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On the Norms of Geometric and Symmetric Geometric Circulant Matrices with the Tribonacci Number

Can KIZILATEŞ[1,*], Naim TUGLU[2]

[1]*Bülent Ecevit University, Department of Mathematics, Zonguldak, Turkey*

[2]*Gazi University, Department of Mathematics, Ankara, Turkey*

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Abstract

In this paper, we study the spectral norms of the geometric circulant matrices $T_{r^*} = Circ_{r^*}(T_0, T_1, \cdots, T_{n-1})$ and the symmetric geometric circulant matrices $ST_{r^*} = SCirc_{r^*}(T_0, T_1, \cdots, T_{n-1})$, where T_n denotes the n^{th} Tribonacci number and r is any complex number.

1. INTRODUCTION

There is no mistrust that circulant and r −circulant matrices have a plenty of applications in many areas such as coding theory, signal processing, image processing, linear forecast and so on.
The n −square r −circulant matrix C_r is the following form

$$C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}.$$

The matrix C_r is determined according to its first row elements and r. Thus, we denote $C_r = Circ_r(c_0, c_1, c_2, \dots, c_{n-1})$. Particularly, for $r = 1$,

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{pmatrix}$$

is called a circulant matrix. Similarly the n −square symmetric r −circulant matrix SC_r is the following form

*Corresponding author, e-mail: cankizilates@gmail.com

$$SC_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & rc_0 \\ c_2 & c_3 & c_4 & \cdots & rc_0 & rc_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_{n-1} & rc_0 & rc_1 & \cdots & rc_{n-3} & rc_{n-2} \end{pmatrix}.$$

The matrix SC_r is determined according to its first row elements and r. Thus, we denote $SC_r = SCirc_r(c_0, c_1, c_2, \dots, c_{n-1})$. Especially when $r = 1$,

$$SC = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \end{pmatrix}$$

is called a symmetric circulant matrix.

Circulant and $r-$circulant matrices with the special numbers have been studied in several papers [1,18]. Solak [1] has given some bounds for the spectral norms of circulant matrices with the Fibonacci and Lucas numbers. Kocer et al. [2] have studied the norms of circulant and semicirculant matrices with the Horadam numbers. In [3], Shen and Cen have given the upper and lower bounds for the spectral norms of $r-$circulant matrices. Bahsi [4] has computed some bounds for the spectral norms of $r-$circulant matrices with the hyperharmonic numbers of the forms $H_r = Circ_r\left(H_0^{(k)}, H_1^{(k)}, \dots, H_{n-1}^{(k)}\right)$, $\widetilde{H}_r = Circ_r\left(H_k^{(0)}, H_k^{(1)}, \dots, H_k^{(n-1)}\right)$. Moreover Bahsi and Solak [5] have studied the norms of $r-$circulant matrices with the hyper-Fibonacci and hyper-Lucas numbers. In [15], Jiang and Zhou have obtained the explicit formula for the spectral norms of even order $r-$circulant matrices. In [6,7,8], Tuglu and Kızılateş have given the spectral norms of circulant and $r-$circulant matrices with the harmonic Fibonacci and hyperharmonic Fibonacci numbers. In [10], Yazlık and Taskara have presented some bounds for the spectral norms of an $r-$circulant matrix with the generalized $k-$Horadam numbers. In [16], Sintunavarat has given upper bound estimation of the spectral norm for $r-$circulant and symmetric $r-$circulant matrices with the Padovan sequence. In [6], Kızılateş and Tuglu have approximated the lower and upper bounds of the spectral norms of geometric circulant matrices with the generalized Fibonacci and hyperharmonic Fibonacci numbers.

Inspired by the recent works, we define a new circulant matrix which is called symmetric geometric circulant matrix and give the upper and lower bounds for the spectral norms of this matrix with the Tribonacci numbers. Moreover, we estimate the lower and upper bounds for the spectral norms of geometric circulant matrices with the Tribonacci numbers.

2. PRELIMINARIES

A Tribonacci sequence $\{T_n\}_{n\geq 0}$ which is a generalized Fibonacci sequence $\{F_n\}$, is defined by

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}$$

with the initial values $T_0 = 0, T_1 = T_2 = 1$.

