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Fixed Point Theorems for (k,l)-Almost Contractions in Cone Metric Spaces over Banach Algebras

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Abstract: In this paper we first introduce the notion of (k,l)- almost contraction in the setting of cone metric spaces over Banach algebras. Next we prove that the class of such mappings contains those of Banach, Kannan and Chatterjea type contractions in this new setting. Morever, by proving a fixed point theorem for such a mapping, we provide some significant extensions of the well known results in the metric fixed point theory.

1. Introduction

The Banach contraction mapping principle is one of the most important tools used in nonlinear analysis. Morever it has many generalizations and extensions. For example, the class of (k,l)-almost contractions [1] generalizes the well known results of Kannan [2], Zamfirescu [3], Ciric [4]. Recently, many significant extensions and results for the class of (k,l)-almost contractions have been introduced by many scholars (see [5–8]). In addition to the above generalizations, many people tried to extend the class of fixed point theorems by replacing real valued metrics with an ordered topological vector space(or an ordered Banach space) valued metrics. For instance, in 2007, the class of cone metric spaces over Banach spaces was introduced by Huang and Zhang [9] to generalize metric spaces as follows:

Definition 1.1. Let E be an ordered Banach space whose order is obtained by a normal cone P of E and $X \neq \emptyset$. A cone metric space over E is given by a pair (X,d) where d is a mapping $d : X \times X \to E$ satisfying

(cm1) $\theta \preceq d(x,y)$ and $d(x,y) = \theta$ if and only if $x = y$;

(cm2) $d(x,y) = d(y,x)$;

(cm3) $d(x,y) \preceq d(x,z) + d(z,y)$

for all $x,y,z \in E$ and for null vector $\theta \in E$.

One can also see from [9] the fixed point theorems and the corresponding definitions such as convercence, Cauchy sequence and contractive mapping in this setting. Afterwards, many works (see [10], [11], [12]) were devoted to studying the fixed point theory for such (tvs)-cone metric spaces to generalize the fixed point theorems in the usual metric spaces. However, in 2010, Du [11] proved that the Banach's contraction principle and many associated fixed point theorems in the setting of (tvs)-cone metric spaces are equivalent to their counterparts in the usual metric spaces. Later, in 2013, Liu and Xu [13] introduced the notion of cone metric space over a Banach algebra by replacing Banach space E in the underlying cone metric with a Banach algebra $\mathscr{A}$ to obtain proper generalizations of the usual fixed point theorems. They also introduced the notion of generalized contraction in this new setting as follows:

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Definition 1.2. For a cone metric space (X,d) over a Banach algebra $\mathscr{A}$, let $T: X \mapsto X$ be a self mapping. T is said to a generalized contraction if there exists a constant vector $k \in \mathscr{A}$ with $\rho(k) \in (0,1)$ such that

$$d(Tx,Ty) \preceq kd(x,y) \text{ for all } x,y \in X \tag{1}$$

where $\rho(k)$ stands for the spectral radius of k.

In addition to the above contraction, Liu and Xu [13] studied fixed point theorems for Kannan [2] and Chatterjea [14] type mappings in this setting by using a solid normal cone in $\mathscr{A}$. They also provided some examples to illustrate that fixed point theorems for such contractive type mappings are proper generalizations of the usual fixed point theorems. Based on the results in this setting, recently, many scholars have paid attention to the fixed point theorems for such contractive type mappings to obtain proper generalizations of the well known results (for example see [15], [16], [17]). In 2014, taking into account some basic results of spectral radius, Xu and Radenovic [17] proved that the main results of [13] can be achieved by omitting the assumption of normality of cone in the underlying Banach algebra.
In this paper we first introduce the notion of (k,l)-almost contraction in the setting of cone metric spaces over Banach algebras. Then we see that the class of such (k,l)-almost contractions contains those of Banach, Kannan and Chatterja type contractions in cone metric spaces over Banach algebras introduced in [13]. Furthermore, we prove a fixed point theorems for (k,l)-almost contractions in the setting of cone metric spaces over Banach algebras without normality condition for solid cone in the underlying Banach algebra.

