_3Y^9(\xi)+(48a_2-432a_4)Y^8(\xi)\\
&+ (15a_1-225a_3)Y^7(\xi)+(304a_4-96a_2)Y^6(\xi)\\
&+ (147a_3-27a_1)Y^5(\xi)+(56a_2-64a_4)Y^4(\xi)+(13a_1-27a_3)Y^3(\xi)\\
&- 8a_2Y^2(\xi)-a_1Y(\xi).
\end{aligned}$$

Using the above derivatives and collecting the coefficient of each power of $Y(\xi)$, setting each of the coefficients to zero, solving the resulting system of algebraic equations by Mathematica we obtain the following results.
Case 1:

$$\begin{aligned}
&a_0=-\frac{c-k_1^2k_2+k_2^3}{3k_2},\quad a_1=0,\quad a_2=-4(k_1^2-k_2^2),\quad a_3=0,\\
&a_4=4(k_1^2-k_2^2),\quad k_1=k_1,\\
&k_2=k_2,\quad l_1=l_1,\quad l_2=0,\\
&c=c.
\end{aligned}$$

Inserting the obtained coefficients into Eq.(4.8) we reach the solution of Eq.(4.5)

$$\begin{aligned}
u_1(x_1,x_2,y_1,y_2,t) &= -\frac{c-k_1^2k_2+k_2^3}{3k_2}+\frac{(k_2^2-k_1^2)}{cosh^2\left[\frac{k_1x_1^{\alpha}+k_2x_2^{\alpha}+l_1y_1^{\alpha}+ct^{\alpha}}{\Gamma(1+\alpha)/2}\right]},\\
u_2(x_1,x_2,y_1,y_2,t) &= -\frac{c-k_1^2k_2+k_2^3}{3k_2}+\frac{(k_2^2-k_1^2)}{sinh^2\left[\frac{k_1x_1^{\alpha}+k_2x_2^{\alpha}+l_1y_1^{\alpha}+ct^{\alpha}}{\Gamma(1+\alpha)/2}\right]}.
\end{aligned}$$

Case 2:

$$\begin{aligned}
&a_0 = a_0, \quad a_1 = 0, \quad a_2 = -4(k_1^2 - k_2^2), \quad a_3 = 0,\\
&a_4 = 4(k_1^2 - k_2^2), \quad k_1 = k_1, \quad k_2 = k_2,\\
&l_1 = \frac{2ck_1 + 6a_0k_1k_2 - 2k_1^3k_2 + 2k_1k_2^3}{3l_2},\\
&l_2 = l_2, \quad c = c.
\end{aligned}$$

Using the foregoing coefficients, we obtain the solutions of Eq.(4.5) as follows:

$$\begin{aligned}
u_3(x_1,x_2,y_1,y_2,t) &= a_0 + \frac{(k_2^2-k_1^2)}{cosh^2\left[\frac{k_1x_1^\alpha+k_2x_2^\alpha+\left(\frac{2ck_1+6a_0k_1k_2-2k_1^3k_2+2k_1k_2^3}{3l_2}\right)y_1^\alpha+l_2y_2^\alpha+ct^\alpha}{\Gamma(1+\alpha)/2}\right]},\\
u_4(x_1,x_2,y_1,y_2,t) &= a_0 + \frac{(k_2^2-k_1^2)}{sinh^2\left[\frac{k_1x_1^\alpha+k_2x_2^\alpha+\left(\frac{2ck_1+6a_0k_1k_2-2k_1^3k_2+2k_1k_2^3}{3l_2}\right)y_1^\alpha+l_2y_2^\alpha+ct^\alpha}{\Gamma(1+\alpha)/2}\right]}.
\end{aligned}$$

Case 3:

$$\begin{aligned}
&a_0 = a_0, \quad a_1 = 0, \quad a_2 = 4k_2^2, \quad a_3 = 0,\\
&a_4 = -4k_2^2, \quad k_1 = 0, \quad k_2 = k_2,\\
&l_1 = 0, \quad l_2 = l_2, \quad c = c.
\end{aligned}$$

From the above coefficients, we obtain the following solutions of Eq.(4.5)

$$\begin{aligned}
u_5(x_1,x_2,y_1,y_2,t) &= a_0 + \frac{k_2^2}{cosh^2\left[\frac{k_2x_2^\alpha+l_2y_2^\alpha+ct^\alpha}{\Gamma(1+\alpha)/2}\right]},\\
u_6(x_1,x_2,y_1,y_2,t) &= a_0 + \frac{k_2^2}{sinh^2\left[\frac{k_2x_2^\alpha+l_2y_2^\alpha+ct^\alpha}{\Gamma(1+\alpha)/2}\right]}.
\end{aligned}$$

