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ERRATUM TO THE PAPER : DISTINGUISHED NORMALIZATION ON NON-MINIMAL NULL HYPERSURFACES

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Abstract. The purpose of this note is to correct a misstatement due to a "minus" sign error in the invariant normalizing differential equation in the above paper. We also normalize the mean curvature function and return to the basic example in section 8.2 accordingly.

1. The invariant normalizing differential equation

In section 6, the non-normalized mean curvature function ν on (M, g, N),

$$\nu = \left[\underline{g}^{\alpha\beta} B^N_{\alpha\beta}\right],$$

and the normalizing differential equation

$$\xi \cdot \psi + \left(\tau^N(\xi) + \frac{\xi \cdot \nu}{\nu}\right)\psi = 0, \tag{1.1}$$

should be replaced respectively by

$$\nu = \frac{1}{\dim M}\left[\underline{g}^{\alpha\beta} B^N_{\alpha\beta}\right], \tag{1.2}$$

and

$$\xi \cdot \psi - \left(\tau^N(\xi) + \frac{\xi \cdot \nu}{\nu}\right)\psi = 0. \tag{1.3}$$

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Remark 1.1. *The proof of invariance of this PDE under any change in normalization remains valid without changing anything in the original proof except the " minus " sign. From this invariance, without reference to any normalization, (it make sense and) we call Eq. (1.3) the invariant normalizing differential equation of the lightlike hypersurface (INDE in short).*

For future use, we introduce the following:
Notation: Smooth (non zero) solutions of the INDE-equation (1.3) will be called normalizing functions (in short NF) of the null hypersurface.

Remark 1.2. *Observe that for $\tau^N = 0$, the mean curvature function ν is a normalizing function (that is solution of Eq. (1.3)).*

2. A distinguished normalization

2.1. **Normalizing constraints and main results.** Using exactly the same steps as in section 8.1 and using (Eq. 1.2), we now get

$$\phi = -(n+1)\frac{\nu}{\psi}, \tag{2.1}$$

and the following expressions for ζ:

(2.2)
$$\forall i = 1\ldots,n, \quad \zeta^i = -B^{ik}\left[\phi\tau^N(X_k) + X_k\cdot\phi\right] = (n+1)B^{ik}\left[\frac{\nu}{\psi}\tau^N(X_k) + X_k\cdot\frac{\nu}{\psi}\right].$$

$$\zeta^0 = \eta(\zeta) = (n+1)\frac{\psi}{2\nu}\left[g_{ij}B^{ik}B^{jl}\Big(\frac{\nu}{\psi}\tau^N(X_k)+X_k\cdot\frac{\nu}{\psi}\Big)\Big(\frac{\nu}{\psi}\tau^N(X_l)+X_l\cdot\frac{\nu}{\psi}\Big)\right]. \tag{2.3}$$

Everything in the prof of the unicity is correct.

Our basic example is now as follows:

2.2. **A basic example: the light-cone $\wedge_0^3 \subset \mathbb{R}_1^4$.** Let us consider the lightcone $\wedge_0^3$ as the immersion

$$\begin{array}{rcl} f : M = \mathbb{R}^3\setminus\{0\} & \longrightarrow & \mathbb{R}_1^4 \\ (x,y,z) & \longmapsto & \left[x,y,z,\varepsilon(x^2+y^2+z^2)^{\frac{1}{2}}\right], \varepsilon = \pm 1. \end{array}$$

Locally, $\wedge_0^3$ is the graph $t = \varepsilon(x^2+y^2+z^2)^{\frac{1}{2}}$ and it is an obvious fact that this is a lightlike hypersurface immersion. (We focus on the future directed connected component i.e $\varepsilon = 1$).