2. Preliminaries

Let us begin by recalling some basic definitions and notions that will be needed to introduce main results in the sequel. Let $\mathscr{A}$ be a real Banach algebra with a multiplicative unit e. It is well known that if the spectral radius of $a \in \mathscr{A}$ defined as follows:

$$\rho(a) := \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} \tag{2}$$

is less than 1, then $e-a$ is invertible (see [18]) and

$$(e-a)^{-1} = \sum_{i=0}^{\infty} a^i. \tag{3}$$

It is also known from [17] that if $\rho(a) < 1$, then $\|a^n\| \to 0$ as $n \to \infty$.
Let P be a subset of $\mathscr{A}$ such that $\{\theta, e\} \subset P$. P is called a cone of $\mathscr{A}$ if the following conditions hold:

(c1) P is closed;

(c2) $\lambda P + \mu P \subset P$ for all non-negative real numbers λ and μ;

(c3) $PP \subset P$ and $P \cap (-P) = \theta$.

For a cone P, a partial ordering $\preceq$ w.r.t. P is defined by $x \preceq y$ iff $y - x \in P$. By $x \prec y$ we understand that $x \preceq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \text{int}P$, where int$P$ denotes the interior of P. A cone P with int$P \neq \emptyset$ is called solid cone. If there exists a positive real number K such that for all $x, y \in \mathscr{A}$

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq K\|y\|, \tag{4}$$

then a cone P is called normal. The least of K's with the above condition is called the normal constant of P.
Throughtout this work we always suppose that $\mathscr{A}$ is a Banach algebra with a multiplicative unit e, P is a solid cone of $\mathscr{A}$ and $\preceq$ stands for the partial ordering induced by P.

Definition 2.1. (See [13]) Let (X,d) be a cone metric space over a Banach space E. If E is replaced with a Banach algebra $\mathscr{A}$, then (X,d) is said to be a cone metric space on a Banach algebra $\mathscr{A}$.

Example 2.2. Let $\mathscr{A}$ be the usual algebra of all real valued continious functions on $X = [0,1]$ which also have continious derivations on X. Endowed with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, $\mathscr{A}$ is a Banach algebra with unit $e = 1$. Morever, $P = \{f \in \mathscr{A} | f(t) \geq 0 \text{ for all } t \in X\}$ is a nonnormal cone. Consider a mapping $d: X \times X \to \mathscr{A}$ defined by $d(x,y)(t) = |x-y|e^t$ for all $x,y \in X$. It is obvious that (X,d) is a cone metric space on the Banach algebra $\mathscr{A}$. For more example see [13, 17].

Definition 2.3. (See [13]) Let (X,d) be a cone metric space on $\mathscr{A}$ and $\{x_n\}$ be a sequence in X. Then

(i) We say that $\{x_n\}$ converges to $x \in X$ if for every $c \gg \theta$ there is a natural number n_0 such that $d(x_n,x) \ll c$ for all $n \geq n_0$. This is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) $\{x_n\}$ is called Cauchy if for all $c \gg \theta$ there is a natural number n_0 such that $d(x_m,x_n) \ll c$ for all $m,n \geq n_0$.

(iii) (X,d) is called complete cone metric if every Cauchy sequence is convergent.

Lemma 2.4. *(See [17]) Let $u \in \mathscr{A}$. For each $c \gg \theta$ if $\theta \preceq u \ll c$, then $u = \theta$.*

Definition 2.5. (see [17]) Let $\{u_n\}$ be a sequence in P. $\{u_n\}$ is called a c-sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geq n_0$.

