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Some Theorems on Compactness and Completeness

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Abstract
In this work, we prove the validity of the converses of some theorems about compactness and completeness. After we give some required basic definitions and theorems, we define monolimit property for sequences and nets, convergent subsequences property for first countable Hausdorff space, convergent subnets property for general Hausdorff space, and also, we show that those properties are equivalent to compactness and sequential compactness. On the other hand, we prove that a metric space is complete iff every totally bounded subset of it is relatively compact. Finally, we give some examples from some abstract spaces and normed spaces for application.

Keywords: Topology, Compactness, Completeness, Sequence, Net, Convergence.

Kompaktlık ve Tamlık Üzerine Bazı Teoremler

Öz
Bu çalışmada, kompaktlık ve tamlık konularındaki bazı teoremlerin terslerinin de doğru olduğunu ispatlayacağız. Çalışma için bize gerekli olan temel tanım ve teoremlere değindikten sonra, diziler ve ağlar için tek limit özelliği, birinci sayılabilir Hausdorff uzaylar için yakınsak alt diziler özelliği, genel Hausdorff uzaylar için ise yakınsak alt ağlar özelliğini tanımlayacağız ve bu özelliklerin kompaktlığa ve dizisel kompaktlığa denk olduğunu göstereceğiz. Bunu yanı sıra, bir metrik uzayın tam olması için bir gerek ve yeter koşulun o metrik uzaydaki tamemen sınırlı her alt kümenin rölatif kompakt olması olduğunu ispatlayacağız. Son olarak ispatladığımız teoremlerin uygulaması için bazı soyut uzaylardan ve bazı normlu uzaylardan örnekler vereceğiz.

Anahtar kelimeler: Topoloji, Kompaktlık, Tamlık, Diziler, Ağlar, Yakınsaklık.

1. Introduction

In topology, some theorems were proved as necessity or sufficiency. However, the converses of them have not been studied yet, especially on compactness and completeness. For example, if every convergent subsequence of a sequence in a sequentially compact space has the same limit, then the original sequence is convergent, see [3]. In this paper, we consider this theorem and we named this property as monolimit property. Also, we define monolimit property for nets in topological spaces and prove that it is necessary and sufficient for sequential compactness.

In the next section, we emphasize on completeness. We know that totally boundedness implies relative compactness in a complete metric space, see [1]. We show that converse of this fact is valid, i.e. a metric space X is complete iff every totally bounded subset of that space is relatively compact.

Finally, we give some examples as application for theorems that we prove.

2. Preliminaries

In this section, we give some required definitions and theorems.

A topology $\mathcal{F}$ over a nonempty set Y is a colloction of some subsets of Y satisfying 3 condition: 1) Emptyset and Y is in $\mathcal{F}$, 2) Any union of subcollection of $\mathcal{F}$ is in $\mathcal{F}$, 2) Any intersection of two sets

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of $\mathcal{F}$ is in $\mathcal{F}$. If a collection $\mathcal{F}$ is a topology on Y, $(Y, \mathcal{F})$ is said to be a topological space. The sets of $\mathcal{F}$ is called opens [2].

Let $\mathcal{F}$ be a topology on Y, $a \in O$ and $O \in \mathcal{F}$. Then, O is said to be a neighborhood of a [2].

$(Y, \mathcal{F})$ is called a Hausdorff topological space if there exist two disjoint neighborhood of any distinct point a and b in Y [2].

$(Y, \mathcal{F})$ is said to be a first countable topological space if there exists countable collection of neighborhoods of any point a such that every neighborhood of a includes a member of that countable collection [3].

Let Y be a set and $d: Y \times Y \to \mathbb{R}$ be a function that satisfies following 3 conditions: 1) $d(a,b) = 0 \Leftrightarrow a = b$, 2) $d(a,b) = d(b,a)$, 3) $d(a,b) \leq d(a,c) + d(c,b)$. Then d is called a metric on Y and The pair (Y, d) is called a metric space [1].

Let (Y, d) be a metric space, $a \in Y$ and $\varepsilon > 0$. Then, the set $U_\varepsilon(a) = \{t \in Y | d(t,a) < \varepsilon\}$ is said to be an open ball with center x and radius ε [1].

Let (Y, d) be a metric space and $A \subset Y$. Then A is called an open set provided there exists a positive number ε_a satisfying the inclusion $U_{\varepsilon_a}(a) \subset A$ for each $a \in A$ [1].

All the open sets of a metric space form a topology. In this case, the topological space generated by given metric is said to be a metrizable space [1].

A metrizable space is both Hausdorff and first countable [2].

