Mathematical Advances in Pure and Applied Sciences, 2018 *Vol.1, No.1, 39-45*
Received: February 2, 2018 *doi: will be added*
Accepted: April 2, 2018
Published online: May 18, 2018

On Some Perfect Codes over Hurwitz Integers

Murat Güzeltepe *†1

1 Sakarya University, Faculty of Arts and Sciences , Department of Mathematics, Sakarya-Turkey

Keywords
Block codes,
Cyclic codes,
Hurwitz metric.

Abstract: The article considers linear codes over Hurwitz integers. The codes are considered with respect to a new Hurwitz metric. This metric is more suitable for (QAM)−type constellations than the Hamming Metric and the Lee metric. Also, one error correcting perfect codes with respect to the Hurwitz metric are defined. The decoding algorithm of these codes is obtained. Moreover, a simple comparison in respect to the average energy for the transmitted signal and the bandwidth occupancy is given.

1. Introduction and Preliminaries

Recently, many researchers in coding theory have investigated some special codes over different fields or rings. Some of these studies are summarized in the following: Güzeltepe defined the Hurwitz metric and obtain some codes over hurwitz integers [1]. Abualrub and Șiap studied constacyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2$ in [2]. In [3], cyclic DNA codes over $\mathbb{F}_2[u]/(u^2-1)$ are obtained. In [5], Yildiz and Karadeniz defined self-dual codes over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$. Yildiz and Karadeniz studied cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ in [6]. Cyclic codes over some finite quaternion integer rings were presented by Özen and Güzeltepe in [7]. Abualrub and Șiap studied cyclic codes over the rings $\mathbb{Z}_2 + u\mathbb{Z}_2$ and $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$ and they found a set of generators for these codes in [4]. On the other hand, it is shown that Hamming and Lee distances have been revealed to be inappropriate metrics to deal with quadrature amplitude modulation (QAM) signal sets and other constellations, [8]. Many authors have studied on codes over different fields and rings to solve this problem up to now. One of the first example was given by Huber in [13]. Huber constructed some constellations by using Gaussian integers. Also, Huber defined the Mannheim metric for linear codes over the constellations. Although, Huber's constellations is of minimal energy, unfortunately, the Mannheim metric is not a true metric that it is proved in [15]. Inspired by Huber's works, T. P. da Nobrega Neto *et al.* constructed linear codes over some quadratic fields, defined linear codes over $A_p[\rho]$ and compared these codes with codes over Gaussian integers $\mathbb{Z}[i]$ in terms of bandwidth occupancy and average power in [12].
Later, C. Martinez *et al.* obtained perfect codes for metrics induced by circulant graphs in [15]. Moreover, they proved that the Mannheim metric is not a true metric in [15]. In [16–20], works on codes over Lipschitz or Hurwitz integers were given.
The present paper is organized as follows. In this section, we give basic definitions and introduce a new Hurwitz weight and a new Hurwitz distance over Hurwitz integers. In Section 2, we construct a new class of perfect codes which can correct errors of Hurwitz weight one and give a decoding algorithm of these codes. In Section 3, we present a simple comparison in terms of bandwidth occupancy, the rate and average power.

In what follows, we consider the following:

Definition 1. *[9] The Hamilton Quaternion Algebra over the Set of the Real numbers* ($\mathbb{R}$)*, denoted by* $H(\mathbb{R})$*, is the associative unital algebra given by the following representation:*

*The corresponding author: mguzeltepe@sakarya.edu.tr
†The work was supported by TÜBİTAK (The Scientific and Technological Research Council of TURKEY) with project number 116F318.

i)$H(\mathbb{R})$ is the free $\mathbb{R}$ module over the symbols $1, \widehat{e}_1, \widehat{e}_2, \widehat{e}_3$, that is, $H(\mathbb{R}) = \{a_0 + a_1\widehat{e}_1 + a_2\widehat{e}_2 + a_3\widehat{e}_3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$;
ii)1 is the multiplicative unit;
iii) $\widehat{e}_1^2 = \widehat{e}_2^2 = \widehat{e}_3^2 = -1$;
iv) $\widehat{e}_1\widehat{e}_2 = -\widehat{e}_2\widehat{e}_1 = \widehat{e}_3$, $\widehat{e}_3\widehat{e}_1 = -\widehat{e}_1\widehat{e}_3 = \widehat{e}_2$, $\widehat{e}_2\widehat{e}_3 = -\widehat{e}_3\widehat{e}_2 = \widehat{e}_1$.

