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DETERMINATION OF A TIME-DEPENDENT POTENTIAL IN A RAYLEIGH-LOVE EQUATION WITH NON-CLASSICAL BOUNDARY CONDITION

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Abstract. Mathematical model of the longitudinal vibration of bars includes higher-order derivatives in the equation of motion under considering the effect of the lateral motion of a relatively thick bar. This paper considers such an inverse coefficient problem of determining time-dependent potential of a linear source together with the unknown longitudinal displacement from a Rayleigh-Love equation (containing the fourth-order space derivative) by using an additional measurement. Existence and uniqueness theorem of the considered inverse coefficient problem is proved for small times by using contraction principle.

1. Introduction

Longitudinal vibrations of elastic bars are often regarded as the classical model in mathematical physics which is described by the second order wave equation under the consideration that the bar is thin and relatively long. More general theories have been formulated considering the effect of lateral movement of a relatively thick rod. If the order is higher than two (pseudo-hyperbolic equation), the equations of longitudinal vibrations can be obtained by taking into account the effects of the lateral motion by which cross section of a long and relatively thick bar becomes variable. Rayleigh [17] and later Love [9] proposed the simplest generalization of the classical forced free model by including the effects of the lateral motion as

$$\rho S\frac{\partial^2 u}{\partial t^2}-\rho\eta^2 I\frac{\partial^4 u}{\partial t^2\partial x^2}-ES\frac{\partial^2 u}{\partial x^2}=0 \tag{1}$$

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where $u = u(x,t)$ is the longitudinal displacement of the rod, ρ is the mass of density, E is the Young modulus, S is the cross sectional rod area, η is the Poisson's coefficient and I is the axial moment of inertia. Since in some papers as [2, 10] the equation (1) is called Boussinesq-Love equation, we call the equation (1) as Rayleigh-Love equation [3].

In this paper, we consider the forced Rayleigh-Love equation with the coefficients in front of the derivatives are equal to 1 for simplicity as

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^4 u}{\partial t^2 \partial x^2} - \frac{\partial^2 u}{\partial x^2} = a(t)u + f(x,t),\ 0 \leq x \leq 1,\ 0 \leq t \leq T, \tag{2}$$

where $a(t)$ is the time-dependent potential and $f(x,t)$ is is continuously distributed transverse force.

For a given function $a(t)$, $0 \leq t \leq T$ the problem which consists the Equation (2) and the initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x),\ 0 \leq x \leq 1, \tag{3}$$

and the boundary conditions

$$u(0,t) = 0,\ u_x(1,t) + du_{xx}(1,t) = 0,\ d > 0,\ 0 \leq t \leq T, \tag{4}$$

for the unknown function $u(x,t)$ is called direct (forward) problem.

The second part of the equation (4) which contains the term $u_{xx}(1,t)$ is called non-classical boundary condition.

Existence and uniqueness of a weak solution of the initial-boundary value problem for a pseudo-hyperbolic equation with non-local boundary conditions is studied in [13] and with dynamical non-local condition in [14]. Moreover, theory of free and forced vibrations of a rigid rod based on the Rayleigh model are investigated in [4].

If the function $a(t)$, $0 \leq t \leq T$ is unknown, finding the pair of solution $\{a(t), u(x,t)\}$ of the problem (2)-(4) with an additional condition

$$u(x_0,t) = h(t),\ x_0 \in (0,1),\ 0 \leq t \leq T, \tag{5}$$

is called inverse problem. The inverse problems for the second order wave equation with different boundary conditions and space dependent coefficients are studied in [11,15] and more recently in [5,6,12]. The inverse problem for the wave equation with time dependent coefficient is investigated in [1] and the time-dependent source function of a time-fractional wave equation with integral condition in a bounded domain is determined in [16]. Nevertheless the inverse problems for the second order wave equation are examined satisfactorily, studies on inverse problems for the pseudo-hyperbolic equations are scarce. The solvability of the problem of determining an unknown coefficient for the fourth-order pseudo-hyperbolic equation is theoretically studied in [18] and the sixth-order linear Boussinesq type equation is theoretically and numerically investigated in [21].

In this paper, we consider an initial boundary value problem for a forced Rayleigh-Love with non-classical boundary condition. Aim of this work is to determine the time-dependent potential together with the unknown longitudinal displacement and prove the existence and uniqueness theorem for small T by using an additional measurement of the displacement at (x_0, t).