Let Y be a set. Then the metric defined by

$$\rho(a,b) = \begin{cases} 0, & a = b, \\ 1, & a \neq b \end{cases}$$

is called the discrete metric on Y [1].

The topology generated by that metric is the power set $\mathcal{P}(Y)$ and it is called the discrete topology on Y [2].

A function defined on $\mathbb{N}$ to Y is called a sequence, where Y is a metric or topological space. A sequence is denoted by (a_n) [1].

A sequence (a_n) is called convergent to a point a in a metric space (Y, d) provided that for each $\varepsilon > 0$, there exists a natural number n_ε such that $d(a_n, a) < \varepsilon$ for every $n > n_\varepsilon$ [1].

A sequence (a_n) is called a Cauchy sequence in a metric space (Y, d) provided that for each $\varepsilon > 0$, there exists a natural number n_ε such that $d(a_n, a_m) < \varepsilon$ for every $n, m > n_\varepsilon$ [1].

In a metric space, any convergent sequence is a Cauchy sequence [1].

A complete space is a metric space that every Cauchy sequence in it is convergent [1].

A topological space is called a compact space if every open cover contains a finite subcover [2].

A topological space is called a sequentially compact space if every sequence contains a convergent subsequence [2].

Let (Y, d) be a metric space, $\varepsilon > 0$ and $a, a_2, \ldots, a_n \in Y$. Then, the finite collection $\{U_\varepsilon(a_1), U_\varepsilon(a_2), \ldots, U_\varepsilon(a_n)\}$ is called an ε-net provided $Y \subset \bigcup_{k=1}^{n} U_\varepsilon(a_k)$. (Y, d) is called totally bounded if it has ε-net for every $\varepsilon > 0$ [1].

Let $(Y, \mathcal{F})$ be a topological space and $A \subset Y$. A is called a relatively compact set if $\overline{A}$ is compact [1].

Let D be a set and $\leq$ be a relation on D. Then, D is said to be a directed set if the relation $\leq$ satisfies following 3 conditions: 1) $\mu \leq \mu$ for each $\mu \in D$, 2) $\mu_1 \leq \mu_2$ and $\mu_2 \leq \mu_3 \Rightarrow \mu_1 \leq \mu_3$, 3) for each μ_1 and μ_2, there exists $\mu \in D$ satisfying $\mu_1 \leq \mu$ and $\mu_2 \leq \mu$ [2].

A net is a function defined on a directed set to a topological space and it is denoted by $(x_\mu)_{\mu \in D}$ [2].

A net $(x_\mu)_{\mu \in D}$ is said to be convergent to a point a in a topological space $(Y, \mathcal{F})$ if for every $O \in \mathcal{F}$ with $a \in O$, there exists $\mu_O \in D$ such that $x_\mu \in O$ for each $\mu \geq \mu_O$ [2].

Let D and E be two directed sets; $(x_\lambda)_{\lambda \in D}$ and $(y_\mu)_{\mu \in E}$ be two nets in a topological space $(Y, \mathcal{F})$. Then, $(y_\mu)_{\mu \in E}$ is called a subnet of the net $(x_\lambda)_{\lambda \in D}$ if there exists a function φ from E to D, such that $y_\mu = x_{\varphi(\mu)}$ for each $\mu \in E$ and for each $\lambda \in D$, there exists $\mu \in E$ with the property $p \geq \mu \Rightarrow \varphi(p) \geq \lambda$ [2].

3. Convergent Subnets Property

We define, in this section, the concept of convergent subnets property and prove that it is equivalent to compactness. For this, first, we define the concept of convergent subsequences property, and then, we define the concept of convergent subnets property.

3.1. Convergent Subsequences Property in A First Countable Topological Space

Definition 1. Let (a_n) be a sequence in a first countable, Hausdorff space $(Y,\mathcal{F})$. Then we state that (a_n) has monolimit property if every convergent subsequence of it has the same limit.

Definition 2. We state that a first countable, Hausdorff space $(Y,\mathcal{F})$ has convergent subsequences property provided that every sequence with the monolimit property is convergent in Y.

Remark 1. It is obvious that if a sequence has the monolimit property and contains at least one convergent subsequence converging to a point a in a space with the convergent subsequences property, then the original sequence converges to the same point a.

Remark 2. If a sequence does not contain a convergent subsequence, then it has monolimit property. Otherwise, we must find at least two subsequences of it converging two distinct points. So, a sequence not containing convergent subsequence has the monolimit property.

