



## ON THE NUMERICAL SOLUTION OF THE KLEIN-GORDON EQUATION BY EXPONENTIAL CUBIC B-SPLINE COLLOCATION METHOD

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**ABSTRACT.** In the present study, we solve initial boundary value problem constructed on nonlinear Klein-Gordon equation. The collocation method on exponential cubic B-spline functions forming a set of basis for the functions defined in the same interval is set up for the numerical approach. The efficiency and validity of the proposed method are determined by computing the error between the numerical and the analytical solutions and relative change of the conserved quantities.

### 1. INTRODUCTION

The nonlinear Klein-Gordon Equation (KGE) with cubic nonlinear term is given in a generalized form

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + P'(u) = 0 \quad (1.1)$$

where  $u = u(x, t)$  and  $P(u)$  is reasonable nonlinear function that is chosen as the potential energy in many cases[1]. The choice of  $P'(u)$  generates various equations such as sine-Gordon, Landau-Ginzburg-Higgs,  $\phi^4$  or reaction Duffing equations arising to model many physical phenomena from quantum mechanics to wave motion [2]. The origin of the KGE is discussed in the Kragh's essay[3]. He mentions in his essay that some earlier types of the KGE was connected with the general relativity theory by significant physicists covering Klein, Fock and Schrödinger. The KGE is absolutely one of the most important equation that the nonlinear Schrödinger equation can be derived from this equation in the canonical form[4]. Galehouse [5] derived the Klein-Gordon equation geometrically using appropriate gauge transformations.

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The scattering theory for the more general form of the KGE (1.1) is proved by Schechter[6]. The same theory for the KGE is developed by following classical procedure of an equivalent equation containing first order time in a particular finite energy normed Hilbert space[7]. Weder has given the proofs of the existence and completeness of the wave operators, the intertwining relations, and the invariance principle in that study[7]. Spectral and scattering theory of the KGE are derived by implementing related eigenfunction expansions in a particular field case to give strong results like in the Schrödinger case [8]. Motion of a KGE in an external field is examined by Tsukanov[9]. The linear response approximation is used to calculate the drift velocity of a kink of the KGE in study.

Since introduced, many types of solutions describing various physical phenomena covering solitary waves and kinks have been found for the KGE. Derivations of solutions of the KGE from the solutions of the wave equation are studied by Chambers[10]. Some bound state type solutions of the KGE for various potentials are set up explicitly by Fleischer and Soff[11]. An exact formal solution for a particular nonlinear KGE is obtained by manipulating the solution of linear Klein-Gordon equation[12]. The envelope soliton and hole solutions for the nonlinear KGE are studied by transforming it to the nonlinear Schrödinger equation[13]. Some solitary wave solutions of a class of field equations for systems with polynomial self-interactions reduce to plane-wave solutions of the KGE[14]. Some bell and kink type solitary wave solutions formed by using some hyperbolic functions for the KGE are constructed with the help of the extended first kind elliptic sub-equation method[15]. Similar solutions have been derived by Akter and Akbar using modified simple equation method[16]. Adomian implemented the decomposition method to the initial boundary value problems for the KGE[17]. The interaction of two solitons that are self-localized and weakly oscillating and damped for the particular form of the KGE are studied by Kudryavtsev[18].

Besides some analytical solutions summarized above, various numerical methods have been applied to derive solutions of the problems related to the KGE. Strauss and Vazquez [19] solved some problems constructed on the KGE by the simplest central second difference method. The equation integrated numerically by using four explicit finite difference methods resulting that the energy conserving scheme is more suitable than the others to simulate the long time behavior of the solutions[20]. A collocation method based on thin plate splines radial basis functions is another proposed method to solve the KGE numerically [21]. The solutions of the non homogeneous form of the KGE are approximated by multiquadratic quasi-interpolation and the integrated radial basis function network schemes[22]. Classical polynomial cubic B-spline functions have also been adapted for the collocation method to derive numerical solutions of the KGE[23, 24, 25, 26, 27].

In the present study, we solve some initial boundary value problems derived for the nonlinear KGE by the collocation method based on non polynomial B-splines, namely exponential cubic B-spline functions.