The paper is organized as follows: In Section 2, we present auxiliary spectral problem of this problem and its properties. In Section 3, we transform the inverse problem (2)-(5) to a fixed-point system and prove the existence and uniqueness of a solution on a sufficiently small time interval by means of the contraction principle.

2. Auxiliary Spectral Problem

The function a is space independent and the boundary conditions are linear and homogeneous. Thus the method of separation of variables is suitable for the problem (2)-(4). The auxiliary spectral problem of the problem (2)-(4) is

$$\begin{aligned} &X''(x) + \lambda X(x) = 0,\ 0 \leq x \leq 1, \\ &X(0) = 0,\ X'(1) - d\lambda X(1) = 0. \end{aligned} \tag{6}$$

This spectral problem arises in many boundary value problems of mathematical modelling. For instance, the problems on vibrations of a homogeneous loaded string, torsional vibrations of a rod with a pulley at one end, heat propagation in a rod with lumped heat capacity at one end, and the current in a cable grounded at one end through a concentrated capacitance or inductance lead to this spectral problem , see [19], [20].

The problem (6) is considered in [7] and has eigenfunctions

$$X_n(x) = \sqrt{2}\sin\left(\sqrt{\lambda_n}x\right),\ n = 0, 1, 2, ... \tag{7}$$

with positive eigenvalues λ_n determined from the equation

$$\cot\sqrt{\lambda} = d\sqrt{\lambda}.$$

The zero index is assigned to an arbitrary eigenfunction. The remaining eigenfunctions are numbered increasing order of eigenvalues. This characteristic equation has any roots outside the positive part of the real axis on the complex plane. The estimate

$$0 < \sqrt{\lambda_n} - \pi n < \frac{1}{d\pi n}$$

is valid starting from some index N.

The system $X_n(x)$, $n=1,2,...$ is biorthogonal to the system

$$Y_n(x)=\frac{\sqrt{2}}{1+d\sin^2\sqrt{\lambda_n}}\left[\sin(\sqrt{\lambda_n}x)-\frac{\sin\sqrt{\lambda_n}}{\sin\sqrt{\lambda_0}}\sin(\sqrt{\lambda_0}x)\right],\ n=1,2,...$$

and the system $X_n(x)$, $n=1,2,...$ forms a Riesz basis in $L_2\,[0,1]$. Also, the system $Y_n(x)$, $n=1,2,...$ is a Riesz basis in $L_2\,[0,1]$ and is complete.

Let the pair $\{a(t),u(x,t)\}$ which belongs to the class $C[0,T]\times C^4(\overline{D}_T)$ satisfies the conditions (3)-(5). Then the pair $\{a(t),u(x,t)\}$ is called a classical solution of the inverse problem (2)-(5). The uniformly convergence of the Fourier series expansion in terms of the system $X_n(x)$, $n=1,2,...$ is important for the classical solution of the inverse problem (2)-(5).

Lemma 1 ([7]). *Let the function $g(x)\in C[0,1]$ and*

$$g(0)=0,\ g(1)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 g(x)\sin(\sqrt{\lambda_0}x)dx=0$$

is satisfied. Then this function can be expanded in a Fourier series in terms of the system $X_n(x)$, $n=1,2,...$ and this expansion is uniformly convergent on $[0,1]$.

Let us introduce the functional space

$$\begin{aligned} B^{3/2}_{2,T} \;=\; & \Bigg\{u(x,t)=\sum_{n=1}^{\infty}u_n(t)X_n(x):\ u_n(t)\in C[0,T], \\ & J_T(u)=\left[\sum_{n=1}^{\infty}\left(\lambda_n^{3/2}\left\|u_n(t)\right\|_{C[0,T]}\right)^2\right]^{1/2}<+\infty\Bigg\}\end{aligned}$$

with the norm $\|u(x,t)\|_{B^{3/2}_{2,T}}\equiv J_T(u)$ which relates the Fourier coefficients of the function $u(x,t)$ by the eigenfunctions $X_n(x)$, $n=1,2,....$ It is shown in [8] that $B^{3/2}_{2,T}$ is Banach space. Obviously $E^{3/2}_T=B^{3/2}_{2,T}\times C[0,T]$ with the norm $\|z\|_{E^{3/2}_T}=\|u(x,t)\|_{B^{3/2}_{2,T}}+\|a(t)\|_{C[0,T]}$ is also Banach space.