Case 4:

$$\begin{aligned}
&a_0 = 0, \quad a_1 = 0, \quad a_2 = -4(k_1^2 - k_2^2), \quad a_3 = 0,\\
&a_4 = 4(k_1^2 - k_2^2), \quad k_1 = k_1, \quad k_2 = k_2,\\
&l_1 = l_1, \quad l_2 = l_2, \quad c = c.
\end{aligned}$$

Inserting the above coefficients into Eq.(4.8), we obtain the following solutions of Eq.(4.5)

$$\begin{aligned}
u_7(x_1,x_2,y_1,y_2,t) &= \frac{(k_2^2-k_1^2)}{cosh^2\left[\frac{k_1x_1^\alpha+k_2x_2^\alpha+l_1y_1^\alpha+l_2x_2+ct^\alpha}{\Gamma(1+\alpha)/2}\right]},\\
u_8(x_1,x_2,y_1,y_2,t) &= \frac{(k_2^2-k_1^2)}{cosh^2\left[\frac{k_1x_1^\alpha+k_2x_2^\alpha+l_1y_1^\alpha+l_2x_2+ct^\alpha}{\Gamma(1+\alpha)/2}\right]}.
\end{aligned}$$

Case 5:

$$\begin{aligned}
&a_0 = -\frac{c - k_1^2k_2 + k_2^3}{3k_2}, \quad a_1 = 0, \quad a_2 = -4(k_1^2 - k_2^2),\\
&a_3 = 0, \quad a_4 = 4(k_1^2 - k_2^2), \quad k_1 = k_1, \quad k_2 = k_2,\\
&l_1 = 0, \quad l_2 = l_2, \quad c = c.
\end{aligned}$$

By using the obtained coefficients, we get the following solutions of Eq.(4.5)

$$\begin{aligned}
u_9(x_1,x_2,y_1,y_2,t) &= -\frac{c-k_1^2k_2+k_2^3}{3k_2} + \frac{(k_2^2-k_1^2)}{cosh^2\left[\frac{k_1x_1^\alpha+k_2x_2^\alpha+l_2y_2^\alpha+ct^\alpha}{\Gamma(1+\alpha)/2}\right]},\\
u_{10}(x_1,x_2,y_1,y_2,t) &= -\frac{c-k_1^2k_2+k_2^3}{3k_2} + \frac{(k_2^2-k_1^2)}{sinh^2\left[\frac{k_1x_1^\alpha+k_2x_2^\alpha+l_2y_2^\alpha+ct^\alpha}{\Gamma(1+\alpha)/2}\right]},
\end{aligned}$$

Case 6:

$$\begin{aligned}
&a_0 = a_0, \quad a_1 = 0, \quad a_2 = -4k_1^2, \quad a_3 = 0,\\
&a_4 = 4k_1^2, \quad k_1 = k_1, \quad k_2 = 0,\\
&l_1 = l_1, \quad l_2 = \frac{2ck_1}{3l_1}, \quad c = c.
\end{aligned}$$

Inserting the obtained coefficients into Eq.(4.8) we reach the solution of Eq.(4.5)

$$u_{11}(x_1,x_2,y_1,y_2,t) = a_0 - \frac{k_1^2}{cosh^2\left[\frac{k_1 x_1^\alpha + l_1 y_1^\alpha + \frac{2ck_1}{3l_1} y_2^\alpha + ct^\alpha}{\Gamma(1+\alpha)/2}\right]},$$
$$u_{12}(x_1,x_2,y_1,y_2,t) = a_0 - \frac{k_1^2}{sinh^2\left[\frac{k_1 x_1^\alpha + l_1 y_1^\alpha + \frac{2ck_1}{3l_1} y_2^\alpha + ct^\alpha}{\Gamma(1+\alpha)/2}\right]},$$

Case 7:

$$a_0 = a_0, \quad a_1 = 0, \quad a_2 = -4k_1^2, \quad a_3 = 0,$$
$$a_4 = 4k_1^2, \quad k_1 = k_1, \quad k_2 = 0,$$
$$l_1 = l_1, \quad l_2 = l_2, \quad c = \frac{3l_1 l_2}{2k_1}.$$

From the foregoing coefficients, the solutions of Eq.(4.5) are obtained as follows:

$$u_{13}(x_1,x_2,y_1,y_2,t) = a_0 - \frac{k_1^2}{cosh^2\left[\frac{k_1 x_1^\alpha + l_1 y_1^\alpha + l_2 y_2^\alpha + \frac{3l_1 l_2}{2k_1} t^\alpha}{\Gamma(1+\alpha)/2}\right]},$$
$$u_{14}(x_1,x_2,y_1,y_2,t) = a_0 - \frac{k_1^2}{sinh^2\left[\frac{k_1 x_1^\alpha + l_1 y_1^\alpha + l_2 y_2^\alpha + \frac{3l_1 l_2}{2k_1} t^\alpha}{\Gamma(1+\alpha)/2}\right]}.$$