In order to solve the initial boundary value problem setup for the nonlinear KGE (1.1) numerically, we first reduce the order of the derivative in time for a particular case assuming  $-P'(u) = u - u^3$  [37].

$$\begin{aligned} v_t &= u_{xx} + u - u^3 \\ u_t &= v \end{aligned} \quad (1.2)$$

To complete the usual classical mathematical statement of the problem, the initial and the boundary conditions are chosen as

$$\begin{aligned} u(x, 0) &= f(x), & a \leq x \leq b \\ v(x, 0) &= g(x), & a \leq x \leq b \end{aligned} \quad (1.3)$$

and

$$u_x(a, t) = u_x(b, t) = 0, v_x(a, t) = v_x(b, t) = 0 \quad (1.4)$$

in the finite problem domain  $[a, b]$ .

## 2. EXPONENTIAL B-SPLINE COLLOCATION METHOD

Let  $\kappa$  be a uniformly distributed points of the finite interval  $[a, b]$ , such as,

$$\kappa : x_m = a + mh, m = 0, 1, \dots, N$$

where  $h = \frac{b-a}{N}$ . Then, the exponential cubic B-splines set

$$B_m(x) = \begin{cases} b_2 \left( (x_{m-2} - x) - \frac{1}{\rho} \sinh(\rho(x_{m-2} - x)) \right) & , [x_{m-2}, x_{m-1}] \\ a_1 + b_1(x_m - x) + c_1 \exp(\rho(x_m - x)) + d_1 \exp(-\rho(x_m - x)) & , [x_{m-1}, x_m] \\ a_1 + b_1(x - x_m) + c_1 \exp(\rho(x - x_m)) + d_1 \exp(-\rho(x - x_m)) & , [x_m, x_{m+1}] \\ b_2 \left( (x - x_{m+2}) - \frac{1}{\rho} \sinh(\rho(x - x_{m+2})) \right) & , [x_{m+1}, x_{m+2}] \\ 0 & , \textit{otherwise} \end{cases} \quad (2.1)$$

where  $m = -1, 0, \dots, N$ ,

$$\begin{aligned} a_1 &= \frac{\rho h \cosh(\rho h)}{\rho h \cosh(\rho h) - \sinh(\rho h)}, \\ b_1 &= \frac{\rho \cosh(\rho h)(\cosh(\rho h) - 1) + \sinh^2(\rho h)}{2(\rho h \cosh(\rho h) - \sinh(\rho h))(1 - \cosh(\rho h))}, \\ b_2 &= \frac{2(\rho h \cosh(\rho h) - \sinh(\rho h))}{1 \exp(-\rho h)(1 - \cosh(\rho h)) + \sinh(\rho h)(\exp(-\rho h) - 1)}, \\ c_1 &= \frac{(\rho h \cosh(\rho h) - \sinh(\rho h))(1 - \cosh(\rho h))}{4 \exp(\rho h)(\cosh(\rho h) - 1) + \sinh(\rho h)(\exp(\rho h) - 1)}, \\ d_1 &= \frac{1 \exp(\rho h)(\cosh(\rho h) - 1) + \sinh(\rho h)(\exp(\rho h) - 1)}{4(\rho h \cosh(\rho h) - \sinh(\rho h))(1 - \cosh(\rho h))}, \end{aligned}$$

with real parameter  $\rho$  forms a basis for the functions defined in  $[a, b]$  [28]. Each exponential cubic B-spline  $B_m(x)$  has two continuous lowest derivatives defined in the interval  $[x_{m-1}, x_{m+2}]$ . Different from polynomial B-splines [29], this basis function set has been chosen as trial functions in various methods to solve different problems

arising in different engineering fields[30, 31, 32, 33, 34, 35]. The approximate solutions  $U$  and  $V$  to  $u$  and  $v$ , respectively, can be written in terms of the exponential B-splines as

$$\begin{aligned}
 U(x, t) &= \sum_{i=-1}^{N+1} \delta_i B_i(x), \\
 V(x, t) &= \sum_{i=-1}^{N+1} \varphi_i B_i(x)
 \end{aligned}
 \tag{2.2}$$

where  $\delta_i$  and  $\varphi_i$  are time dependent parameters to be determined from the complementary conditions. The nodal values of  $U$ ,  $V$  and their lowest two derivatives at the nodes can be found from (2.2) as