The set of Lipschitz integers $H(\mathbb{Z})$, which is defined by $H(\mathbb{Z}) = \{a_0 + a_1\widehat{e}_1 + a_2\widehat{e}_2 + a_3\widehat{e}_3 : a_0, a_1, a_2, a_3 \in \mathbb{Z}\}$, is a subset of $H(\mathbb{R})$, where $\mathbb{Z}$ denotes the set of all integers. If $q = a_0 + a_1\widehat{e}_1 + a_2\widehat{e}_2 + a_3\widehat{e}_3$ is a Lipschizt integer then, its conjugate quaternion is $\overline{q} = a_0 - (a_1\widehat{e}_1 + a_2\widehat{e}_2 + a_3\widehat{e}_3)$. The norm of q is $N(q) = q\overline{q} = a_0^2 + a_1^2 + a_2^2 + a_3^2$. The units of $H(\mathbb{Z})$ are $\pm1, \pm\widehat{e}_1, \pm\widehat{e}_2, \pm\widehat{e}_3$.

Definition 2. *[10] The set of all Hurwitz integers is*

$$\begin{aligned}\mathscr{H} &= \left\{a_0 + a_1\widehat{e}_1 + a_2\widehat{e}_2 + a_3\widehat{e}_3 \in H(\mathbb{R}) : a_0, a_1, a_2, a_3 \in \mathbb{Z} \text{ or } a_0, a_1, a_2, a_3 \in \mathbb{Z} + \tfrac{1}{2}\right\} \\ &= H(\mathbb{Z}) \cup H\left(\mathbb{Z} + \tfrac{1}{2}\right).\end{aligned}$$

It can be checked that $\mathscr{H}$ is closed under quaternion multiplication and addition, so that it forms a subring of the ring of all quaternions.

Definition 3. *If the norm of a Lipschitz integer q is a prime integer, then it is called a prime Lipschitz integer.*

Definition 4. *Let π be a prime Lipschitz integer in $H(\mathbb{Z})$. If there exist $\delta, q_1, q_2 \in \mathscr{H}$ such that $q_1 - q_2 = \pi\delta$, then q_1 and q_2 are right congruent modulo π and it is denoted by $q_1 \equiv_r q_2 \pmod{\pi}$.*

Hence, we can consider the quotient ring of the Hurwitz integers modulo this equivalence relation, which we denote as

$$\mathscr{H}_\pi = \{q \pmod{\pi} \mid q \in \mathscr{H}\}.$$

This set coincides with the quotient ring of the Hurwitz integers over the left ideal generated by π, which we denote as $\langle\pi\rangle$. The commutative property of multiplication does not hold over $\mathscr{H}_\pi$ since the product of two Lipschitz integers are not commutative in general. Note that if π is a prime Lipschitz integer then, the ring $\mathscr{H}_\pi$ has no zero divisors. To see this, let $q_1q_2 \equiv 0 \pmod{\pi}$, $q_1, q_2 \in \mathscr{H}_\pi$, and without loss of generality $N(q_1) < N(q_2)$. In this case, $N(q_1)N(q_2) \equiv 0 \mod N(\pi)$ and therefore we get $N(q_1) = 1$, $N(q_2) = N(\pi)$. This contradicts the fact that $N(q_2) < N(\pi)$.
In the following definition, we introduce a new Hurwitz weight and a new Hurwitz metric.

Definition 5. *Let π be a prime Lipschitz integer, $\gamma = a_0 + a_1\widehat{e}_1 + a_2\widehat{e}_2 + a_3\widehat{e}_3 + a_4w \in \mathscr{H}_\pi$ and let*

$$\begin{aligned}A &= |a_0| + |a_1| + |a_2| + |a_3| + a_4\overline{a_4}, \\ B &= \left|a_0'\right| + \left|a_1'\right| + \left|a_2'\right| + \left|a_3'\right| + a_4'\overline{a_4'},\end{aligned}$$

where $a_0 + a_1\widehat{e}_1 + a_2\widehat{e}_2 + a_3\widehat{e}_3 + a_4w = a_0' + a_1'\widehat{e}_1 + a_2'\widehat{e}_2 + a_3'\widehat{e}_3 + a_4'\overline{w}$ and some $a_0, a_1, a_2, a_3,\ a_0', a_1', a_2', a_3 \in \mathbb{Z}$, $a_4, a_4' \in \{\pm1, \pm\widehat{e}_1, \pm\widehat{e}_2, \pm\widehat{e}_3\}$. Here, the symbol $|\cdot|$ denotes the absolute value.
Then, we define the Hurwitz weight of $\gamma = a_0 + a_1\widehat{e}_1 + a_2\widehat{e}_2 + a_3\widehat{e}_3 + a_4w$ as