Case 8:

$$a_0 = a_0, \quad a_1 = 0, \quad a_2 = -4(k_1^2 - k_2^2),$$
$$a_3 = 0, \quad a_4 = 4(k_1^2 - k_2^2), \quad k_1 = k_1, \quad k_2 = k_2,$$
$$l_1 = l_1, \quad l_2 = l_2,$$
$$c = -\frac{6a_0 k_1 k_2 - 2k_1^3 k_2 + 2k_1 k_2^3 - 3l_1 l_2}{2k_1}$$

By means of the obtained coefficients, solutions of Eq.(4.5) are in the form:

$$u_{15}(x_1,x_2,y_1,y_2,t) = a_0 + \frac{(k_2^2 - k_1^2)}{cosh^2\left[\frac{k_1 x_1^\alpha + k_2 x_2^\alpha + l_1 y_1^\alpha + l_2 y_2^\alpha - \left(\frac{6a_0 k_1 k_2 - 2k_1^3 k_2 + 2k_1 k_2^3 - 3l_1 l_2}{2k_1}\right) t^\alpha}{\Gamma(1+\alpha)/2}\right]},$$
$$u_{16}(x_1,x_2,y_1,y_2,t) = a_0 + \frac{(k_2^2 - k_1^2)}{sinh^2\left[\frac{k_1 x_1^\alpha + k_2 x_2^\alpha + l_1 y_1^\alpha + l_2 y_2^\alpha - \left(\frac{6a_0 k_1 k_2 - 2k_1^3 k_2 + 2k_1 k_2^3 - 3l_1 l_2}{2k_1}\right) t^\alpha}{\Gamma(1+\alpha)/2}\right]}.$$

Case 9:

$$a_0 = a_0, \quad a_1 = 0, \quad a_2 = -4(k_1^2 - k_2^2),$$
$$a_3 = 0, \quad a_4 = 4(k_1^2 - k_2^2), \quad k_1 = k_1, \quad k_2 = k_2,$$
$$l_1 = l_1, \quad l_2 = l_2,$$
$$c = -\frac{12a_0 k_1 k_2 + 131k_1^3 k_2 - 131k_1 k_2^3 - 6l_1 l_2}{4k_1}$$

By using the obtained coefficients, we get the following solutions of Eq.(4.5)

$$u_{17}(x_1,x_2,y_1,y_2,t) = a_0 + \frac{(k_2^2 - k_1^2)}{cosh^2\left[\frac{k_1 x_1^\alpha + k_2 x_2^\alpha + l_1 y_1^\alpha + l_2 y_2^\alpha - \left(\frac{12a_0 k_1 k_2 + 131k_1^3 k_2 - 131k_1 k_2^3 - 6l_1 l_2}{4k_1}\right) t^\alpha}{\Gamma(1+\alpha)/2}\right]},$$
$$u_{18}(x_1,x_2,y_1,y_2,t) = a_0 + \frac{(k_2^2 - k_1^2)}{sinh^2\left[\frac{k_1 x_1^\alpha + k_2 x_2^\alpha + l_1 y_1^\alpha + l_2 y_2^\alpha - \left(\frac{12a_0 k_1 k_2 + 131k_1^3 k_2 - 131k_1 k_2^3 - 6l_1 l_2}{4k_1}\right) t^\alpha}{\Gamma(1+\alpha)/2}\right]}.$$

Remark 2: Although two cases were offered for the solutions of space−time fractional Fokas equation by using modified Kudryashov method [11], nine cases of solutions are arised by extended Kudryashov method. As well as the obtained solitary wave solutions show similarity with solutions which are obtained by Kudryashov method, increment values of parameters can effect wavelenght and speed of the wave.

5. Conclusion

In this study, we have proposed extended Kudryashov method to solve nonlinear fractional differential equations with the help of Mathematica. By this way, degree of the auxilary polynomials are increased and more solutions are provided an opportunity for some models. The space-time Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZKBBM) equation and the space-time fractional Fokas equation are taken to indicate the effectiveness of the proposed method. Besides the Kudryashov method, more traveling wave solution cases are obtained. In addition, change in the parameters effects both the wavelenght and speed of the wave. Eventually, the method is influential and suitable for solving other types of space-time fractional differential equations in which the homogenous balance principle is satisfied.

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