Now we define $K_n(a, b, c)$ in the following form:

$$K_n(a, b, c) = aT_{n-1}^2 + bT_{n-2}^2 + cT_{n-3}^2 .$$

Let

$$K_n = K_n(1, -2, -3),$$
$$K_n^{(1)} = K_n(|r|^6(|r|^6 - 6),\ |r|^2(3|r|^8 - 1),\ -|r|^4),$$

$$K_n^{(2)}=K_n(|r|^{2n-2}(1-6|r|^6),\ |r|^{2n}(3-|r|^8),\ -|r|^{2n+6}),$$
$$K_n^{(3)}=K_n(|r|^{2n+4},\ |r|^{2n+2}(-2|r|^2+1),\ -|r|^{2n+8}),$$
$$K_n^{(4)}=K_n(|r|^{2n+12},\ |r|^{2n+10},\ |r|^{2n+8}).$$

be special cases of K_n which are considered throughout this paper.

Some authors have studied the Tribonacci sequence and its various properties. For example in [17], Li et al. have studied the determinants, norms of circulant and left circulant matrices with the Tribonacci and generalized Lucas numbers. Furthermore, they gave the squares of sums of Tribonacci numbers as follows:

$$\sum_{k=1}^{n}T_k^2=\frac{1+4T_nT_{n+1}-(T_{n+1}-T_{n-1})^2}{4}.$$

In [18], Rabinowitz has studied the third order linear recurrences and gave various summation formulas for Tribonacci numbers. For example, Rabinowitz [18] pointed out that the following formula was proved by Zeitlin

$$\begin{aligned}-(1-2x-3x^2-6x^3+x^4+x^6)\sum_{k=0}^{n}T_k^2\,x^k=\;&T_{n+1}^2x^{n+1}+(T_{n+2}^2-2T_{n+1}^2)x^{n+2}\\&+(T_{n+3}^2-2T_{n+2}^2-3T_{n+1}^2)x^{n+3}\\&+(T_{n+4}^2-2T_{n+3}^2-3T_{n+2}^2-6T_{n+1}^2)x^{n+4}\\&-T_{n-1}^2x^{n+5}-T_n^2x^{n+6}-x+x^2+x^3+x^4.\end{aligned}\tag{1}$$

Especially, if we take $x=|r|^{-2}$ and $x=|r|^2$ in (1) respectively, we have

$$\sum_{k=0}^{n-1}\left(\frac{T_k}{|r|^k}\right)^2=\frac{|r|^{10}K_{n+2}+|r|^8K_{n+3}+|r|^6K_{n+4}+K_{n+1}^{(1)}+|r|^{2n+4}(1+|r|^2+|r|^4-|r|^6)}{-|r|^{2n}(|r|^{12}-2|r|^{10}-3|r|^8-6|r|^6+|r|^4+1)},\tag{2}$$

and

$$\sum_{k=0}^{n-2}T_k^2\,|r|^{2k}=\frac{|r|^{2n}K_{n+1}+|r|^{2n+2}K_{n+2}+|r|^{2n+4}K_{n+3}+K_n^{(2)}-|r|^2(1-|r|^2-|r|^4-|r|^6)}{-(1-2|r|^2-3|r|^4-6|r|^6+|r|^8+|r|^{12})},\tag{3}$$

for

$$|r|(|r|^{12}-2|r|^{10}-3|r|^8-6|r|^6+|r|^4+1)\neq 0$$

and

$$(1-2|r|^2-3|r|^4-6|r|^6+|r|^8+|r|^{12})\neq 0.$$

Definition 1. [6] An $n\times n$ matrix C_{r^*} is called a geometric circulant matrix if it is the following form

$$C_{r^*}=\begin{pmatrix}c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1}\\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2}\\ r^2c_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ r^{n-1}c_1 & r^{n-2}c_2 & r^{n-3}c_3 & \cdots & rc_{n-1} & c_0\end{pmatrix}.$$

We denote it for brevity by $C_{r^*} = Circ_{r^*}(c_0, c_1, c_2, \dots, c_{n-1})$. Note that for $r = 1$, geometric circulant matrix turns into circulant matrix given in [14]. In fact, in [14], the author calculated the spectral norms of the circulant matrices with the Tribonacci numbers.

Based on definition of the geometric circulant matrix, we define symmetric geometric circulant matrix.