$$W_{hur}(\gamma) = \begin{cases} A, & A \le B \\ B, & B < A \end{cases}$$

Here and thereafter, w will denote $\frac{1}{2}(1 + \widehat{e}_1 + \widehat{e}_2 + \widehat{e}_3)$
Also, we define the Hurwitz distance d_{hur} between α and β as

$$d_{hur}(\alpha, \beta) = W_{hur}(\gamma),$$

where $\gamma = \alpha - \beta \pmod{\pi}$.

It is possible to show that $d_{hur}(\alpha, \beta)$ is a metric. We only show that the triangle inequality holds since the other conditions are straightforward. For this, let α, β, and γ be any three elements of $\mathscr{H}_\pi$. We have
i) $d_{hur}(\alpha, \beta) = W_{hur}(\delta_1) = |a_0| + |a_1| + |a_2| + |a_3| + |a_4|$, where $\alpha - \beta = \delta_1 = a_0 + a_1\widehat{e}_1 + a_2\widehat{e}_2 + a_3\widehat{e}_3 + a_4w \ (mod\ \pi)$ is an element of $\mathscr{H}_\pi$, and $|a_0| + |a_1| + |a_2| + |a_3| + |a_4|$ is minimum.

ii) $d_{hur}(\alpha, \gamma) = W_{hur}(\delta_2) = |b_0| + |b_1| + |b_2| + |b_3| + |b_4|$, where $\alpha - \gamma = \delta_2 = b_0 + b_1\widehat{e}_1 + b_2\widehat{e}_2 + b_3\widehat{e}_3 + b_4w \ (mod\ \pi)$ is an element of $\mathscr{H}_\pi$, and $|b_0| + |b_1| + |b_2| + |b_3| + |b_4|$ is minimum.

iii) $d_{hur}(\gamma,\beta)=W_{hur}(\delta_3)=|c_0|+|c_1|+|c_2|+|c_3|+|c_4|$, where $\gamma-\beta=\delta_3=c_0+c_1\widehat{e}_1+c_2\widehat{e}_2+c_3\widehat{e}_3+c_4w\ (mod\ \pi)$ is an element of $\mathscr{H}_\pi$, and $|c_0|+|c_1|+|c_2|+|c_3|+|c_4|$ is minimum.
Thus, $\alpha-\beta=\delta_2+\delta_3\ (\bmod\ \pi)$. However, $W_{hur}(\delta_2+\delta_3)\geq W_{hur}(\delta_1)$ since $W_{hur}(\delta_1)=|a_0|+|a_1|+|a_2|+|a_3|+|a_4|$ is minimum. Therefore,

$$d_{hur}(\alpha,\beta)\leq d_{hur}(\alpha,\gamma)+d_{hur}(\gamma,\beta).$$

Note that the Hurwitz weight W_{hur} is not the Hurwitz weight w_H defined in [1]. To see this, the Hurwitz weight of the element $q=\overline{w}=\frac{1}{2}-\frac{\widehat{e}_1}{2}-\frac{\widehat{e}_2}{2}-\frac{\widehat{e}_3}{2}$ is $W_{hur}=1$ and the Hurwitz weight w_H of the same element is $w_H(q)=2$. In the following section, we construct one error correcting perfect codes with respect to this new metric. In this aspect, this present paper has an important advantage according to the paper given in [1]. Also, the dimension of these perfect code is not only $n-k=1$ but also $n-k=t$. In this aspect, this present paper has an important advantage according to the papers given in [17, 19, 20].

2. One Hurwitz Error Correcting Perfect Codes over $\mathscr{H}_\pi$

In this section, we obtain perfect codes correcting errors of the Hurwitz weight one over $\mathscr{H}_\pi$. Recall that, the size of $\mathscr{H}_\pi$ is equal to p^2, where $N(\pi)=p$. Also, a Hurwitz error of weight one takes on one of the twenty four values $\pm1,\pm\widehat{e}_1,\pm\widehat{e}_2,\pm\widehat{e}_3,\pm w,\pm\overline{w},\pm\widehat{e}_1w,\pm\widehat{e}_2w,\pm\widehat{e}_3w,\pm\widehat{e}_1\overline{w},\pm\widehat{e}_2\overline{w},\pm\widehat{e}_3\overline{w}$ at the position l, $0\leq l\leq n$, for the length n. The number of error vectors of the Hurwitz weight one including the vector of all zeros over $\mathscr{H}_\pi$ is $24n+1$, where n denotes the length.
Let π be a prime in $H(\mathbb{Z})$ and let $N(\pi)=p\geq5$ be an odd prime in $\mathbb{Z}$. Then, there does naturally exist a partition of $\mathscr{H}_\pi$ as follows:

$$\mathscr{H}_\pi=\{0\}\cup G_1\cup G_2\cup\cdots\cup G_{\left(p^2-1\right)/24}.$$

