Available online: June 19, 2018

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 68, Number 1, Pages 528–537 (2019)
DOI: 10.31801/cfsuasmas.434525
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

ON ANDERSSON'S INEQUALITY FOR HYPERBOLICALLY CONVEX FUNCTIONS

NASHAT FARIED, MOHAMED S. S. ALI, AND ZEINAB M. YEHIA

Abstract. In this paper, we use Andersson's Inequality for ordinary convex functions to prove a similar inequalities concerning hyperbolically convex functions.

1. Introduction

One of the powerful properties of functions, which play a very important role in many areas of mathematics both pure and applied, is convexity. An arbitrary function f defined on an interval I is said to be convex if each point on the chord between $(u, f(u))$ and $(v, f(v))$ is above the graph of f for any $u, v \in I$. In fact, there are families of real functions $\{F(x)\}$ which are not topologically equivalent to the family $\{L(x)\}$ of all non vertical line segments terminating on $x = u$ and $x = v$. So that, there are many activities concerning to generalize the notion of a convex function to other classes of functions. But many properties of convex functions are satisfied for these general functions. Generalized convex functions were first defined and systematically investigated by Beckenbach [1] and studied furthermore by Beckenbach and Bing [2]. More generally, let $\{F(x)\}$ be a family of real functions $F(x)$ defined in an interval I, such that for given points $p_1 : (u_1, v_1)$ and $p_2 : (u_2, v_2)$, $u_1, u_2 \in I$ with $u_1 < u_2$, there is a unique member of $\{F(x)\}$ through p_1 and p_2. Functions $f(x)$ dominated by $\{F(x)\}$ are said to be convex relative to $\{F(x)\}$. In this work, we concern with one of these generalizations in the sense of Beckenbach by replacing the family of linear functions with a family of hyperbolic functions,

$$H(x) = A \cosh px + B \sinh px,$$

where A, B arbitrary constants and p is a fixed positive constant.

Received by the editors: September 22, 2017; Accepted: February 15, 2018.

2010 *Mathematics Subject Classification.* Primary 52A41,52A99; Secondary 26B25,26D15.

Key words and phrases. Generalized convex functions, hyperbolically convex functions, supporting functions, integral inequalities.

2. Definitions and Preliminary Results

In this section, we introduce the basic definitions and results which will be used later. For more informations see [3, 4, 5, 6].

Definition 1. *A function $f : I \to \mathbb{R}$ is said to be sub H-function on I or hyperbolically convex function, if for any arbitrary closed subinterval $[u, v]$ of I the graph of $f(x)$ for $x \in [u, v]$ lies nowhere above the function, determined by the equation:*
$H(x) = H(x, u, v, f) = A \cosh px + B \sinh px$; $p > 0$
where A and B are chosen such that $H(u) = f(u)$, and $H(v) = f(v)$.
Equivalently, for all $x \in [u, v]$

$$f(x) \le H(x) = \frac{f(u)\sinh p(v-x) + f(v)\sinh p(x-u)}{\sinh p(v-u)}. \tag{1}$$

Remark 2. *The hyperbolically convex functions possess a number of properties analogous to those of convex functions.*
For example: If $f : I \to \mathbb{R}$ is Hyperbolically convex function, then for any $u, v \in I$, the inequality $f(x) \ge H(x)$ holds outside the interval $[u, v]$.

Definition 3. *Let $f : I \to \mathbb{R}$ be a hyperbolically convex function.*
A function $S_u(x) = A \cosh px + B \sinh px$, is said to be supporting function for $f(x)$ at the point $u \in I$ if:

(1) $S_u(u) = f(u)$,
(2) $S_u(x) \le f(x)$ $\forall x \in I$.

That is, if $f(x)$ and $S_u(x)$ agree at $x = u$, the graph of $f(x)$ does not lie under the support curve.