3. Solution of the Inverse Problem

Let $a(t)$, $t\in[0,T]$ is an unknown function. Since the function $a(t)$ is time dependent, seeking the solution of the problem (2)-(5) in the following form is suitable:

$$u(x,t)=\sum_{n=1}^{\infty}u_n(t)X_n(x) \tag{7}$$

where $u_n(t)=\int_0^1 u(x,t)Y_n(x)dx$, $k=1,2,....$

From the equation (2) and initial condition (3), we obtain

$$\begin{cases} (1+\lambda_n)u_n''(t)+\lambda_n u_n(t)=F_n(t;a,u), \\ u_n(0)=\varphi_n, \quad u_n'(0)=\psi_n, \end{cases} \quad n=1,2,.. \tag{8}$$

where $F_n(t;a,u)=a(t)u_n(t)+f_n(t)$, $f_n(t)=\int_0^1 f(x,t)Y_n(x)dx$, $\varphi_n=\int_0^1 \varphi(x)Y_n(x)dx$, $\psi_n=\int_0^1 \psi(x)Y_n(x)dx$, $n=1,2,....$
Solving the problem (8), we get

$$u_n(t)=\varphi_n\cos(\beta_n t)+\frac{1}{\beta_n}\psi_n\sin(\beta_n t)+\frac{1}{\beta_n}\int_0^t F_n(\tau;a,u)\sin(\beta_n(t-\tau))\,d\tau \tag{9}$$

where $\beta_n=\sqrt{\frac{\lambda_n}{1+\lambda_n}}$.

Substituting (9) into (7), the second component of the pair $\{a(t),u(x,t)\}$ is

$$u(x,t)=\sum_{n=1}^{\infty}\left[\varphi_n\cos(\beta_n t)+\frac{1}{\beta_n}\psi_n\sin(\beta_n t)\right.$$
$$\left.+\frac{1}{\beta_n}\int_0^t F_n(\tau;a,u)\sin(\beta_n(t-\tau))\,d\tau\right]X_n(x) \tag{10}$$

Using the additional condition (5), from the equation (2) we get

$$a(t)=\frac{1}{h(t)}\left[h''(t)-f(x_0,t)+\sum_{n=1}^{\infty}(\lambda_n u_n''+\lambda_n u_n)\sin\sqrt{\lambda_n}x_0\right].$$

Since $\lambda_n u_n''+\lambda_n u_n=F_n(t;a,u)-u_n''$, we obtain the first component of the pair as

$$a(t)=\frac{1}{h(t)}\left[h''(t)-f(x_0,t)+\sum_{n=1}^{\infty}\big(F_n(t;a,u)+\varphi_n\beta_n^2\cos(\beta_n t)\right.$$
$$\left.+\beta_n\psi_n\sin(\beta_n t)+\beta_n\int_0^t F_n(\tau;a,u)\sin(\beta_n(t-\tau))\,d\tau\big)\sin\sqrt{\lambda_n}x_0\right] \tag{11}$$

by using the equality (9).

Thus, the solution of the inverse problem (2)-(5) is reduced to the solution of system (10)-(11) with respect to the unknown functions $\{a(t),u(x,t)\}$.

Let us present $z=[a(t),u(x,t)]^T$ and investigate the existence of a unique solution of the operator equation

$$z=\Phi(z). \tag{12}$$

The operator Φ is determined in the set of functions z and has the form $[\phi_1,\phi_2]^T$, where

$$\begin{aligned}\phi_1(z)=\frac{1}{h(t)}\Bigg[h''(t)-f(x_0,t)+\sum_{n=1}^{\infty}\big(F_n(t;a,u)+\varphi_n\beta_n^2\cos(\beta_n t)\\ +\beta_n\psi_n\sin(\beta_n t)+\beta_n\int_0^t F_n(\tau;a,u)\sin(\beta_n(t-\tau))\,d\tau\Big)\sin\sqrt{\lambda_n}x_0\Bigg],\end{aligned} \tag{13}$$

$$\begin{aligned}\phi_2(z)=\sum_{n=1}^{\infty}\Bigg[\varphi_n\cos(\beta_n t)+\frac{1}{\beta_n}\psi_n\sin(\beta_n t)\\ +\frac{1}{\beta_n}\int_0^t F_n(\tau;a,u)\sin(\beta_n(t-\tau))\,d\tau\Bigg]X_n(x).\end{aligned} \tag{14}$$

We need to verify that Φ maps $E_T^{3/2}$ onto itself continuously. In other words, we have to demonstrate $\phi_1(z)\in C[0,T]$ and $\phi_2(z)\in B_{2,T}^{3/2}$ for arbitrary $z=[a(t),u(x,t)]^T$ with $a(t)\in C[0,T]$, $u(x,t)\in B_{2,T}^{3/2}$.