Here,

$$|G_1|=|G_2|=\cdots=\left|G_{\left(p^2-1\right)/24}\right|=24$$

and $tG_{i_1}\neq G_{i_2}$ for all $i_1\neq i_2$, $1\leq i_1,i_2\leq\left(p^2-1\right)/24$ and $t\in\mathscr{A}$. Here and thereafter $\mathscr{A}$ will denote the set

$$\{\pm1,\pm\widehat{e}_1,\pm\widehat{e}_2,\pm\widehat{e}_3,\pm w,\pm\overline{w},\pm\widehat{e}_1w,\pm\widehat{e}_2w,\pm\widehat{e}_3w,\pm\widehat{e}_1\overline{w},\pm\widehat{e}_2\overline{w},\pm\widehat{e}_3\overline{w}\}.$$

For example, let $\pi=2+\widehat{e}_1+\widehat{e}_2+\widehat{e}_3$. Then, we get

$$G_1=\{\pm1,\pm\widehat{e}_1,\pm\widehat{e}_2,\pm\widehat{e}_3,\pm w,\pm\overline{w},\pm\widehat{e}_1w,\pm\widehat{e}_2w,\pm\widehat{e}_3w,\pm\widehat{e}_1\overline{w},\pm\widehat{e}_2\overline{w},\pm\widehat{e}_3\overline{w}\}$$

and

$$G_2=\{\pm(1\pm\widehat{e}_1),\pm(1\pm\widehat{e}_2),\pm(1\pm\widehat{e}_3),\pm(\widehat{e}_1\pm\widehat{e}_2),\pm(\widehat{e}_1\pm\widehat{e}_3),\pm(\widehat{e}_2\pm e_3)\}.$$

Hence, it is obtained that

$$\mathscr{H}_{2+\widehat{e}_1+\widehat{e}_2+\widehat{e}_3}=\{0\}\cup G_1\cup G_2$$

and

$$\left|\mathscr{H}_{2+\widehat{e}_1+\widehat{e}_2+\widehat{e}_3}\right|=|G_1|+|G_2|+1=49.$$

Theorem 1. *Let C be a code of length $n=\frac{(p^2)^{n-k}-1}{24}$ and let $p=\pi\overline{\pi}\geq5$, where $\pi\in H(\mathbb{Z})$ is a prime Lipschitz integer and k denotes the dimension of the code. Assume that a partition of $\mathscr{H}_\pi$ is $\mathscr{H}_\pi=\{0\}\cup G_1\cup G_2\cup\cdots\cup G_{\frac{p^2-1}{24}}$. Then, the code C defined by the parity check matrix*

$$H_{(n-k)\times n}=\left(H_0^*|\,H_1^*|\,H_2^*|\,H_3^*|\,H_4^*|\,H_5^*|\right),$$

where

$$H_0^*=\begin{pmatrix} g_i^1 & 0 & \cdots & 0\\ 0 & g_i^1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & g_i^1\end{pmatrix},\quad H_1^*=\begin{pmatrix} g_i^1 & 0 & 0 & 0 & 0 & \cdots & 0\\ G_j & g_i^1 & 0 & 0 & 0 & \cdots & 0\\ 0 & G_j & g_i^1 & 0 & 0 & \cdots & 0\\ 0 & 0 & G_j & g_i^1 & 0 & \cdots & 0\\ 0 & 0 & 0 & G_j & g_i^1 & & \vdots\\ \vdots & \vdots & \vdots & \vdots & & & g_i^1\\ 0 & 0 & 0 & 0 & \cdots & 0 & G_j\end{pmatrix},$$

$$H_2^* = \begin{pmatrix} g_i^1 & 0 & 0 & 0 & 0 & \cdots & 0 & & g_i^1 \\ 0 & g_i^1 & 0 & 0 & 0 & \cdots & 0 & & 0 \\ G_j & 0 & g_i^1 & 0 & 0 & \cdots & 0 & & 0 \\ 0 & G_j & 0 & \ddots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & 0 & G_j & & & \vdots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & g_i^1 & & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & G_j & & G_j \end{pmatrix},$$