Example 1. Given a finite set Y equipped with a Hausdorff topology. One can easily see that the unique Hausdorff topology on a finite set is the discrete topology $\mathcal{P}(Y)$. We now show that this space has convergent subsequences property. Let (a_n) be a sequence in Y with monolimit property. We prove that it is convergent. Since Y is finite, there exists a subsequence $\left(a_{k_n}\right)$ of (a_n) and there exists a point a in Y such that $a_{k_n}=a$ for every $n\in\mathbb{N}$. On the other hand, if a point b in Y satisfies $a_{m_n}=b$ for a subsequence $\left(a_{m_n}\right)$, then $a=b$ because (a_n) has the monolimit property. Consequently, we has proven that the sequence (a_n) is eventually equal to the point a. Then, it is convergent to that point.

Conversely, we consider the discrete topology on a set Y with the convergent subsequences property. We now prove that Y is finite. Assume the contrary. Let Y be an infinite set. Then, there exists an infinite sequence (a_n) satisfying the condition $n_1\neq n_2\Rightarrow a_{n_1}\neq a_{n_2}$. It is well known that a sequence is convergent iff it is eventually equal to a point in the discrete topology on a set. So, the infinite sequence (a_n) has no convergent subsequence. By this fact, it has monolimit property. However, it does not converge a point. It is a contradiction. Then, Y must be a finite set.

Theorem 1. In order that a first countable, Hausdorff space $(Y,\mathcal{F})$ be compact, it is necessary and sufficient that it has convergent subsequences property.

Proof. For the necessity, see [3]. We only prove the sufficiency. Assume that this space has the convergent subsequences property. We show that it is compact. In a first countable space, compactness and sequential compactness are equivalent, see [3]. Then, we show that this space is sequentially compact by contradiction. Assume that $(Y,\mathcal{F})$ has the convergent subsequences property but is not sequentially compact. Then, there exists a sequence in Y with no convergent subsequence. So, it has monolimit property. By the hypotesis, it must be convergent. It is a contridiction. Consequently, the space $(Y,\mathcal{F})$ is sequentially compact, i.e., it is compact.

With the above theorem, we have the following corollary.

Corollary 1. For a first countable, Hausdorff space $(Y,\mathcal{F})$ the following propositions are equivalent:

1. Y is compact,
2. Y is sequentially compact,
3. Y is both totally bounded and complete (if Y is a metric space),
4. Y has the convergent subsequences property.

Remark 3. In Example 1, we have proven that a set equipped with the discrete topology has the convergent subsequences property if and only if it is finite. It is well known that a set equipped with the discrete topology is compact if and only if it is finite. Corollary 1 supports this fact.

3.2. Convergent Subnets Property in an Arbitrary Topological Space

In a space not satisfying first countability, sequences are not sufficient for some characterizations. For example, sequential compactness does not require compactness. That's why we will define convergent subsequences property by using nets. We will call this feature convergent subnets property later. First, we define monolimit property for nets.

Definition 3. Let D be a directed set and $(x_\mu)_{\mu\in D}$ be a net in a Hausdorff space $(Y,\mathcal{F})$. Then we state that $(x_\mu)_{\mu\in D}$ has monolimit property if every convergent subnet of it has the same limit.

Definition 4. We state that a Hausdorff space $(Y,\mathcal{F})$ has convergent subnets property provided that every net with the monolimit property is convergent in Y.

Remark 4. Remark 1 and Remark 2 is valid if we use the word "net" instead of "sequence".

Theorem 2. In order that a Hausdorff space $(Y,\mathcal{F})$ be compact, it is necessary and sufficient that it has convergent subnets property.

The above theorem can be proven by using nets instead of sequences similar to proof of Theorem 1 (for the equivalency of 1 and 2 in the following corollary, see [2]).

Corollary 1. For a Hausdorff space $(Y,\mathcal{F})$, the following propositions are equivalent:

1. Y is compact,
2. Every net in Y contains a convergent subnet,
3. Y has the convergent subnets property.

4. A Criterion for Completeness of a Metric Space

Completeness of a metric space is that every Cauchy sequence is convergent in that space. In this section, we give another criterion for completeness via compactness. Indeed, we give a relation between completeness and compactness. In a complete metric space, every totally bounded subset is relatively compact. We show that this requirement is sufficient for completeness.

Theorem 3. A metric space (Y,d) is complete if and only if every totally bounded subset of it is relatively compact.