We will use the following assumptions on the data of problem (2)-(5):

$$(A_1)\textbf{:}\ \varphi(x)\in C^3[0,1],\ \begin{cases}\varphi(0)=\varphi''(0)=0,\ \varphi'(1)+d\varphi''(1)=0,\\ \varphi(1)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1\varphi(x)\sin(\sqrt{\lambda_0}x)dx=0,\end{cases},$$

$$(A_2)\textbf{:}\ \psi(x)\in C^2[0,1],\ \begin{cases}\psi(0)=\psi''(0)=0,\ \psi'(1)+d\psi''(1)=0,\\ \psi(1)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1\psi(x)\sin(\sqrt{\lambda_0}x)dx=0,\end{cases},$$

$$(A_3)\textbf{:}\ \begin{cases}f(x,t)\in C(\overline{D}_T),f_x,f_{xx},f_{xxx}\in C[0,1],\forall t\in[0,T]\\ f(0,t)=f_{xx}(0,t)=0,\ f_x(1,t)+df_{xx}(1,t)=0,\\ f(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 f(x,t)\sin(\sqrt{\lambda_0}x)dx=0,\end{cases},$$

(A_4)**:** $h(t) \in C^2[0,T]$, $h(0) = \varphi(x_0)$, $h'(0) = \psi(x_0)$, $h(t) \neq 0$.

By using integration by parts under the assumptions (A$_1$)-(A$_4$), it easy to see that

$$\begin{aligned}\varphi_n &= \frac{1}{\lambda_n^{3/2}} \frac{-\sqrt{2}}{1 + d\sin^2\sqrt{\lambda_n}} \int_0^1 \varphi'''(x)\cos(\sqrt{\lambda_n}x)dx,\\ \psi_n &= \frac{1}{\lambda_n} \frac{-\sqrt{2}}{1 + d\sin^2\sqrt{\lambda_n}} \int_0^1 \varphi''(x)\sin(\sqrt{\lambda_n}x)dx,\\ f_n(t) &= \frac{1}{\lambda_n^{3/2}} \frac{-\sqrt{2}}{1 + d\sin^2\sqrt{\lambda_n}} \int_0^1 f_{xxx}(x,t)\cos(\sqrt{\lambda_n}x)dx.\end{aligned}$$

From these equalities, we have

$$\begin{array}{c}\sum_{n=1}^{\infty} |\varphi_n| \leq C_2 \left\|\varphi'''(x)\right\|_{L_2[0,1]},\\ \sum_{n=1}^{\infty} |\psi_n| \leq C_1 \left\|\psi''(x)\right\|_{L_2[0,1]},\\ \sum_{n=1}^{\infty} |f_n(t)| \leq C_2 \left\|f_{xxx}(x,t)\right\|_{L_2(D_T)},\end{array} \tag{15}$$

by using Cauchy-Schwartz inequality and Bessel inequality where $C_1 = \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2}\right)^{1/2}$ and $C_2 = \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^3}\right)^{1/2}$.

First, let us show that $\phi_1(z) \in C[0,T]$. Under the assumptions (A$_1$)-(A$_4$), considering the estimates (15) and $\frac{1}{\sqrt{2}} < \beta_n < 1$, we obtain from (13)

$$\max_{0\leq t\leq T} |\phi_1(t)| \leq R_1(T) + R_2(T) \left\|a(t)\right\|_{C[0,T]} \left\|u(x,t)\right\|_{B_{2,T}^{3/2}} \tag{16}$$

where $R_1(T) = \frac{1}{\|h(t)\|_{C[0,T]}}(\|h''(t)\|_{C[0,T]} + \|f(x_0,t)\|_{C[0,T]} + C_2 \|\varphi'''(x)\|_{L_2[0,1]} + C_1 \left\|\psi''(x)\right\|_{L_2[0,1]} + C_2(1+T) \|f_{xxx}(x,t)\|_{L_2(D_T)}))$, $R_2(T) = \frac{C_2(1+T)}{\|h(t)\|_{C[0,T]}}$. Since the right hand side is bounded, $\phi_1(z)$ is continuous in $[0,T]$.