$$H_3^* = \begin{pmatrix} g_i^1 & g_i^1 & g_i^1 & g_i^1 & g_i^1 & \cdots & g_i^1 & & g_i^1 \\ G_{j_1} & G_{j_1} & G_{j_1} & G_{j_1} & G_{j_1} & \cdots & G_{j_1} & & G_{j_1} \\ G_{j_2} & 0 & 0 & 0 & 0 & \cdots & 0 & & G_{j_2} \\ 0 & G_{j_2} & 0 & \ddots & 0 & \cdots & 0 & \cdots & G_{j_3} \\ 0 & 0 & 0 & 0 & & & \vdots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & 0 & & G_{j_{n-k-1}} \\ 0 & 0 & 0 & 0 & \cdots & 0 & G_{j_2} & & 0 \end{pmatrix},$$

$$H_4^* = \begin{pmatrix} g_i^1 & g_i^1 & & g_i^1 & & g_i^1 \\ 0 & G_{j_1} & & G_{j_1} & & G_{j_1} \\ G_{j_1} & 0 & & G_{j_2} & & G_{j_2} \\ G_{j_2} & G_{j_2} & \cdots & \vdots & \cdots & \vdots \\ G_{j_3} & G_{j_3} & & G_{j_{n-k-2}} & & \\ \vdots & \vdots & & 0 & & G_{j_{n-k-1}} \\ G_{j_{n-k-1}} & G_{j_{n-k-1}} & & G_{j_{n-k-1}} & & 0 \end{pmatrix}, \quad H_5^* = \begin{pmatrix} g_i^1 \\ G_{j_1} \\ \vdots \\ G_{j_{n-k}} \end{pmatrix}$$

is one Hurwitz error correcting perfect code over $\mathscr{H}_\pi$, except $n-k=1$ and $p=5$. Here, $g_i^1, g_i^2, \cdots, g_i^{24} \in G_i$, $1 \le i \le 24$, and $\mathscr{H}_\pi = \bigcup_j G_j$ and $G_{j_1} \cap G_{j_2} = \emptyset$ for all $j_1 \neq j_2$, $1 \le j_1, j_2 \le (p^2-1)/24$.

Proof. By the sphere-packing, we get

$$(p^2)^k(24n+1) = p^{2k}(24\frac{(p^2)^{n-k}-1}{24}+1) = (p^2)^n,$$

where $p = N(\pi) \ge 5$.
On the other hand, assume that we have the partition of $\mathscr{H}_\pi$ as $\mathscr{H}_\pi = \{0\} \cup G_1 \cup G_2 \cup \cdots \cup G_{(p^2-1)/24}$. By multiplying all error vectors of the Hurwitz weight 1 by the parity check matrix H, the syndromes are distinct altogether due to the fact that $tG_{j_1} \neq G_{j_2}$ for all $j_1 \neq j_2$, $1 \le j_1, j_2 \le (p^2-1)/24$ and $t \in \mathscr{A}$. Let $n-k=1$ and $p=5$. Then, the dimension k of the code C becomes 0 but it is not feasible. This completes the proof. □

Let us assume that an error of the Hurwitz weight 1 occurs in location l. Decoding is straightforward. Take the received vector $r = c+e$ and compute the syndrome S of r as $S = (rH^T)^T$. The syndrome S is equal to the product of θ and the column l of the parity check matrix H, where $\theta \in \mathscr{A}$. The location of the error is l, and the value of the error is $\theta \in \mathscr{A}$.

Example 1. *Let $\pi = 2+\widehat{e}_1+\widehat{e}_2+\widehat{e}_3$. If we select $n-k=1$, then in general the parity check matrix $H_{1\times n}$ is obtained as follows:*

$$H_{1\times n} = \left[g_i^1\right] = \left[\begin{array}{cccc} g_1^1, & g_2^1, & \cdots, & g^1_{(p^2-1)/24} \end{array}\right]. \tag{1}$$

In this case, the parity check matrix H is chosen as

$$H_{1\times 2} = \left[\begin{array}{cc} 1, & \widehat{e}_1+\widehat{e}_2 \end{array}\right].$$