Definition 4. *The hypergeometric function ${}_1F_2(a; b, c; x)$ is defined as*

$${}_1F_2(a; b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!(b)_n(c)_n} x^n$$

where,

$$(a)_n = a(a+1)...(a+n-1).$$

Theorem 5. *A function $f : I \to \mathbb{R}$ is hyperbolically convex function on I if and only if there exists a supporting function for $f(x)$ at each point x in I.*

Proposition 6. *If $f : I \to \mathbb{R}$ is a differentiable hyperbolically convex function, then the supporting function for $f(x)$ at the point $u \in I$ has the form*

$$S_u(x) = f(u) \cosh p(x-u) + \frac{f'(u)}{p} \sinh p(x-u).$$

Theorem 7. *A function $f : I \to \mathbb{R}$ is convex if and only if there is an increasing function $g : (a, b) \to \mathbb{R}$ and a point $c \in (a, b)$ such that for all $x \in (a, b)$,*

$$f(x) - f(c) = \int_c^x g(t)dt.$$

Theorem 8. *If $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ are both non-negative, increasing(decreasing), and convex, then $h(x) = f(x)g(x)$ also preserve these three properties.*

Theorem 9. *If $f : I \to \mathbb{R}$ is differentiable function, then f is convex if and only if f' is increasing.*

Theorem 10. ***(Andersson's inequality)*** *Let $F_1(x), F_2(x), ..., F_n(x)$ be convex functions, defined in $0 \leq x \leq 1$, and for which*

$$F_p(x) \geq 0,\ F_p(0) = 0,\ \ p = 1, 2, ..., n.$$

$$If \int_0^1 F_p(x)dx = \alpha_p,\ then$$

$$\int_0^1 F_1(x)F_2(x)...F_n(x)dx \geq \frac{2^n}{n+1}\alpha_1\alpha_2...\alpha_n.$$

3. Main Results

In this section, we derive a similar results to Andersson's inequality for hyperbolically convex functions through three theorems the fundamental difference between them depends on the different intervals of integration and the conditions imposed on the convex functions.

Theorem 11. *Let $k_1(x), k_2(x), ..., k_n(x)$ be convex functions, defined in $0 \leq x \leq 1$, for which*

$$k_r(x) \geq 0,\ k_r(0) = 0,\ \ r = 1, 2, ..., n,$$

and let $f(x)$ be differentiable hyperbolically convex function defined on $[0, 1] \subseteq R$, such that: $f(0) \geq 0$, $f'(0) = 0$ and

$$\int_0^1 k_r(x)dx = \alpha_r,$$

then,

$$\int_0^1 f(x)\prod_{r=1}^{n} k_r(x)dx \geq \frac{2^n f(0)}{n+1}\prod_{r=1}^{n}\alpha_r.$$

Proof. As $f(x)$ is hyperbolically convex function, then from Definition 3, it follows that:

$$f(x) \geq S_0(x) \quad \forall x \in [0, 1].$$

Since $f(x)$ is differentiable and $f'(0) = 0$, then from Proposition 6, the supporting function $S_0(x)$ for $f(x)$ at the point $0 \in [0, 1]$ can be written in the form

$$S_0(x) = f(0)\cosh px.$$

Consequently,

$$f(x) \geq f(0)\cosh px \quad \forall x \in [0, 1]. \tag{2}$$

As $\prod_{r=1}^{n} k_r(x) \geq 0$, by using 2, one has:

$$f(x)\prod_{r=1}^{n} k_r(x) \geq f(0)\cosh px \prod_{r=1}^{n} k_r(x).$$

As $\cosh px \geq 1$, then

$$f(x)\prod_{r=1}^{n} k_r(x) \geq f(0)\prod_{r=1}^{n} k_r(x),$$

$$\int_0^1 f(x)\prod_{r=1}^{n} k_r(x)dx \geq f(0)\int_0^1 \prod_{r=1}^{n} k_r(x)dx.$$

Now by using Theorem 10, we get

$$\int_0^1 f(x)\prod_{r=1}^{n} k_r(x)dx \geq \frac{2^n f(0)}{n+1}\prod_{r=1}^{n} \alpha_r.$$