Start (the normalization) with the tentative null vector field

$$N = x\partial_x + y\partial_y + z\partial_z - t\partial_t, \tag{2.4}$$

and let P and D denote the projection morphism of the tangent bundle TM onto $\mathscr{S}(N)$ and the Levi-Civita connection on $\mathbb{R}_1^4$ respectively. Then, where

$$\xi = \frac{1}{2t^2}\Big(x\partial_x + y\partial_y + z\partial_z + t\partial_t\Big) = \frac{1}{2t^2}\xi_0, \tag{2.5}$$

and

$$\begin{aligned} D_X\xi &= D_X\Big(\frac{1}{2t^2}\xi_0\Big) = X\cdot\Big(\frac{1}{2t^2}\Big)\xi_0 + \frac{1}{2t^2}D_X\xi_0 \\ &= 2t^2X\cdot\Big(\frac{1}{2t^2}\Big)\xi + \frac{1}{2t^2}X \\ &= 2t^2X\cdot\Big(\frac{1}{2t^2}\Big)\xi + \frac{1}{2t^2}\Big(PX + \eta(X)\xi\Big) \\ &= \frac{1}{2t^2}PX + \Big[2t^2X\cdot\Big(\frac{1}{2t^2}\Big) + \frac{1}{2t^2}\eta(X))\Big]\xi, \end{aligned}$$

for all $X \in \Gamma(T\wedge_0^3)$. Comparing with the Weingarten formula

$$D_X\xi = -\overset{\star}{A}_\xi X - \tau^N(X)\xi,$$

yields

$$\overset{\star}{A}_\xi X = -\frac{1}{2t^2}PX \text{ and } \tau^N(X) = -\Big[2t^2X\cdot\Big(\frac{1}{2t^2}\Big) + \frac{1}{2t^2}\eta(X)\Big]\ \forall X \in \Gamma(T\wedge_0^3), \tag{2.6}$$

It follows that the mean curvature fonction ν on $\wedge_0^3 \subset \mathbb{R}_1^4$ is given by

$$\nu = -\frac{1}{3t^2}. \tag{2.7}$$

Also,

$$\begin{aligned} \eta(X) &= \langle N\ ,\ X\rangle = xX^x + yX^y + zX^z + tX^t \\ &= \frac{1}{2}d(x^2+y^2+z^2+t^2)(X) = \frac{1}{2}d(2t^2)(X) \\ &= 2tdt(X), \end{aligned}$$

that is

$$\eta = 2tdt. \tag{2.8}$$

So, using Eq. (2.6) we get

$$\tau^N = \frac{dt}{t}. \tag{2.9}$$

We derive the following INDE-equation:

$$\Bigg[x\partial_x + y\partial_y + z\partial_z + t\partial_t\Bigg]\psi + \psi = 0. \tag{2.10}$$

The (only) time dependent solutions of this PDE are given by the following family of functions on $\wedge_0^3 \subset \mathbb{R}_1^4$:

$$\psi(x,y,z,t) = \frac{K}{t}, \qquad t>0,\ K\in\mathbb{R}. \tag{2.11}$$

So, let us prescribe the calibrated divergence function to be given by

$$\psi(x,y,z,t) = \frac{1}{t}, \qquad t>0. \tag{2.12}$$

It follows that

$$\phi(x,y,z,t) = -3\frac{\nu}{\psi} = -3\frac{-1}{3t^2}t = \frac{1}{t}. \tag{2.13}$$

The rank 2 screen distribution $\mathscr{S}(N)$ associated to this normalization is spanned by

$$X_1 = y\partial_x - x\partial_y \quad \text{and} \quad X_2 = z\partial_x - x\partial_z.$$

It follows (Eq. 2.2), (Eq. 2.3) and using (Eq. 2.7) and (Eq. 2.8) that

$$\zeta^1 = \zeta^2 = \zeta^0 = 0, \quad \text{that is} \quad \zeta = 0.$$

Finally, we get the normalizing null vector field

$$\widetilde{N} = \phi N + \zeta = \frac{1}{t}\Big(x\partial_x + y\partial_y + z\partial_z - t\partial_t\Big). \tag{2.14}$$

The corresponding normalized radical (characteristic) null vector field is then given by

$$\widetilde{\xi} = \frac{1}{2t}\Big(x\partial_x + y\partial_y + z\partial_z + t\partial_t\Big). \tag{2.15}$$

Remark 2.1.

Observe that taking the integration constant K in Eq.(2.11) to be $\sqrt{2}$ leads to the Blaschke type normalization on lightlike hypersurfaces as presented in [4].

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