Lemma 2.6. *(See [17]) If $\{u_n\}$ and $\{v_n\}$ are two c-sequences in P, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence for positive real numbers α and β.*

Lemma 2.7. *(See [17]) Let $\{u_n\} \subset P$ be a c-sequence and $k \in P$. Then $\{ku_n\}$ is a c-sequence in P.*

Lemma 2.8. *(See [17]) The following items are equivalent:*

(i) $\{u_n\}$ is a c-sequence.

(ii) For each $c \gg \theta$ there is $n_0 \in \mathbb{N}$ such that $u_n \prec c$ whenever $n \geq n_0$.

(iii) For each $c \gg \theta$ there is $n_1 \in \mathbb{N}$ such that $u_n \preceq c$ whenever $n \geq n_1$.

Lemma 2.9. *(See [17]) Let (X,d) be a cone metric space on $\mathscr{A}$. If (X,d) is a complete cone metric space and $\{x_n\} \subset X$ is a sequence that converges to $x \in X$, then the following assertions are true:*

(i) $\{d(x_n,x)\}$ is a c-sequence.

(ii) $\{d(x_n,x_{n+m})\}$ is a c-sequence for all $m \in \mathbb{N}$.

Lemma 2.10. *(See [15]) Let (X,d) be a cone metric space on $\mathscr{A}$ and $h \in \mathscr{A}$. If $\rho(h) < 1$, then $\{u_n\}$ with $u_n = h^n$ is a c-sequence.*

Theorem 2.11. *(see [13, 17]) Let (X,d) be a cone metric space over $\mathscr{A}$ with a solid cone P and $T : X \to X$ be a mapping. If there is a $k \in P$ with $\rho(k) < 1$ such that*

$$d(Tx,Ty) \preceq kd(x,y) \text{ for all } x,y \in X, \tag{5}$$

then T has a unique fixed point in X.

3. Main Results

Definition 3.1. (See Theorem 2.3 in [13]) Let (X,d) be a cone metric space over $\mathscr{A}$ and let $T : X \to X$ be a mapping. If there is $\alpha \in P$ with $0 < \rho(\alpha) < \frac{1}{2}$ such that for all $x,y \in X$

$$d(Tx,Ty) \preceq \alpha[d(x,Tx) + d(y,Ty)], \tag{6}$$

then T is said to be a Kannan type contraction in the setting of cone metric spaces with Banach algebras.

Definition 3.2. (See Theorem 2.2 in [13]) Let (X,d) be a cone metric space over a Banach algebra $\mathscr{A}$ and let $T : X \to X$ a mapping. If there is $\alpha \in P$ with $0 < \rho(\alpha) < \frac{1}{2}$ such that for all $x,y \in X$

$$d(Tx,Ty) \preceq \alpha[d(x,Ty) + d(y,Tx)], \tag{7}$$

then T is called a Chatterjea contraction type in the setting of cone metric spaces with Banach algebras.

Definition 3.3. Let (X,d) be a cone metric space over $\mathscr{A}$ and let $T : X \to X$ a mapping. If there is $k \in P$ with $0 < \rho(k) < 1$ and some $l \succeq \theta$ such that for all $x,y \in X$

$$d(Tx,Ty) \preceq kd(x,y) + ld(y,Tx), \tag{8}$$

then we call T a (k,l)-almost contraction in the setting of cone metric spaces with Banach algebras.

Note that when we take into account the symmetry property of d, we have that the condition (8) implies the following dual one:

$$d(Tx,Ty) \preceq kd(x,y) + ld(x,Ty), \text{for all } x,y \in X. \tag{9}$$

Consequently, it is necessary to investigate both (8) and (9) in order for checking the (k,l)-almost contractiveness of a self mapping T in cone metric space (X,d) on $\mathscr{A}$.