Hence, the claim. □

Theorem 12. *Let $k_1(x), k_2(x), ..., k_n(x)$ be convex functions, defined in $0 \leq x \leq \frac{1}{p}\sinh^{-1}(1)$, for which*

$$k_r(x) \geq 0,\ k_r(0) = 0,\ k_r'(x) \leq 0,\ r = 1, 2, ..., n,$$

and let $f(x)$ be differentiable hyperbolically convex function defined on $[0, \frac{1}{p}\sinh^{-1}(1)] \subseteq R$, such that: $f(0) \geq 0$, $f'(0) = 0$ and

$$\int_0^{\frac{1}{p}\sinh^{-1}(1)} k_r(x)\cosh px dx = \alpha_r,$$

then,

$$\int_0^{\frac{1}{p}\sinh^{-1}(1)} f(x)\prod_{r=1}^{n} k_r(x)dx \geq \frac{2^n f(0)p^{n-1}}{(n+1)}\prod_{r=1}^{n} \alpha_r.$$

Proof. As $f(x)$ is hyperbolically convex function, from Definition 3, we get:

$$f(x) \geq S_0(x) \quad \forall x \in [0, \frac{1}{p}\sinh^{-1}(1)].$$

Since $f(x)$ is differentiable and $f'(0) = 0$, then from Proposition 6, the supporting function $S_0(x)$ for $f(x)$ at the point $0 \in [0, \frac{1}{p}\sinh^{-1}(1)]$ can be written in the form

$$S_0(x) = f(0)\cosh px.$$

Consequently,

$$f(x) \geq f(0)\cosh px \quad \forall x \in [0, \frac{1}{p}\sinh^{-1}(1)]. \tag{3}$$

As $\prod_{r=1}^{n} k_r(x) \geq 0$, by using 3, one has:

$$\int_0^{\frac{1}{p}\sinh^{-1}(1)} f(x) \prod_{r=1}^{n} k_r(x)dx \geq f(0) \int_0^{\frac{1}{p}\sinh^{-1}(1)} \prod_{r=1}^{n} k_r(x) \cosh pxdx. \tag{4}$$

Using the following substitution

$$t = \sinh px, \tag{5}$$

and let

$$h_r(t) = k_r(x), \tag{6}$$

then it follows that:

$$\int_0^{\frac{1}{p}\sinh^{-1}(1)} k_r(x) \cosh pxdx = \frac{1}{p} \int_0^1 h_r(t)dt = \alpha_r,$$

$$h_r(0) = 0, \; h_r(t) \geq 0 \;\; \forall t \in [0,1],$$

and we will use the following notations,

$$k'_r(x) = \frac{d}{dx}[k_r(x)], \; k''_r(x) = \frac{d^2}{dx^2}[k_r(x)],$$

$$\dot{h_r}(t) = \frac{d}{dt}[h_r(t)], \; \ddot{h_r}(t) = \frac{d^2}{dt^2}[h_r(t)].$$

Hence,

$$k'_r(x) = \dot{h_r}(t) p \cosh px, \tag{7}$$

$$k''_r(x) = \ddot{h_r}(t)p^2 \cosh^2 px + \dot{h_r}(t)p^2 \sinh px,$$

$$\ddot{h_r}(t) = \frac{1}{p^2 \cosh^2 px}[k''_r(x) - \dot{h_r}(t)p^2 \sinh px].$$

From 7, then

$$\ddot{h_r}(t) = \frac{1}{p^2 \cosh^2 px}[k''_r(x) - pk'_r(x) \tanh px].$$

Since,

$$k'_r(x) \leq 0.$$

Then $k''_r(x) - pk'_r(x) \tanh px \geq 0$. It follows that $\ddot{h_r}(t) \geq 0$. Hence $\dot{h_r}(t)$ is an increasing function in $[0,1]$. Using Theorem 9, one obtains that $h_r(t)$ is a convex function in $[0,1]$.

Then $h_r(t), r = 1, 2, ..., n$ satisfy all assumptions of Theorem 10 in the interval $[0,1]$. Hence,

$$\int_0^1 \prod_{r=1}^{n} h_r(t)dt \geq \frac{2^n p^n}{n+1} \prod_{r=1}^{n} \alpha_r. \tag{8}$$

Now using 5, 6 and 8, then 4 turns out to:

$$\begin{aligned}\int_0^{\frac{1}{p}\sinh^{-1}(1)} f(x)\prod_{r=1}^{n} k_r(x)dx &\geq \frac{f(0)}{p}\int_0^1 \prod_{r=1}^{n} h_r(t)dt \\ &\geq \frac{2^n f(0)p^{n-1}}{(n+1)}\prod_{r=1}^{n}\alpha_r.\end{aligned}$$