Here,

$$G_1 = \{\pm 1, \pm\widehat{e}_1, \pm\widehat{e}_2, \pm\widehat{e}_3, \pm w, \pm\overline{w}, \pm\widehat{e}_1 w, \pm\widehat{e}_2 w, \pm\widehat{e}_3 w, \pm\widehat{e}_1\overline{w}, \pm\widehat{e}_2\overline{w}, \pm\widehat{e}_3\overline{w}\},$$
$$G_2 = \{\pm(1\pm\widehat{e}_1), \pm(1\pm\widehat{e}_2), \pm(1\pm\widehat{e}_3), \pm(\widehat{e}_1\pm\widehat{e}_2), \pm(\widehat{e}_1\pm\widehat{e}_3), \pm(\widehat{e}_2\pm e_3)\}$$

and $g_1^1 = 1,\ g_2^1 = \widehat{e}_1 + \widehat{e}_2$*. The code* C *defined by the above parity check matrix* H *is one Hurwitz error correction perfect code. For the decoding, for example, take the codeword* $c = (-\widehat{e}_1 - \widehat{e}_2, 1)$*,* $e = (\overline{w}, 0)$*, and* $r = c + e = (\widehat{e}_3, 1) = (-\widehat{e}_1 - \widehat{e}_2 + \overline{w}, 1) \pmod{2 + \widehat{e}_1 + \widehat{e}_2 + \widehat{e}_3}$*. Then, the syndrome* S *of* r *is equal to*

$$S = rH^T = \overline{w} = \widehat{e}_1 + \widehat{e}_2 + \widehat{e}_3 \pmod{2 + \widehat{e}_1 + \widehat{e}_2 + \widehat{e}_3}.$$

Hence, we see that the error occurs in location 1 since $\overline{w} = \overline{w}.g_1^1 = \overline{w}$ *and the value of the error is* $\overline{w}$ *since* $\overline{w} = \theta g_1^1 = \theta.1 = \theta$*.*

Example 2. *Let* $\pi = 2 + \widehat{e}_1$*. If we take* $n - k = 2$*, then in general the parity check matrix* $H_{2\times n}$ *is obtained as follows:*

$$H_{2\times n} = \begin{bmatrix} g_i^1 & 0 & g_i^1 \\ 0 & g_i^1 & G_j \end{bmatrix},$$

where $1 \le i \le 24$*,* $1 \le j \le (p^2-1)/24$*. So, we get*

$$\mathscr{H}_{2+\widehat{e}_1} = \{0\} \cup G_1.$$

Hence, the parity check matrix H *is chosen as*

$$H_{2\times 26} = \begin{bmatrix} g_1^1 & 0 & g_1^1 & g_1^1 & \cdots & g_1^1 \\ 0 & g_1^1 & g_1^1 & g_1^2 & \cdots & g_1^{24} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & g_1^2 & g_1^3 & \cdots & g_1^{24} \end{bmatrix},$$

where

$$G_1 = \mathscr{A} = \{\pm 1, \pm\widehat{e}_1, \pm\widehat{e}_2, \pm\widehat{e}_3, \pm w, \pm\overline{w}, \pm\widehat{e}_1 w, \pm\widehat{e}_2 w, \pm\widehat{e}_3 w, \pm\widehat{e}_1\overline{w}, \pm\widehat{e}_2\overline{w}, \pm\widehat{e}_3\overline{w}\}.$$

Let us assume that

$$g_1^2 = -1, g_1^3 = \hat{e}_1, g_1^4 = -\hat{e}_1, g_1^5 = \hat{e}_2, g_1^6 = -\hat{e}_2, g_1^7 = \hat{e}_3, g_1^8 = -\hat{e}_3, g_1^9 = w, g_1^{10} = -w, g_1^{11} = \bar{w}, \\ g_1^{12} = -\bar{w}, \cdots, g_1^{24} = -\hat{e}_3\bar{w}.$$

Then the parity check matrix H *becomes as follows:*

$$H_{2\times 26} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & -1 & \widehat{e}_1 & -\widehat{e}_1 & \widehat{e}_2 & -\widehat{e}_2 & \widehat{e}_3 & -\widehat{e}_3 & w & -w & \overline{w} & -\overline{w} & \widehat{e}_1 w & \cdots & -\widehat{e}_3\overline{w} \end{bmatrix}.$$

Let the codeword c

$$c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

and let the error vector e

$$e = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \widehat{e}_3\overline{w} & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then the syndrome of the received vector $r = c + e$ *is computed as follows:*

$$S = \left(rH^T\right)^T = \begin{pmatrix} \widehat{e}_3\overline{w} \\ \widehat{e}_3 \end{pmatrix} = \begin{pmatrix} 4 - \widehat{e}_1\overline{w} \\ \widehat{e}_1 - w \end{pmatrix} \pmod{2 + \widehat{e}_1}.$$