Proof. For the necessity, see [1]. We prove the sufficiency. Assume that every totally bounded subset of Y is relatively compact. Given a Cauchy sequence (a_n) in Y and $\varepsilon>0$. Then, there exists a natural number n_ε such that $d(a_n,a_m)<\varepsilon$ for every $n,m>n_\varepsilon$. This implies that the ball $U_\varepsilon(a_{n_\varepsilon+1})$ contains the set $\{a_{n_\varepsilon+1},a_{n_\varepsilon+2},a_{n_\varepsilon+3},\ldots\}$. So, the ε-net $\bigcup_{k=1}^{n_\varepsilon+1}U_\varepsilon(a_k)$ contains all the points of the sequence (a_n). This shows us that the set S of all the terms of the sequence (a_n) is a totally bounded set in Y. By the hypotesis, S is relatively compact i.e., the closure $\overline{S}$ is compact. The sequence (a_n) is a Cauchy sequence in the compact set $\overline{S}$ and we know that every compact set is complete. Then, (a_n) is convergent in $\overline{S}\subset Y$. This ends the proof.

Example 2. The rationals $\mathbb{Q}$ is not a complete space bacause the subset of all the rational between 0 and 1 is totally bounded but is not relatively compact.

Example 3. The space $(0,1)$ is not complete. This is why the set $(0,1)$ is totally bounded but the closure is itself and it is not compact.

Example 4. Let Y be nonempty set with discrete metric. Given a totally bounded subset S. Then, the 1-net of it must be finite and contains S. This shows us that any totally bounded subset of Y must be finite. On the other hand, every finite set in a metric space is compact, so is relatively compact. Consequently, any set equipped with discrete metric is complete, by Theorem 3.

Example 5. Let Y be all the polynomials on $[0,1]$ with uniform metric

$$d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$$

and S be the subset of Y defined as follows

$$S = \left\{ \sum_{k=0}^{n} \frac{x^k}{k!} \middle| n \in \mathbb{N} \right\}.$$

Given $\varepsilon > 0$. Since the series

$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

is convergent, then there exists an integer n_ε such that

$$\sum_{k=n_\varepsilon+1}^{n} \frac{1}{k!} < \varepsilon$$

for each $n > n_\varepsilon$. So, we have

$$d\left(\sum_{k=0}^{n} \frac{x^k}{k!}, \sum_{k=0}^{n_\varepsilon} \frac{x^k}{k!} \right) = \max_{x\in[0,1]} \left| \sum_{k=0}^{n} \frac{x^k}{k!} - \sum_{k=0}^{n_\varepsilon} \frac{x^k}{k!} \right| = \max_{x\in[0,1]} \sum_{k=n_\varepsilon+1}^{n} \frac{x^k}{k!} = \sum_{k=n_\varepsilon+1}^{n} \frac{1}{k!} < \varepsilon$$

Thus, the set

$$\bigcup_{m=1}^{n_\varepsilon} U_\varepsilon \left(\sum_{k=0}^{m} \frac{x^k}{k!} \right)$$

is an ε-net of the set S. This proves that S is totally bounded in Y. On the other hand, S is also a sequence in $\overline{S}$ and $\overline{S}$ must be included in the space Y. We now show that the sequence doesn't contain a convergent subsequence in $\overline{S}$. Assume the contrary. Let a subsequence of S converge a function f in $\overline{S}$ and we denote that subsequence by

$$S^* = \left\{ \sum_{k=0}^{m_n} \frac{x^k}{k!} \middle| n \in \mathbb{N} \right\}.$$

Note that f must be a polynomial because $f \in \overline{S} \subset Y$. Since S^* is convergent to f in the sense of d, it is uniformly convergent. Then we can exchange limit and derivative:

$$f'(x) = \frac{d}{dx}\left(\lim_{n\to\infty} \sum_{k=0}^{m_n} \frac{x^k}{k!} \right) = \lim_{n\to\infty} \sum_{k=0}^{m_n} \frac{d}{dx}\left(\frac{x^k}{k!}\right) = \lim_{n\to\infty} \sum_{k=1}^{m_n} \frac{x^{k-1}}{(k-1)!} = \lim_{n\to\infty} \sum_{k=0}^{m_n-1} \frac{x^k}{k!} = f(x)$$

Then, the function f must be a solution the differential equation $y' = y$. So, there exists a constant $c \in \mathbb{R}$ such that $f(x) = ce^x$. Since f is a polynomial, then c must be 0, i.e., $f(x) = 0$. However, a positive series never converges to 0. The last shows us S doesn't contain a convergent subsequence, and so, $\overline{S}$ is not compact i.e, S is not relatively compact. Consequently, S is a totally bounded set that is not relatively compact in Y. By Theorem 3, Y is not a complete metric space.

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