$$\begin{aligned}
 U_i = U(x_i, t) &= \frac{s - \rho h}{2(\rho h c - s)} \delta_{i-1} + \delta_i + \frac{s - \rho h}{2(\rho h c - s)} \delta_{i+1}, \\
 U'_i = U'(x_i, t) &= \frac{\rho(1 - c)}{2(\rho h c - s)} \delta_{i-1} + \frac{\rho(c - 1)}{2(\rho h c - s)} \delta_{i+1} \\
 U''_i = U''(x_i, t) &= \frac{\rho^2 s}{2(\rho h c - s)} \delta_{i-1} - \frac{\rho^2 s}{\rho h c - s} \delta_i + \frac{\rho^2 s}{2(\rho h c - s)} \delta_{i+1}.
 \end{aligned}
 \tag{2.3}$$

$$\begin{aligned}
 V_i = V(x_i, t) &= \frac{s - \rho h}{2(\rho h c - s)} \phi_{i-1} + \phi_i + \frac{s - \rho h}{2(\rho h c - s)} \phi_{i+1}, \\
 V'_i = V'(x_i, t) &= \frac{\rho(1 - c)}{2(\rho h c - s)} \phi_{i-1} + \frac{\rho(c - 1)}{2(\rho h c - s)} \phi_{i+1} \\
 V''_i = V''(x_i, t) &= \frac{\rho^2 s}{2(\rho h c - s)} \phi_{i-1} - \frac{\rho^2 s}{\rho h c - s} \phi_i + \frac{\rho^2 s}{2(\rho h c - s)} \phi_{i+1}.
 \end{aligned}
 \tag{2.4}$$

where  $c = \cosh \rho h$  and  $s = \sinh \rho h$ .

The time discretization of the system (1.2) by usual forward finite difference and the Crank-Nicolson method leads to

$$\begin{aligned}
 \frac{V^{n+1} - V^n}{\Delta t} - \frac{U^{n+1}_{xx} + U^n_{xx}}{2} - \frac{U^{n+1} + U^n}{2} + \frac{(U^3)^{n+1} + (U^3)^n}{U^{n+1} - U^n - \frac{V^{n+1} + V^n}{2}} = 0 \\
 \frac{U^{n+1} - U^n}{\Delta t} - \frac{V^{n+1} + V^n}{2} = 0
 \end{aligned}
 \tag{2.5}$$

where  $U^{n+1} = U(x, (n + 1)\Delta t)$  represents the solution at the  $(n + 1)$ .th time level. Linearizing the term  $(U^3)^{n+1}$  in (2.5) as [35]

$$(U^3)^{n+1} = 3U^{n+1}(U^2)^n - 2(U^3)^n
 \tag{2.6}$$

yields the time-integrated system:

$$\begin{aligned}
 \frac{V^{n+1} - V^n}{\Delta t} - \frac{U^{n+1}_{xx} + U^n_{xx}}{2} - \frac{U^{n+1} + U^n}{2} + \frac{3U^{n+1}(U^2)^n - (U^3)^n}{U^{n+1} - U^n - \frac{V^{n+1} + V^n}{2}} = 0 \\
 \frac{U^{n+1} - U^n}{\Delta t} - \frac{V^{n+1} + V^n}{2} = 0
 \end{aligned}
 \tag{2.7}$$





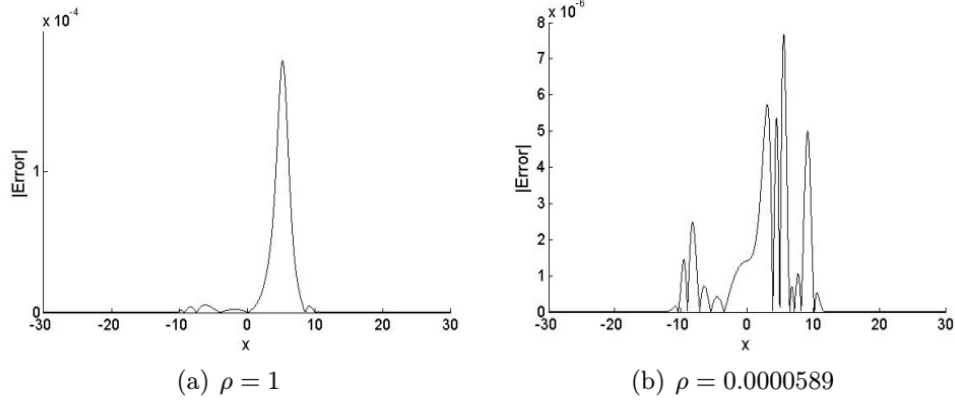


FIGURE 1. Comparison of error norms  $L_\infty$  for various values of  $\rho$

The conservation laws defined in [36]

$$E = \frac{1}{2} \int_a^b [V^2 + U_x^2 - U^2 + \frac{1}{2}U^4] dx$$

$$P = \frac{1}{2} \int_a^b VU_x dx$$

are also computed for each time step to observe if they are preserved.