It is shown that the error occurs in the 11th component since the syndrome S *is equal to the product* $\widehat{e}_3\overline{w}$ *and the 11th column of the parity check matrix* H*, that is,*

$$S = \widehat{e}_3\overline{w} \begin{pmatrix} 1 \\ w \end{pmatrix}.$$

The value of the error is computed as $\widehat{e}_3\overline{w}$*. Hence, the codeword is obtained by* $c = r - e$*.*

3. Comparison Between Codes over $\mathscr{H}_\pi$, Codes over $\mathbb{Z}[\rho]$ and Codes over $\mathbb{Z}[i]$

In this section, we give a comparison between codes over $\mathscr{H}$, codes over $A_p[\rho]$ and codes over $\mathbb{Z}[i]$ in terms of average energy, the code rate and bandwidth occupancy. Note that codes over $\mathbb{Z}[\rho]$ and codes over $\mathbb{Z}[i]$ were presented in [12, 13], respectively. We first give a comparison between the average energy of codes over over $\mathscr{H}_\pi$ and the average energy of codes over $A_p[\rho]$. Let $\pi = 2+\widehat{e}_1+\widehat{e}_2+\widehat{e}_3$ and let $\alpha = 5+3\rho$. We show that the average energy for the transmitted signal, considering constellations with the same cardinality, is smaller in the case of $\mathscr{H}_\pi$ than in the case of $A_p[\rho]$, see Table I.
Note that the average energy is calculated as:

$$\mathscr{E} = \frac{1}{M}\sum_{s=0}^{M-1} N(\alpha_s),$$

where $N(\alpha_s)$ is in signal space and it has a magnitude (distance from the origin) and M denotes the cardinality of constellation.

Table I: Comparison between codes over $\mathscr{H}_\pi$ and $A_p[\rho]$.

Alphabet	Base ring	Average energy
$GF(49)$	$\mathscr{H}$	1.47
$GF(49)$	$A_p[\rho]$	7.22

We second give a comparison between the average energy of codes over over $\mathscr{H}_\pi$ and the average energy of codes over $\mathbb{Z}[i]_\alpha$. Let $\pi = 2+\widehat{e}_1$ and let $\alpha = 4+i3$. We show that the average energy for the transmitted signal, considering constellations with the same cardinality, is the smaller in the case of $\mathscr{H}_\pi$ than in the case of $\mathbb{Z}[i]_\alpha$, see Table II.

Table II: Comparison between codes over $\mathscr{H}_\pi$ and $\mathbb{Z}[i]_\alpha$.

Alphabet	Base ring	Average energy
$GF(25)$	$\mathscr{H}$	0.96
$GF(25)$	$\mathbb{Z}[i]_\alpha$	4.16

Bandwidth is one of the most important parameter of analog/digital communication systems. Various modulation and coding techniques have developed to provide bandwidth efficiency up to now. It is known from the communication theory, if we increase the codewords numbers (with the same dimension), we get higher channel capacity required bandwidth [21].
We now compare the rate and bandwidth occupancy of the codes over $\mathscr{H}_\pi$ with the codes over $A_p[\rho]$, $\mathbb{Z}[i]_\alpha$, when the alphabets considered have the same cardinality. The codes over $A_p[\rho]$ presented in [12] and the OMEC codes presented in [13] can be generalized to the lengths $n = \frac{p^2-1}{6}$ and $n = \frac{p^2-1}{4}$, respectively. Let $p \equiv 1 \pmod{12}$. Then we have $p \equiv 1 \pmod 6$ and $p \equiv 1 \pmod 4$. In this case, a code C_1 over $\mathscr{H}_\pi$ has length $n_1 = \frac{p^2-1}{24}$, a code C_2 over $A_p[\rho]$ has length $n_2 = \frac{p^2-1}{6}$ and a code C_3 over $\mathbb{Z}[i]_\alpha$ has length $n_3 = \frac{p^2-1}{4}$. Hence, if the dimension k_1, k_2 and k_3 of the codes C_1,C_2 and C_3 equal to k, then the rate R_1 of C_1 is greater than the rate R_2 of C_2 and the rate R_3 of C_3 since $R_1 = \frac{k_1}{n_1} = \frac{24k}{p^2-1}$, $R_2 = \frac{k_2}{n_2} = \frac{6k}{p^2-1}$ and $R_3 = \frac{k_3}{n_3} = \frac{4k}{p^2-1}$. For example, let $p = 13$ and $k = 1$. Then we get $R_1 = \frac{1}{7}$, $R_2 = \frac{1}{28}$ and $R_3 = \frac{1}{42}$. It is shown that the bandwidth occupancy of the code $C_1$is better than the bandwidth occupancy of the code C_2 and the bandwidth occupancy of the code C_3.