Now, let us show that $\phi_2(z) \in B_{2,T}^{3/2}$, i.e. we need to show

$$J_T(\phi_2) = \left[\sum_{n=1}^{\infty} \left(\lambda_n^{3/2} \left\|\phi_{2n}(t)\right\|_{C[0,T]}\right)^2\right]^{1/2} < +\infty,$$

where

$$\phi_{2n}(t) = \varphi_n \cos(\beta_n t) + \frac{1}{\beta_n}\psi_n \sin(\beta_n t) + \frac{1}{\beta_n}\int_0^t F_n(\tau; a, u)\sin(\beta_n(t-\tau))\,d\tau.$$

After some manipulations under the assumptions (A$_1$)-(A$_4$), we get

$$\left[\sum_{n=1}^{\infty}\left(\lambda_n^{3/2}\left\|\phi_{2n}(t)\right\|_{C[0,T]}\right)^2\right]^{1/2} \le \widetilde{R}_1(T)+\widetilde{R}_2(T)\left\|a(t)\right\|_{C[0,T]}\left\|u(x,t)\right\|_{B^3_{2,T}} \quad (17)$$

where $\widetilde{R}_1(T) = 4\left\|\varphi'''(x)\right\|_{L_2[0,1]} + 4\sqrt{2}\left\|\psi''(x)\right\|_{L_2[0,1]} + 4\sqrt{2}T\left\|f_{xxx}(x,t)\right\|_{L_2(D_T)}$, $\widetilde{R}_2(T) = 4\sqrt{2}TC_2$.

Thus $J_T(\phi_2) < +\infty$ and ϕ_2 is belongs to the space $B^{3/2}_{2,T}$.

Now, let z_1 and z_2 be any two elements of $E^{3/2}_T$. We know that $\left\|\Phi(z_1)-\Phi(z_2)\right\|_{E^{3/2}_T} = \left\|\phi_1(z_1)-\phi_1(z_2)\right\|_{C[0,T]} + \left\|\phi_2(z_1)-\phi_2(z_2)\right\|_{B^{3/2}_{2,T}}$. Here $z_i = [a^i(t), u^i(x,t)]^T$, $i = 1,2$.

Under the assumptions (A_1)-(A_4) and considering (16)-(17), we obtain

$$\left\|\Phi(z_1)-\Phi(z_2)\right\|_{E^{3/2}_T} \le A(T)C(a^1,u^2)\left\|z_1-z_2\right\|_{E^{3/2}_T}$$

where $A(T) = C_2\left(\frac{(1+T)}{\left\|h(t)\right\|_{C[0,T]}} + 4\sqrt{2}T\right)$ and $C(a^1,u^2)$ is the constant includes the norms of $\left\|a^1(t)\right\|_{C[0,T]}$ and $\left\|u^2(x,t)\right\|_{B^{3/2}_{2,T}}$.

$A(T)$ has limit zero as T tends to zero. Thus the operator Φ is contraction mapping which maps $E^{3/2}_T$ onto itself continuously for sufficient small T. According to the Banach fixed point theorem the solution of the operator equation (12) exists and unique.

Thus, we proved the following theorem:

Theorem 2 (Existence and uniqueness)**.** *Let the assumptions (A_1)-(A_4) be satisfied. Then, the inverse problem (2)-(5) has unique solution for small T.*

4. Conclusion

The inverse problems for pseudo-hyperbolic equations connected with recovery of the coefficient are scarce. The paper considers the problem of determining the time-dependent coefficient for the pseudo-hyperbolic equation with homogeneous boundary conditions and an additional measurement. The existence and uniqueness of a solution on a sufficiently small time interval are proved by means of the contraction principle. The fixed-point system is presented via Fourier series. Such a form of the system brings along computations that are technically simpler than the system in the case of the usual variational approach. The numerical method of the inverse problem (2)-(5) will be considered with a suitable finite difference scheme discretization as a future work.

Declaration of Competing Interests. The author has no competing interests to declare.