Proposition 3.4. *A mapping $T: X \to X$ satisfying (6) is a (k,l)-almost contraction in the setting of cone metric spaces over Banach algebras.*

Proof. By the condition (6) and the triangle rule for d, we have

$$\begin{aligned} d(Tx,Ty) &\preceq \alpha[d(x,Tx)+d(y,Ty)] \\ &\preceq \alpha\{[d(x,y)+d(y,Tx)]+d(y,Tx)+d(Tx,Ty)]\}, \end{aligned} \tag{10}$$

which implies

$$(e-\alpha)d(Tx,Ty) \preceq \alpha d(x,y)+2\alpha d(y,Tx). \tag{11}$$

Since $\rho(\alpha) < \frac{1}{2}$, $e-\alpha$ is invertible, we get from the above inequality

$$d(Tx,Ty) \preceq \alpha(e-\alpha)^{-1}d(x,y)+2(e-\alpha)^{-1}\alpha d(y,Tx) \tag{12}$$

for all $x,y \in X$. Since $\rho(\alpha(e-\alpha)^{-1}) < 1$ and $\theta \preceq 2(e-\alpha)^{-1}\alpha$, by letting $k = \alpha(e-\alpha)^{-1}$ and $l = 2(e-\alpha)^{-1}\alpha$, we see that (8) holds. By the symmetry property of (6) with respect x and y, we also see that (9) holds. □

Proposition 3.5. *A mapping $T: X \to X$ satisfying (7) is a (k,l)-almost contraction in the setting of cone metric spaces with Banach algebras.*

Proof. By considering $d(x,Ty) \preceq d(x,y)+d(y,Tx)+d(Tx,Ty)$ together with (7), we have

$$(e-\alpha)d(Tx,Ty) \preceq \alpha d(x,y)+2\alpha d(y,Tx). \tag{13}$$

Since $e-\alpha$ is invertible, we have

$$d(Tx,Ty) \preceq (e-\alpha)^{-1}\alpha d(x,y)+2(e-\alpha)^{-1}\alpha d(y,Tx), \tag{14}$$

Morever, since $\rho\left((e-\alpha)^{-1}\alpha\right) < 1$ and $\theta \preceq 2(e-\alpha)^{-1}\alpha$, (14) is nothing else than (8) with $k = (e-\alpha)^{-1}\alpha$ and $l = 2(e-\alpha)^{-1}\alpha$. By the symmetry property of (7), (7) also implies (9). □

Theorem 3.6. *Let (X,d) be a complete cone metric space over $\mathscr{A}$. If $T: X \to X$ is a (k,l)-almost contraction, then T has at least one fixed point in X. Additionally, if T satisfies*

$$d(Tx,Ty) \preceq kd(x,y)+ld(x,Tx) \text{ for all } x,y \in X, \tag{15}$$

then it has unique fixed point, and for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. For arbitray $x_0 \in X$, let $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ with $n \geq 1$. By (8), we obtain

$$d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1}) \preceq kd(x_n,x_{n-1}), \tag{16}$$

which implies that

$$d(x_n,x_{n+1}) \preceq k^n d(x_0,x_1).$$

Therefore, for all $m,n \in \mathbb{N}$ with $m \geq n$, we get

$$\begin{aligned} d(x_n,x_m) &\preceq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{m-1},x_m) \\ &\preceq (k^n+k^{n+1}+k^{n+2}+\cdots+k^{m-1})d(x_0,x_1) \\ &\preceq k^n(e+k+k^2+\cdots+k^{m-n-1})d(x_0,x_1) \\ &\preceq k^n(e-k)^{-1}d(x_0,x_1). \end{aligned} \tag{17}$$

Thus, by considering Lemma 2.7 and Lemma 2.10, we easily see that $\{x_n\}$ is Cauchy sequence. Since (X,d) is complete, there is a point $x \in X$ such that $x_n \to x$. Morever we have

$$\begin{aligned} d(x,Tx) &\preceq d(x,x_{n+1})+d(x_{n+1},Tx) \\ &= d(x,x_{n+1})+d(Tx_n,Tx) \\ &\preceq d(x,x_{n+1})+kd(x_n,x)+ld(x,x_{n+1}) \\ &\preceq (e+l)d(x,x_{n+1})+kd(x_n,x). \end{aligned}$$