4. Conclusion

In this paper, we define a new Hurwitz metric and construct linear codes over $\mathscr{H}_\pi$ with respect to this metric.We show that the average energy for the transmitted signal is smaller in the case of $\mathscr{H}$ than in the case of $\mathbb{Z}[\rho]$ and $\mathbb{Z}[i]$. Moreover, the bandwidth occupancy of codes over $\mathscr{H}$ is better than the bandwidth occupancy of codes over $\mathbb{Z}[\rho]$ and $\mathbb{Z}[i]$.

References

[1] M. Güzeltepe, *Codes over Hurwitz integers*, Discrete Math., 2013 (**313**): 704-714.

[2] T. Abualrub, İ. Siap, *Constacyclic codes over* $\mathbb{F}_2+u\mathbb{F}_2$, J. Franklin Inst., 2009 (**346**): 520-529.

[3] İ. Șiap, T. Abualrub, A. Ghrayeb, *Cyclic DNA codes over the ring* $\mathbb{F}_2[u]/(u^2-1)$ *based on the deletion distance*, J. Franklin Inst., 2009 (**346**): 731-740.

[4] T. Abualrub, İ. Ṣiap, *Cyclic codes over the rings* $\mathbb{Z}_2 + u\mathbb{Z}_2$ *and* $\mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$, Des. Codes Cryptogr., 2007(**42**): 273-287.

[5] B. Yildiz, S. Karadeniz, *Self-dual codesover* $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$, J. Franklin Inst., 2010 (**347**): 1888-1894.

[6] B. Yildiz, S. Karadeniz, *Cyclic codes over* $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$, Des. Codes Cryptogr. 2011 (**58**): 221-234.

[7] M. Özen, M. Güzeltepe, *Cyclic codes over some finite quaternion integer rings*, J. Franklin Inst., 2011 (**348**): 1312-1317.

[8] J. H. van Lint, *Nonexistence theorems for perfect error-correcting codes*, Computers in Algebra and Number Theory, SIAM-AMS Proceedings, 1971 (**4**).

[9] G. Davidoff, P. Sarnak, and A. Valette., *Elementary number theory, group theory, and Ramanujan graphs*, Cambridge University Pres, 2003.

[10] J. H. Conway, D. A. Smith, *On quaternions and octonions*, A K Peters, 2003.

[11] C. Martinez, R. Beivide and E. Gabidulin, *Perfect codes from Cayley graphs over Lipschitz integers*, IEEE Trans. Inf. Theory, 2009 (**55**): 3552-3562.

[12] T. P. da N. Neto, J. C. Interlando., *Lattice constellation and codes from quadratic number fields*, IEEE. Trans. Inform. Theory, 2001(**47**): 1514-1527.

[13] K. Huber., *Codes over Gaussian integers*, IEEE Trans. Inform.Theory, 1994 (**40**): 207-216.

[14] K. Huber., *Codes Over Eisenstein-Jacobi integers*, AMS. Contemp. Math., 2004 (**158**): 165-179.

[15] C. Martinez, R. Beivide and E. Gabidulin., *Perfect codes for metrics induced by circulant graphs*, IEEE. Trans. Inform. Theory, 2007 (**53**): 3042-3052.

[16] M. Güzeltepe, *The Macwilliams identity for Lipschitz weight enumerators*, G.U. J Sci. 2016 (**29**): 869-877.

[17] M. Güzeltepe, O. Heden, *Perfect Mannheim, Lipschitz and Hurwitz weight codes*, Math. Commun., 2014 (**19**): 253-276.

[18] O. Heden, M. Güzeltepe, *On perfect* $1-$ *error-correcting codes*", Math. Commun., 2015 (**20**): 23-35.

[19] O. Heden, M. Güzeltepe, *Perfect 1-error-correcting Lipschitz weight codes*, Math. Commun., 2016 (**21**): 23-30.

[20] M. Güzeltepe, A. Altınel, *Perfect 1-error-correcting Hurwitz weight codes*, Math. Commun., 2017 (**22**): 265-272.

[21] J. G. Proakis, M. Salehi, *Commun. Syst. Engrg.*, Second Edition, Prentice Hall.