Hence, the claim. □

Theorem 13. *Let $k_1(x), k_2(x), ..., k_n(x)$ be convex functions, defined in $0 \leq x \leq \frac{\pi}{2p}$, for which*

$$k_r(x) \geq 0,\ k_r(0) = 0,\quad r = 1, 2, ..., n,$$

and let $f(x)$ be differentiable hyperbolically convex function defined on $[0, \frac{\Pi}{2p}] \subseteq R$, such that: $f(0) \geq 0, f'(0) = 0$ and

$$\int_0^{\frac{\pi}{2p}} k_r(x)dx = \alpha_r.$$

then, one has the following sharp inequality:

$$\int_0^{\frac{\pi}{2p}} f(x)\prod_{r=1}^{n} k_r(x)dx \geq \frac{2^{2n-1}}{n+1} f(0)(\frac{p}{\pi})^{n-1}(\prod_{r=1}^{n}\alpha_r)\,{}_1F_2(\frac{1}{2}+\frac{n}{2};\frac{1}{2},\frac{3}{2}+\frac{n}{2};\frac{\pi^2}{16}). \quad (9)$$

Proof. We have four steps in this proof. Let M denote the class of convex functions of the theorem.

Step 1. If $k \in M$, then k is increasing. Since $k(x)$ is convex in $[0, \frac{\pi}{2p}]$ and $k(0) = 0$, then from Theorem 7 there is an increasing function $h : [0, \frac{\pi}{2p}] \to R$, such that

$$k(x) = \int_0^x h(t)dt,\ x \in [0, \frac{\pi}{2p}]. \quad (10)$$

Now, suppose that $h(t_0) < 0$ for some $t_0 \in [0, \frac{\pi}{2p}]$. As h is increasing, then $h(t) < 0$ for all $t \in [0, t_0]$, therefore, $\int_0^x h(t)dt < 0,\ x \in [0, t_0]$. From 10 it follows, $k(x) < 0$, $x \in [0, \frac{\pi}{2p}]$, which contradicts the fact that $k(x) \geq 0,\ x \in [0, \frac{\pi}{2p}]$. Thus, $h(t) \geq 0$ for all $t \in [0, \frac{\pi}{2p}]$.

Now, let $x_1, x_2 \in [0, \frac{\pi}{2p}]$, if $x_1 \leq x_2$, then using 10, one has:

$$0 \leq \int_{x_1}^{x_2} h(t)dt = \int_0^{x_2} h(t)dt - \int_0^{x_1} h(t)dt = k(x_2) - k(x_1).$$

Hence, $k(x)$ is an increasing function.

For the next steps, let

$$k_r^*(x) = \frac{2\alpha_r x}{(\frac{\pi}{2p})^2},\ 0 \leq x \leq \frac{\pi}{2p}, \quad (11)$$

and

$$\phi_r(x) = \int_0^x [k_r^*(s) - k_r(s)]ds. \tag{12}$$

Step 2. M is closed under multiplication. From 11, it follows that: k_r^* is non-negative, increasing, and convex function satisfies $k_r^*(0) = 0$. Thus, $k_r^* \in M$, $r = 1, 2, ..., n$. Hence, from Theorem 8, M is closed under multiplication.

Step 3. $\Phi_r(x) \geq 0,\ x \in [0, \frac{\Pi}{2p}]$. Using 11 and 12, one concludes that

$$\int_0^{\frac{\pi}{2p}} k_r^*(x)dx = \int_0^{\frac{\pi}{2p}} k_r(x)dx = \alpha_r, \tag{13}$$

$$\Phi_r(0) = 0,\ and\ \Phi_r(\frac{\pi}{2p}) = 0. \tag{14}$$

From the convexity of $k_r(x)$, it follows that the graph of $k_r(x)$ must intersect the straight line of $k_r^*(x)$ in a unique point q as shown in figure 1. If x lies in $[0, b]$, then obviously from 12 it follows that

$$\Phi_r(x) \geq 0,\ 0 \leq x \leq b.$$

Otherwise, if $b \leq x \leq \frac{\pi}{2p}$, one has:

$$\int_x^{\frac{\pi}{2p}} k_r(s)ds \geq \int_x^{\frac{\pi}{2p}} k_r^*(s)ds.$$

Using 13, one obtains:

$$\int_0^{\frac{\pi}{2p}} k_r^*(s)ds - \int_x^{\frac{\pi}{2p}} k_r^*(s)ds \geq \int_0^{\frac{\pi}{2p}} k_r(s)ds - \int_x^{\frac{\pi}{2p}} k_r(s)ds$$

$$\int_0^x [k_r^*(s)ds - k_r(s)]ds \geq 0.$$

Thus, $\Phi_r(x) \geq 0,\ b \leq x \leq \frac{\pi}{2p}$. Hence, $\Phi_r(x) \geq 0,\ x \in [0, \frac{\pi}{2p}]$.

As $f(x)$ is differentiable hyperbolically convex function and $f'(0) = 0$, according to Proposition 6, for convenience, we denote

$$f^*(x) = f(0)\cosh px, \tag{15}$$

as the supporting function for $f(x)$ at the point $0 \in [0, \frac{\pi}{2p}]$. But, from Definition 3, one obtains:

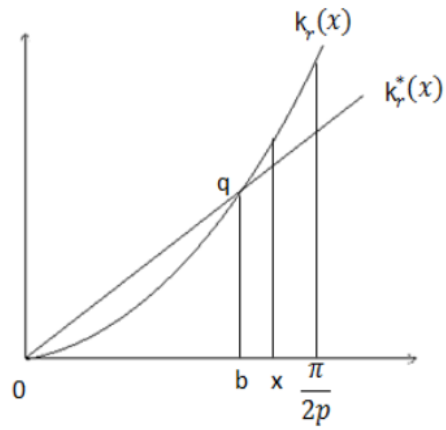
$$f(x) \geq f^*(x),\ x \in [0, \frac{\pi}{2p}], \tag{16}$$

hence we can go to: Step 4. We show that if $G_1(x), G_2(x) \in M$, then

$$\int_0^{\frac{\pi}{2p}} G_1(x)G_2(x)f(x)dx \geq \int_0^{\frac{\pi}{2p}} G_1^*(x)G_2(x)f^*(x)dx. \tag{17}$$

From Step 2, we observe:

$$G_1(x)G_2(x) \geq 0,\ x \in [0, \frac{\pi}{2p}].$$



Using 16, one has

$$\int_0^{\frac{\pi}{2p}} G_1(x)G_2(x)f(x)dx \geq \int_0^{\frac{\pi}{2p}} G_1(x)G_2(x)f^*(x)dx. \quad (18)$$

Let

$$S_1(x) = G_2(x)f^*(x). \quad (19)$$

Using 12 and 14, it follows that

$$\begin{aligned}\int_0^{\frac{\pi}{2p}} G_1(x)S_1(x)dx &= \int_0^{\frac{\pi}{2p}} S_1(x)[G_1^* - (G_1^* - G_1)]dx \\ &= \int_0^{\frac{\pi}{2p}} S_1(x)G_1^*(x)dx - \int_0^{\frac{\pi}{2p}} S_1(x)d\Phi_1(x) \\ &= \int_0^{\frac{\pi}{2p}} G_1^*(x)S_1(x)dx + \int_0^{\frac{\pi}{2p}} \Phi_1(x)dS_1(x).\end{aligned}$$

Since $dS_1 \geq 0$, and from Step 3, we infer that

$$\int_0^{\frac{\pi}{2p}} \Phi_1(x)dS_1(x) \geq 0.$$

Thus

$$\int_0^{\frac{\pi}{2p}} G_1(x)S_1(x)dx \geq \int_0^{\frac{\pi}{2p}} G_1^*(x)S_1(x)dx. \quad (20)$$

Hence, from 18, 19 and 20, we get the required inequality 17.
Now, we prove the main inequality 9. From step 2, we have

$$k_n(x), k_1(x)k_2(x)...k_{n-1}(x) \in M.$$

Thus, using 17, one has

$$\int_0^{\frac{\pi}{2p}} k_1(x)...k_{n-1}(x)k_n(x)f(x)dx \geq \int_0^{\frac{\pi}{2p}} k_1(x)...k_{n-1}(x)k_n^*(x)f^*(x)dx.$$