The KGE equation has an analytical solution representing a travelling wave of the form

$$u(x, t) = \tanh\left(\frac{(x - ct)}{\sqrt{2(1 - c^2)}}\right) \quad (3.1)$$

where  $c$  is the velocity of wave[36]. The initial condition is generated by substituting  $t = 0$  into the analytical solution. In this test problem, the parameters are chosen as  $\Delta t = 0.005$ ,  $h = 0.02$ ,  $c = 0.5$  and various values of the exponential B-spline parameter  $\rho$  in the interval  $[a, b] = [-30, 30]$  for  $t \in [0, 30]$ . The maximum error comparison at the simulation terminating time  $t = 30$  shows that the results are better in two decimal digits when  $\rho = 0.0000589$ , Fig 1(a) and 1(b).

The maximum errors are tabulated for various values of space and time step lengths and the exponential B-spline parameter  $\rho$  at time  $t = 10$  in Table 1. A comparison with Boz's work, based on polynomial B-spline collocation, shows that the exponential B-spline generates the results one decimal digit better even when the same parameters are used.

**Table 1:**  $L_\infty \times 10^3$  error norms for  $c = 0.5$ ,  $t = 10$ ,  $-30 \leq x \leq 30$

Time		Present ( $\rho = 1$ )	Present (various $\rho$ )	Ref: [37]
$h$	$\Delta t$			
0.2	0.05	17.5000	3.6155( $\rho = 0.0000012$ )	11.0155
0.1	0.02	4.5179	0.2525( $\rho = 0.0000053$ )	2.8275
0.05	0.01	1.1333	0.0583( $\rho = 0.0000151$ )	0.6611
0.02	0.005	0.1790	0.0076( $\rho = 0.0000589$ )	0.0466

The analytical values of the conservation laws initially are calculated as  $E = -13.91133798$  and  $P = -0.5443310539$ . The relative changes of the conservation laws defined as

$$C_P(t) = \left| \frac{P_t - P_0}{P_0} \right| \tag{3.2}$$

$$C_E(t) = \left| \frac{E_t - E_0}{E_0} \right|$$

where  $C_P(t)$  and  $C_E(t)$  are the relative changes of conservation laws at the time  $t$  with respect to the reference initial values  $E_0$  and  $P_0$ . We record the computed values of relative changes in Table 2. Even when the results obtained various values when  $\rho = 1$  can be recovered by changing it, Table 1, a significant recovery is not observed in the relative changes of conservation laws, Table 2.

**Table 2:** Relative changes of conservative laws for various values of  $\rho$

Time		( $\rho = 1$ )	( $\rho = 1$ )	(various $\rho$ )	(various $\rho$ )
$h$	$\Delta t$	$C_P(10)$	$C_E(10)$	$C_P(10)$	$C_E(10)$
0.2	0.05	$3.9512 \times 10^{-5}$	$1.3011 \times 10^{-6}$	$3.6516 \times 10^{-5}(\rho = 0.0000012)$	$2.2376 \times 10^{-7}$
0.1	0.02	$2.4573 \times 10^{-6}$	$8.6386 \times 10^{-8}$	$2.3098 \times 10^{-6}(\rho = 0.0000053)$	$2.5001 \times 10^{-8}$
0.05	0.01	$2.9692 \times 10^{-7}$	$2.6278 \times 10^{-9}$	$2.8803 \times 10^{-7}(\rho = 0.0000151)$	$4.5722 \times 10^{-9}$
0.02	0.005	$3.6163 \times 10^{-8}$	$4.9621 \times 10^{-10}$	$3.5804 \times 10^{-8}(\rho = 0.0000589)$	$6.8026 \times 10^{-10}$

#### 4. CONCLUSION

In the study, we derive a collocation method based on exponential B-spline functions combined with strong stable Crank-Nicolson implicit method. In order to check the validity and accuracy of the method, we solve an initial boundary value problem constructed with nonlinear Klein-Gordon equation. The accuracy of the method is measured by computing the maximum error norm. Comparison with the results obtained by polynomial B-splines in [37] shows that the proposed method is capable and generates more accurate results. The relative changes of the lowest two conservation laws are also good indicators of good agreement of the numerical results with analytical ones. Different from polynomial B-splines, the change of parameter in the exponential B-splines provides recovery in the results.

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