Definition 2. An $n \times n$ matrix SC_{r^*} is called a symmetric geometric circulant matrix if it is the following form

$$SC_{r^*} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & rc_0 \\ c_2 & c_3 & c_4 & \cdots & rc_0 & r^2c_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_{n-1} & rc_0 & r^2c_1 & \cdots & r^{n-2}c_{n-3} & r^{n-1}c_{n-2} \end{pmatrix}.$$

We denote it for brevity by $SC_{r^*} = SCirc_{r^*}(c_0, c_1, c_2, \dots, c_{n-1})$. Note that for $r = 1$, geometric circulant matrix turns into symmetric circulant matrix.

Definition 3. Let $A = (a_{ij})$ be any $m \times n$ matrix. The Euclidean norm of matrix A is

$$\|A\|_E = \sqrt{\left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)}.$$

Definition 4. Let $A = (a_{ij})$ be any $m \times n$ matrix. The spectral norm of matrix A is

$$\|A\|_2 = \sqrt{\max_{1 \le i \le n} \lambda_i(A^H A)},$$

where $\lambda_i(A^H A)$ is eigenvalue $A^H A$ and A^H is conjugate transpose of matrix A.

The following relations between Euclidean norm and spectral norm

$$\frac{1}{\sqrt{n}}\|A\|_E \le \|A\|_2 \le \|A\|_E\,, \tag{4}$$
$$\|A\|_2 \le \|A\|_E \le \sqrt{n}\,\|A\|_2\,, \tag{5}$$

are well known.

Definition 5. Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. Then their Hadamard product is the $m \times n$ matrix of elementwise products $A \circ B = (a_{ij}\, b_{ij})$.

Lemma 1. [12] Let A and B be two $m \times n$ matrices. Thus

$$\|A \circ B\|_2 \le r_1(A) c_1(B),$$

where

$$r_1(A) = \max_{1 \le i \le m} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2}, \quad c_1(B) = \max_{1 \le j \le n} \sqrt{\sum_{i=1}^{m} |b_{ij}|^2}.$$

3. MAIN RESULTS

Theorem 1. Let $T_{r^*} = Circ_{r^*}(T_0, T_1, T_2, \dots, T_{n-1})$ be an $n \times n$ geometric circulant matrix.
i) If $|r| > 1$, then

$$\frac{1}{2}\sqrt{1 + 4T_{n-1}T_n - (T_n - T_{n-2})^2} \le \|T_{r^*}\|_2 \le \frac{1}{2}\sqrt{\frac{(|r|^2 - |r|^{2n})(1 + 4T_{n-1}T_n - (T_n - T_{n-2})^2)}{1 - |r|^2}}$$

ii) If $|r| < 1$, and $|r|^{2n}(|r|^{12} - 2|r|^{10} - 3|r|^8 - 6|r|^6 + |r|^4 + 1) \neq 0$, then

$$\sqrt{\frac{|r|^{10}K_{n+2} + |r|^8 K_{n+3} + |r|^6 K_{n+4} + K_{n+1}^{(1)} + |r|^{2n+4}(1 + |r|^2 + |r|^4 - |r|^6)}{-|r|^{2n}(|r|^{12} - 2|r|^{10} - 3|r|^8 - 6|r|^6 + |r|^4 + 1)}} \le \|T_{r^*}\|_2$$

and

$$\|T_{r^*}\|_2 \le \frac{1}{2}\sqrt{(n-1)(1 + 4T_{n-1}T_n - (T_n - T_{n-2})^2)}\,.$$

Proof. From the definition of T_{r^*}, we have

$$T_{r^*} = \begin{pmatrix} T_0 & T_1 & T_2 & \cdots & T_{n-2} & T_{n-1} \\ rT_{n-1} & T_0 & T_1 & \cdots & T_{n-3} & T_{n-2} \\ r^2T_{n-2} & rT_{n-1} & T_0 & \cdots & T_{n-4} & T_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1}T_1 & r^{n-2}T_2 & r^{n-3}T_3 & \cdots & rT_{n-1} & T_0 \end{pmatrix}.$$