Let $h_n = (e+l)d(x,x_{n+1})+kd(x_n,x)$. Using Lemma 2.7 and Lemma 2.9 together with Lemma 2.6, we see that $\{h_n\}$ is a c-sequence, implying that for each $c \gg \theta$, there is $n_0 \in \mathbb{N}$ such that $d(x,Tx) \preceq h_n \ll c$ for $n \geq n_0$. Considering Lemma 2.4 we obtain $x = Tx$. Now suppose that T satisfies (15) and $Tx^* = x^*$ for $x^* \in X$. Thus by using

$$d(x,x^*) = d(Tx,Tx^*) \preceq kd(x,x^*)+Ld(x,Tx),$$

which implies that

$$(e-k)d(x,x^*) \preceq \theta. \tag{18}$$

Since $k \in P$ with $\rho(k) < 1$ implies that $e-k$ is invertible and $(e-k)^{-1} \in P$, we derive from (18) that $d(x,x^*) \preceq \theta$. It says that $x = x^*$. Therefore T has a unique fixed point. □

Remark 3.7. Since d holds the symmetry rule, we have that the condition (15) holds if and only if

$$d(Tx,Ty) \preceq kd(x,y) + ld(y,Ty) \text{ for all } x,y \in X \tag{19}$$

satisfies. Thus it is necessary to check whether both (15) and (19) hold in concrete applications.

Corollary 3.8. *(see [1]) Suppose that (X,d) is a complete metric space and $T : X \to X$ is a mapping satisfying*

$$d(Tx,Ty) \preceq \delta d(x,y) + Ld(y,Ty) \textit{ for all } x,y \in X \tag{20}$$

for $\delta \in (0,1)$ and some $L \geq 0$. Then

1) $F(T) = \{x \in X : Tx = x\} \neq \emptyset$;

2) For arbitrary $x_0 \in X$, the Picard iteration $x_{n+1} = Tx_n$ for $n \geq 0$ converges to some $x \in F(T)$;

3) The following estimates

$$d(x_n,x) \leq \frac{\delta^n}{1-\delta} d(x_0,x_1), n = 0,1,2,...,$$

$$d(x_n,x) \leq \frac{\delta}{1-\delta} d(x_{n-1},x_n), n = 0,1,2,...,$$

Proof. By letting $\mathscr{A} = \mathbb{R}$, $k = \delta$ and $l = L$, the proof appears as a special case of that of Theorem 3.6. □

Example 3.9. Let $X = [0,1] \times [0,1]$ and consider the usual Banach algebra $\mathscr{A} = \mathbb{R}^2$ endowed with the standart norm and pointswise multiplication. For a mapping $d : X \times X \to \mathscr{A}$ defined by $d((x_1,y_1),(x_2,y_2)) = (|x_2 - x_1|, |y_2 - y_1|)$, it is obvious that (X,d) is a complete cone metric space over $\mathscr{A}$ with solid cone $P = \{(a,b)|a \geq 0 \text{ and } b \geq 0\}$. Then a mapping $T : X \to X$ defined by

$$f(x,y) = \begin{cases} \left(\frac{x}{3}, \frac{2y}{3}\right) & \text{if} \quad 0 \leq x \leq 1 \quad \text{and} \quad 0 \leq y \leq \frac{1}{2} \\ \left(\frac{x}{3}, \frac{2y}{3} + \frac{1}{3}\right) & \text{if} \quad 0 \leq x \leq 1 \quad \text{and} \quad \frac{1}{2} < y \leq 1 \end{cases}$$

is a (k,l)-almost contraction where $k = \left(\frac{1}{3}, \frac{2}{3}\right) \in P$ with $\rho(k) < 1$ and $l = (0,6) \succeq \theta$. The set of fixed points of T is $\{(0,0),(0,1)\}$.

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