Again, $k_{n-1}(x), k_1(x)...k_{n-2}(x)k_n^*(x) \in M$. Hence, from 20 it follow that

$$\int_0^{\frac{\pi}{2p}} k_1(x)...k_{n-1}(x)k_n^*(x)f^*(x)dx \geq \int_0^{\frac{\pi}{2p}} k_1(x)...k_{n-1}^*(x)k_n^*(x)f^*(x)dx$$

Repeating the above argument and using 20 each time, then from 11 and 15 one obtains:

$$\begin{aligned}\int_0^{\frac{\pi}{2p}} k_1(x)...k_n(x)f(x)dx &\geq \int_0^{\frac{\pi}{2p}} k_1^*(x)...k_n^*(x)f^*(x)dx \\ &= \frac{2^n f(0)}{(\frac{\pi}{2p})^{2n}}(\prod_{r=1}^{n}\alpha_r)\int_0^{\frac{\pi}{2p}} x^n \cosh pxdx.\end{aligned}$$

Using the following substitution

$$z = px,$$

we get

$$\begin{aligned}\int_0^{\frac{\pi}{2p}} f(x)\prod_{r=1}^{n}k_r(x)dx &\geq 2^n(\frac{2p}{\pi})^{2n}f(0)(\prod_{r=1}^{n}\alpha_r)\frac{1}{p^{n+1}}\int_0^{\frac{\pi}{2}} z^n\cosh zdz \\ &= \frac{2^n}{n+1}f(0)(\frac{2p}{\pi})^{2n}(\prod_{r=1}^{n}\alpha_r)(\frac{\pi}{2})^{1+n}\ {}_1F_2(\frac{1}{2}+\frac{n}{2};\frac{1}{2},\frac{3}{2}+\frac{n}{2};\frac{\pi^2}{16}) \\ &= \frac{2^{2n-1}}{n+1}f(0)(\frac{p}{\pi})^{n-1}(\prod_{r=1}^{n}\alpha_r){}_1F_2(\frac{1}{2}+\frac{n}{2};\frac{1}{2},\frac{3}{2}+\frac{n}{2};\frac{\pi^2}{16}),\end{aligned}$$

and the theorem is proved. We obtain equality if

$$f(x) = f(0)\cosh px,\ k_r(x) = \frac{2\alpha_r x}{(\frac{\pi}{2p})^2},\ r = 1, 2, ..., n.$$

□

References

[1] Beckenbach, E. F., Generalized convex functions, *Bulletin of the American Mathematical Society*, vol. 43, no. 6, (1937), 363-371.

[2] Beckenbach E. F. and Bing, R. H., On generalized convex functions, *Transactions of the American Mathematical Society*, vol. 58, (1945) , 220-230.

[3] Ali, Mohamed S. S., On certain properties for two classes of generalized convex functions,. *Abstract and Applied Analysis*, vol. 2016, (2016) , Article ID 4652038, 7 pages.

[4] Andersson B. J., An inequality for Convex Functions, *Nordisk Matematisk Tidskrift.*, 6, (1968), 25-26.

[5] Roberts A. W. and Varberg, D. E., *Convex Functions*, Academic Press, London, Uk, 1973.

[6] Daalhuis, Olde and Adri, B., Hypergeometric function, in Olver, Frank W. J.; Lozier, Daniel M.; Boisvert, Ronald F.; Clark, Charles W., NIST Handbook of Mathematical Functions, Cambridge University Press, 2010.

Current address: Nashat Faried: Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt.

E-mail address: nashatfaried@sci.asu.edu.eg & n_faried@hotmail.com

ORCID Address: http://orcid.org/0000-0002-9593-9699

Current address: Mohamed S. S. Ali: Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt.

E-mail address: mss_ali5@yahoo.com

ORCID Address: http://orcid.org/0000-0002-2219-6038

Current address: Zeinab M. Yehia: Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt.

E-mail address: zeinabyehia@edu.asu.edu.eg

ORCID Address: https://orcid.org/0000-0002-0263-9049