i) From $|r| > 1$ and definition of Euclidean norm, we have

$$\begin{aligned}\|T_{r^*}\|_E^2 &= \sum_{k=0}^{n-1}(n-k)\,T_k^2 + \sum_{k=1}^{n-1} k\,|r^{n-k}|^2\,T_k^2 \\ &\ge \sum_{k=0}^{n-1}(n-k)\,T_k^2 + \sum_{k=0}^{n-1} k\ \ T_k^2 \\ &= n\sum_{k=0}^{n-1} T_k^2 \\ &= n\left(\frac{1 + 4T_{n-1}T_n - (T_n - T_{n-2})^2}{4}\right),\end{aligned}$$

thus

$$\frac{1}{\sqrt{n}}\|T_{r^*}\|_E \ge \frac{1}{2}\sqrt{1 + 4T_{n-1}T_n - (T_n - T_{n-2})^2}.$$

From (4), we have

$$\frac{1}{2}\sqrt{1 + 4T_{n-1}T_n - (T_n - T_{n-2})^2} \le \|T_{r^*}\|_2.$$

On the other hand, let the matrices A and B be defined by

$$A=\begin{pmatrix} T_0 & 1 & 1 & \cdots & 1 & 1 \\ r & T_0 & 1 & \cdots & 1 & 1 \\ r^2 & r & T_0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \cdots & r & T_0 \end{pmatrix}, \tag{6}$$

and

$$B=\begin{pmatrix} T_0 & T_1 & T_2 & \cdots & T_{n-2} & T_{n-1} \\ T_{n-1} & T_0 & T_1 & \cdots & T_{n-3} & T_{n-2} \\ T_{n-2} & T_{n-1} & T_0 & \cdots & T_{n-4} & T_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ T_1 & T_2 & T_3 & \cdots & T_{n-1} & T_0 \end{pmatrix}, \tag{7}$$

where $T_{r^*}=A\circ B$. Then we obtain

$$\begin{aligned} r_1(A) &= \max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}\left|a_{ij}\right|^2} \\ &= \sqrt{|r|^2+|r^2|^2+\cdots+|r^{n-1}|^2} \\ &= \sqrt{\frac{|r|^2-|r|^{2n}}{1-|r|^2}}, \end{aligned}$$

and

$$\begin{aligned} c_1(B) &= \max_{1\le j\le n}\sqrt{\sum_{i=1}^{n}\left|b_{ij}\right|^2} \\ &= \sqrt{\sum_{k=0}^{n-1}T_k^2} \\ &= \sqrt{\frac{1+4T_{n-1}T_n-(T_n-T_{n-2})^2}{4}}. \end{aligned}$$

From Lemma 1, we have

$$\|T_{r^*}\|_2\le\frac{1}{2}\sqrt{\frac{(|r|^2-|r|^{2n})(1+4T_{n-1}T_n-(T_n-T_{n-2})^2)}{1-|r|^2}}.$$

Thus, we have

$$\frac{1}{2}\sqrt{1+4T_{n-1}T_n-(T_n-T_{n-2})^2}\le\|T_{r^*}\|_2\le\frac{1}{2}\sqrt{\frac{(|r|^2-|r|^{2n})(1+4T_{n-1}T_n-(T_n-T_{n-2})^2)}{1-|r|^2}}$$

ii) From $|r|<1$, we have

$$
\begin{aligned}
\|T_{r^*}\|_E^2 &= \sum_{k=0}^{n-1}(n-k)\,T_k^2 + \sum_{k=1}^{n-1} k\,|r^{n-k}|^2\,T_k^2 \\
&\geq \sum_{k=0}^{n-1}(n-k)|r^{n-k}|^2\,T_k^2 + \sum_{k=0}^{n-1} k|r^{n-k}|^2\,T_k^2 \\
&= n\,|r|^{2n}\sum_{k=0}^{n-1}\left(\frac{T_k}{|r|^k}\right)^2.
\end{aligned}
$$

From (2) and (4), we have

$$
\sqrt{\frac{|r|^{10}K_{n+2} + |r|^8 K_{n+3} + |r|^6 K_{n+4} + K_{n+1}^{(1)} + |r|^{2n+4}(1+|r|^2+|r|^4-|r|^6)}{-|r|^{2n}(|r|^{12}-2|r|^{10}-3|r|^8-6|r|^6+|r|^4+1)}} \leq \|T_{r^*}\|_2.
$$