References

[1] Aliev, Z. S., Mehraliev, Y. T., An inverse boundary value problem for a second-order hyperbolic equation with nonclassical boundary conditions, *Doklady Mathematics*, 90(1) (2014), 513-517. https://doi.org/10.1134/S1064562414050135

[2] Chen, G., Shubin W., Existence and nonexistence of global solutions for the generalized IMBq equation, *Nonlinear Analysis: Theory, Methods & Applications* 36(8) (1999), 961-980. https://doi.org/10.1016/S0362-546X(97)00710-4

[3] Fedotov, I., Shatalov, M., Marais, J., Hyperbolic and pseudo-hyperbolic equations in the theory of vibration, *Acta Mechanica*, 227(11) (2016), 3315-3324. https://doi.org/10.1007/s00707-015-1537-6

[4] Fedotov, I. A., Polyanin, A. D., Shatalov, M. Yu., Theory of free and forced vibrations of a rigid rod based on the Rayleigh model, *Doklady Physics*, 52 (2007) 607-612. https://doi.org/10.1134/S1028335807110080

[5] Imanuvilov, O., Yamamoto, M., Global uniqueness and stability in determining coefficients of wave equations, *Comm. Part. Diff. Equat.*, 26 (2001), 1409-1425. https://doi.org/10.1081/PDE-100106139

[6] Isakov, V., Inverse Problems for Partial Differential Equations. Applied Mathematical Sciences, Springer, New York (NY), 2006.

[7] Kapustin, N. Yu, Evgenii, I. M., A remark on the convergence problem for spectral expansions corresponding to a classical problem with spectral parameter in the boundary condition, *Differential Equations*, 37(12) (2001), 1677-1683. https://doi.org/10.1023/A:1014406921176

[8] Khudaverdiyev, K. I., Alieva, A. G., On the global existence of solution to one-dimensional fourth order nonlinear Sobolev type equations, *Appl. Math. Comput.*, 217(1) (2010), 347-354. https://doi.org/10.1016/j.amc.2010.05.067

[9] Love, A. E. H., A Treatise on the Mathematical Theory of Elasticity. Cambridge University Press, 2013.

[10] Mehraliyev, Y. T., Huseynova, A. F., On solvability of an inverse boundary value problem for pseudo hyperbolic equation of the fourth order, *Journal of Mathematics Research*, 7.2 (2015), 101-109. http://dx.doi.org/10.5539/jmr.v7n2p101

[11] Namazov, G. K., Inverse Problems of the Theory of Equations of Mathematical Physics, Baku, Azerbaijan, 1984. (in Russian).

[12] Prilepko, A. I., Orlovsky, D. G., Vasin, I. A., Methods for Solving Inverse Problems in Mathematical Physics, CRC Press, 2000.

[13] Pul'kina, L., A problem for a -pseudohyperbolic equation with nonlocal boundary condition, *Journal of Mathematical Sciences*, 219(2) (2016), 245-252. https://doi.org/10.1007/s10958-016-3102-9

[14] Pul'kina, L., A problem with dynamic nonlocal condition for pseudohyperbolic equation, *Russian Mathematics*, 60(9) (2016), 38-45. https://doi.org/10.3103/S1066369X16090048

[15] Romanov, V.G., Inverse Problems of Mathematical Physics, VNU Science Press BV, Utrecht, Netherlands, 1987.

[16] Šišková, K., Slodička, M., Recognition of a time-dependent source in a time-fractional wave equation, *Applied Numerical Mathematics*, 121 (2017), 1-17. https://doi.org/10.1016/j.apnum.2017.06.005

[17] Strutt, J. W., Rayleigh B. The Theory of Sound. Dover, 1945.

[18] Tekin, I., Reconstruction of a time-dependent potential in a pseudo-hyperbolic equation, *UPB Scientific Bulletin-Series A-Applied Mathematics and Physics*, 81(2) (2019), 115-124.

[19] Tikhonov, A. N., Samarskii, A. A., Equations of Mathematical Physics, Moscow, 1972. (in Russian)

[20] Vladimirov, V. S., Collection of Tasks on Equations of Mathematical Physics. Moscow: Nauka, 1982. (in Russian)

[21] Yang, H., An inverse problem for the sixth-order linear Boussinesq-type equation, *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, 82(2) (2020), 27-36.