Since $T_{r^*} = A \circ B$ for A and B matrices are as in (6) and (7), we obtain

$$
\begin{aligned}
r_1(A) &= \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}\left|a_{ij}\right|^2} \\
&= \sqrt{T_0^2 + n - 1} \\
&= \sqrt{n-1},
\end{aligned}
$$

and

$$
\begin{aligned}
c_1(B) &= \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}\left|b_{ij}\right|^2} \\
&= \sqrt{\sum_{k=0}^{n-1} T_k^2} \\
&= \sqrt{\frac{1+4T_{n-1}T_n - (T_n - T_{n-2})^2}{4}}.
\end{aligned}
$$

Hence, from Lemma 1, we have

$$
\|T_{r^*}\|_2 \leq \frac{1}{2}\sqrt{(n-1)(1+4T_{n-1}T_n - (T_n - T_{n-2})^2)}.
$$

Thus, we have proved the theorem.

Theorem 2. Let $ST_{r^*} = SCirc_{r^*}(T_0, T_1, T_2, \ldots, T_{n-1})$ be an $n \times n$ symmetric geometric circulant matrix.
i) If $|r| > 1$, then

$$
\frac{1}{2}\sqrt{1+4T_{n-1}T_n - (T_n - T_{n-2})^2} \leq \|ST_{r^*}\|_2 \leq \frac{1}{2}\sqrt{\frac{(1-|r|^{2n})(1+4T_{n-1}T_n - (T_n - T_{n-2})^2)}{1-|r|^2}}
$$

ii) If $|r| < 1$, and $(1 - 2|r|^2 - 3|r|^4 - 6|r|^6 + |r|^8 + |r|^{12}) \neq 0$, then

$$\sqrt{\frac{|r|^{2n+6}K_{n+3} + K_{n+2}^{(3)} - K_n^{(4)} - |r|^4(1 - |r|^2 - |r|^4 - |r|^6)}{-(1 - 2|r|^2 - 3|r|^4 - 6|r|^6 + |r|^8 + |r|^{12})}} \leq \|ST_{r^*}\|_2$$

and

$$\|ST_{r^*}\|_2 \leq \frac{1}{2}\sqrt{n(1 + 4T_{n-1}T_n - (T_n - T_{n-2})^2)}\,.$$

Proof. From the definition of ST_{r^*}, we have

$$ST_{r^*} = \begin{pmatrix} T_0 & T_1 & T_2 & \cdots & T_{n-2} & T_{n-1} \\ T_1 & T_2 & T_3 & \cdots & T_{n-1} & rT_0 \\ T_2 & T_3 & T_4 & \cdots & rT_0 & r^2T_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ T_{n-1} & rT_0 & r^2T_1 & \cdots & r^{n-2}T_{n-3} & r^{n-1}T_{n-2} \end{pmatrix}.$$

i) From $|r| > 1$, we have

$$\begin{aligned} \|ST_{r^*}\|_E^2 &= \sum_{k=0}^{n-1}(k+1)\,T_k^2 + \sum_{k=0}^{n-2}\big(n - (k+1)\big)\,|r^{k+1}|^2\,T_k^2 \\ &\geq \sum_{k=0}^{n-1}(k+1)\,T_k^2 + \sum_{k=0}^{n-2}(n - (k+1))\,T_k^2 \\ &= n\sum_{k=0}^{n-1} T_k^2\,, \end{aligned}$$

thus

$$\frac{1}{\sqrt{n}}\|ST_{r^*}\|_E \geq \frac{1}{2}\sqrt{1 + 4T_{n-1}T_n - (T_n - T_{n-2})^2}\,,$$

from (4), we have

$$\frac{1}{2}\sqrt{1 + 4T_{n-1}T_n - (T_n - T_{n-2})^2} \leq \|ST_{r^*}\|_2.$$

On the other hand, let the matrices D and E be defined by

$$D = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & r \\ 1 & 1 & 1 & \cdots & r & r^2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & r & r^2 & \cdots & r^{n-2} & r^{n-1} \end{pmatrix}, \tag{8}$$

$$E = \begin{pmatrix} T_0 & T_1 & T_2 & \cdots & T_{n-2} & T_{n-1} \\ T_1 & T_2 & T_3 & \cdots & T_{n-1} & T_0 \\ T_2 & T_3 & T_4 & \cdots & T_0 & T_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ T_{n-1} & T_0 & T_1 & \cdots & T_{n-3} & T_{n-2} \end{pmatrix}, \tag{9}$$

That is, $ST_{r^*} = D \circ E$. Then we obtain

$$\begin{aligned} r_1(D) &= \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} \left|d_{ij}\right|^2} \\ &= \sqrt{1 + |r|^2 + \cdots + |r^{n-1}|^2} \\ &= \sqrt{\frac{1-|r|^{2n}}{1-|r|^2}}, \end{aligned}$$

and

$$\begin{aligned} c_1(E) &= \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} \left|e_{ij}\right|^2} \\ &= \sqrt{\sum_{k=0}^{n-1} T_k^2} \\ &= \sqrt{\frac{1 + 4T_{n-1}T_n - (T_n - T_{n-2})^2}{4}}. \end{aligned}$$

From Lemma 1, we have

$$\|ST_{r^*}\|_2 \le \frac{1}{2}\sqrt{\frac{(1-|r|^{2n})(1 + 4T_{n-1}T_n - (T_n - T_{n-2})^2)}{1-|r|^2}}.$$

Thus, we have

$$\frac{1}{2}\sqrt{1 + 4T_{n-1}T_n - (T_n - T_{n-2})^2} \le \|ST_{r^*}\|_2 \le \frac{1}{2}\sqrt{\frac{(1-|r|^{2n})(1 + 4T_{n-1}T_n - (T_n - T_{n-2})^2)}{1-|r|^2}}.$$

ii) From $|r| < 1$, we have

$$\begin{aligned} \|ST_{r^*}\|_E^2 &= \sum_{k=0}^{n-1} (k+1)\, T_k^2 + \sum_{k=0}^{n-2} \big(n-(k+1)\big)|r^{k+1}|^2\, T_k^2 \\ &\ge \sum_{k=0}^{n-1} (k+1)|r^{k+1}|^2\, T_k^2 + \sum_{k=0}^{n-2} (n-(k+1))|r^{k+1}|^2\, T_k^2 \\ &= n\left(|r|^{2n}T_{n-1}^2 + |r|^2 \sum_{k=0}^{n-2} |r^k|^2\, T_k^2\right). \end{aligned}$$

From (3) and (4), we have

$$\sqrt{\frac{|r|^{2n+6}K_{n+3} + K_{n+2}^{(3)} - K_n^{(4)} - |r|^4(1-|r|^2-|r|^4-|r|^6)}{-(1-2|r|^2-3|r|^4-6|r|^6+|r|^8+|r|^{12})}} \le \|ST_{r^*}\|_2.$$

On the other hand, $ST_{r^*} = D \circ E$ for D and E matrices are as in (8) and (9). Then we obtain

$$r_1(D) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} \left|d_{ij}\right|^2}$$
$$= \sqrt{n},$$

and

$$c_1(E) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} \left|e_{ij}\right|^2}$$
$$= \sqrt{\sum_{k=0}^{n-1} T_k^2}$$
$$= \sqrt{\frac{1 + 4T_{n-1}T_n - (T_n - T_{n-2})^2}{4}}.$$

From Lemma 1, we have

$$\|ST_{r^*}\|_2 \le \frac{1}{2}\sqrt{n(1 + 4T_{n-1}T_n - (T_n - T_{n-2})^2)}\,.$$

Thus, we have proved the theorem.

4. NUMERICAL EXAMPLES

In this section, we present some numerical examples for the spectral norms of geometric circulant and symmetric geometric circulant matrices with the Tribonacci numbers.

We obtain upper and lower bounds for the spectral norms of some geometric circulant matrices T_{r^*}, with the aid of Theorem 1.

Table 1. *Some lower and upper bounds for the spectral norms of T_{r^*} for $n = 5,10$ and $|r| > 1$*

n	r	Lower Bound	$\|T_{r^*}\|_2$	Upper Bound
	1.05	4.69041576	8.402033780	10.62931143
	1.1	4.69041576	8.849230428	12.04007873
	1.3	4.69041576	11.12108364	19.63834913
5	1.5	4.69041576	14.26340057	31.22986852
	2.0	4.69041576	27.42748232	86.48699325
	3.0	4.69041576	96.93973984	402.9392014
	5.0	4.69041576	655.8764131	2991.955882
	1.05	96.50388593	190.9787184	375.3707634
	1.1	96.50388593	207.0636156	494.6594645
	1.3	96.50388593	294.7644139	1594.462022
10	1.5	96.50388593	434.0993489	4975.713953
	2.0	96.50388593	1373.819407	57053.63275
	3.0	96.50388593	24136.90407	2014709.130
	5.0	96.50388593	2050365.697	2014709.130

Table 2*. Some lower and upper bounds for the spectral norms of* T_{r^*} *for* $n = 5{,}10$ *and* $|r| <$

n	r	Lower Bound	$\|T_{r^*}\|_2$	Upper Bound
	0.99	4.624738794	7.924867122	9.380831520
	0.95	4.369958503	7.641418221	9.380831520
	0.9	4.067715355	7.324250001	9.380831520
5	0.5	2.066284407	5.917138485	9.380831520
	0.1	0.400500948	5.513050311	9.380831520
	0.05	0.200062529	5.491902324	9.380831520
	0	0	5.475122251	9.380831520
	0.99	95.14223158	174.4546153	289.5116578
	0.95	89.85816321	165.1016853	289.5116578
	0.9	83.58428773	155.2357764	289.5116578
10	0.5	42.08280145	121.7992637	289.5116578
	0.1	8.111977421	115.2524333	289.5116578
	0.05	4.051494664	115.0687989	289.5116578
	0	0	114.9788979	289.5116578

We obtain lower and upper bounds for the spectral norms of some symmetric geometric circulant matrices ST_{r^*}, with the aid of Theorem 2.

Table 3. *Some lower and upper bounds for the spectral norms of* ST_{r^*} *for* $n = 5{,}10$ *and* $|r| > 1$

n	r	Lower Bound	$\|ST_{r^*}\|_2$	Upper Bound
	1.05	4.69041576	8.220239450	11.61818666
	1.1	4.69041576	8.487190502	12.92143552
	1.3	4.69041576	10.26132665	20.19070966
5	1.5	4.69041576	13.84131204	31.58013122
	2.0	4.69041576	35.23793521	86.61408660
	3.0	4.69041576	167.2233277	402.9664998
	5.0	4.69041576	1263.130476	2991.959558
	1.05	96.50388593	187.4666567	387.5773601
	1.1	96.50388593	209.8077569	503.9851045
	1.3	96.50388593	585.2339956	1597.379774
10	1.5	96.50388593	1953.780197	4976.649710
	2.0	96.50388593	24335.66094	57053.71440
	3.0	96.50388593	895639.3634	2014709.133
	5.0	96.50388593	86972284.25	192370832.3

Table 4*. Some lower and upper bounds for the spectral norms of* ST_{r^*} *for* $n = 5{,}10$ *and* $|r| < 1$

n	r	Lower Bound	$\|ST_{r^*}\|_2$	Upper Bound
	0.99	4.479192799	7.960742675	10.48808848
	0.95	3.712555843	7.817564142	10.48808848
	0.9	2.913462696	7.665706410	10.48808848
5	0.5	0.330718914	7.058029285	10.48808848
	0.1	0.010051945	6.910052528	10.48808848
	0.05	0.002503154	6.906491360	10.48808848
	0	0	6.905344296	10.48808848
	0.99	87.65078325	175.6894269	305.1702826
	0.95	59.13733005	171.9043109	305.1702826
	0.9	35.43967690	169.1320420	305.1702826
10	0.5	0.392535610	164.1475325	305.1702826
	0.1	0.010051947	163.7610623	305.1702826
	0.05	0.002503154	163.7548164	305.1702826
	0	0	163.7528611	305.1702826

5. CONCLUSION

In this paper we give some bounds for the spectral norms of geometric circulant and symmetric geometric circulant matrices with the Tribonacci numbers. In [19], the author introduced the Quadrapell numbers. They are defined by the recurrence relation for $n \geq 4$

$$D_n = D_{n-2} + 2D_{n-3} + D_{n-4}$$

with initial values $D_0 = D_1 = D_2 = 1$ and $D_3 = 2$.
It would be interesting to study the spectral norms of geometric circulant and symmetric geometric circulant matrices with the Quadrapell numbers.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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